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Some Results on Equi Independent Equitable Dominating Sets in Graph

S. K. Vaidya[1*], N. J. Kothari[2]

[1]Department of Mathematics, Saurashtra University, Rajkot 360005, Gujarat, India

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Abstract

A subset D of $V(G)$ is called an equitable dominating set if for every $v \in V(G)-D$, there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u)-deg(v)| \leq 1$. A subset D of $V(G)$ is called an equitable independent set if for any $u \in D$, $v \notin N^e(u)$ for all $v \in D-\{u\}$ where, $N^e(u)=\{v \in V(G) / v \in N(u), |deg(u)-deg(v)| \leq 1\}$. An equitable dominating set D is said to be an equi independent equitable dominating set if it is also an equitable independent set. The minimum cardinality of an equi independent equitable dominating set is called equi independent equitable domination number which is denoted by i^e. We investigated an equi independent equitable domination number for some special graphs.

Keywords: Equi independent equitable domination number; Equitable domination number; Domination number.

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1. Introduction

Throughout this work, the term graph we mean finite, connected, undirected and simple graph G with vertex set $V(G)$ and edge set $E(G)$. For any undefined term we rely upon West [9] and Haynes *et al*. [5]. For every vertex $v \in V(G)$ the open neighbourhood set $N(v)$ is the set of all vertices adjacent to v in G. That is, $N(v)=\{u \in V(G) / uv \in E(G)\}$. The closed neighbourhood set $N[v]$ of v is defined as $N[v]=N(v) \cup \{v\}$. A set $D \subseteq V(G)$ is called a dominating set if every vertex in $V(G)-D$ is adjacent to at least one vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. A subset D of $V(G)$ is an independent set if no two vertices in D are adjacent. A dominating set D which is also an independent set is called an independent dominating set. The independent domination number $i(G)$ is the minimum cardinality of an independent dominating set. The concept of an independent domination was formalized by Berge [2] and Ore [6] while the definition of an independent domination number and the notation

* *Corresponding author*: samirkvaidya@yahoo.co.in

$i(G)$ were introduced by Cockayne and Hedetniemi [3]. A survey on the concept of an independent domination can be found in Goddard and Henning [4] while applications of dominating sets in computer network is well studied by Basavanagoud and Hosamani [1]. A subset D of $V(G)$ is called an equitable dominating set if for every $v \in V(G)-D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|\deg(u)-\deg(v)| \leq 1$. The minimum cardinality of such an equitable dominating set is equitable domination number of G which is denoted by γ^e. A vertex $u \in V(G)$ is *degree* equitable adjacent or equitable adjacent with a vertex $v \in V(G)$ if $|\deg(u)-\deg(v)| \leq 1$ for $uv \in E(G)$. An equitable dominating set D is said to be a minimal equitable dominating set if no proper subset of D is an equitable dominating set. Swaminathan and Dharmalingam [7] have derived following necessary and sufficient condition for minimal equitable dominating set.

Theorem 1.1: An equitable dominating set D is minimal if and only if for every vertex $u \in D$ one of the following holds.

(i) Either $N(u) \cap D = \varnothing$ or $|\deg(v) - \deg(u)| \geq 2$ for all $N(u) \cap D$.

(ii) There exists a vertex $v \in V(G)-D$ such that $N(v) \cap D = \{u\}$ and $|\deg(v) - \deg(u)| \leq 1$.

A vertex $v \in V(G)$ is called equitable isolate if $|\deg(v)-\deg(u)| \geq 2$ for every $u \in N(v)$. It is obvious that, if $v \in V(G)$ is an equitable isolate and D is any equitable dominating set then $v \in D$. Obviously isolated vertices are equitable isolates. Hence $I_s \subseteq I_e \subseteq D$ for every equitable dominating set D, where I_s and I_e are the set of all isolated vertices and set of all equitable isolates of G respectively. As reported in [7], a graph G has a unique minimal equitable dominating set if and only if the set of all equitable isolates form an equitable dominating set.

The equitable neighbourhood of v denoted by $N^e(v)$ is defined as $N^e(v) = \{u \in V(G) / u \in N(v), |\deg(u)-\deg(v)| \leq 1\}$. The cardinality of $N^e(v)$ is denoted by $\deg_G^e(v)$. $\Delta^e(G) = \max_{v \in V(G)} |N^e(v)|$, and $\delta^e(G) = \min_{v \in V(G)} |N^e(v)|$ are known as maximum and minimum equitable degree of graph G respectively.

Remark 1.2: $\delta^e(G) \leq \delta(G)$ and $\Delta^e(G) \leq \delta(G)$.

Remark 1.3: In regular graphs and $(k, k+1)$ bi-regular graphs, $\delta^e(G) = \delta(G)$ and $\Delta^e(G) = \delta(G)$.

Remark 1.4: $\Delta^e(G) = \delta^e(G) = 0$ for $K_{1,n}$ for $n \geq 3$.

Swaminathan and Dharamlingam [7] have also introduced the concept of equitable independent set. According to them a subset D of $V(G)$ is called an equitable independent set if for any $u \in D$, $v \notin N^e(u)$ for all $v \in D-\{u\}$. The maximum cardinality of an equitable independent set is denoted by β^e.

Remark 1.5: Every independent set is an equitable independent set.

Remark 1.6: [7] Let D be a maximal equitable independent set. Then D is a minimal equitable dominating set.

Motivated by the concept of equitable dominating set and equitable independent set a new concept was conceived by Swaminathan and Dharamlingam [7] while it was formalized and named as equi independent equitable dominating set by Vaidya and Kothari [8].

Definition 1.7: An equitable dominating set D is said to be equi independent equitable dominating set if it is also equitable independent set. The minimum cardinality of an equi independent equitable dominating is called equi independent equitable domination number which denoted by i^e .

Illustration 1.8: In Fig. 1 and $D=\{v,v_1,v_5,v_8,u_3,u_7,u_9\}$ is equitable independent set as well as equitable dominating set for closed helm CH_9 with $i^e(CH_9)=7$.

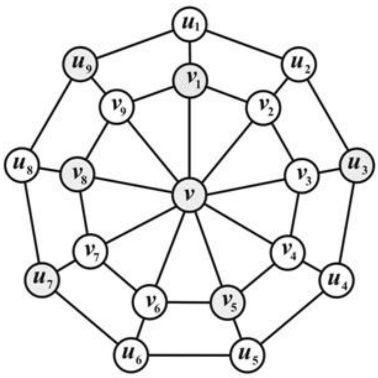

Fig. 1

2. Main Results

Definition 2.1: The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined to be the graph obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2 and joining i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2 .

Definition 2.2: The graph $P_n \odot K_1$ is known as *comb.*

Theorem 2.3 $$i^e(P_n \odot K_1) = \gamma^e(P_n \odot K_1) = \begin{cases} 2 & \text{for } n = 2 \\ 3 & \text{for } n = 3 \\ 4 & \text{for } n = 4 \\ i^e(P_{n-4}) + n & \text{for } n \geq 5 \end{cases}$$

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of path P_n and $u_1, u_2, \ldots, u_n$ are pendant vertices of comb $P_n \Theta K_1$.

Case 1: $n = 2$

In this case $N^e(u_1) = \{v_1\}$, $N^e(u_2) = \{v_2\}$, $N(v_1) = \{u_1, v_2\}$, $N^e(v_2) = \{v_1, u_2\}$. Therefore $D = \{u_1, u_2\}$ is an equi independent equitable dominating set with minimum cardinality. Hence, $i^e(P_2 \Theta K_1) = \gamma^e(P_2 \Theta K_1) = 2$.

Case 2: $n = 3$

In $P_3 \Theta K_1$, vertex u_2 is an equitable isolates and vertices u_1, u_3 are pendant vertices. Then $D = \{u_2, v_1, v_3\}$ is an equi independent equitable dominating set of $P_3 \Theta K_1$ with minimum cardinality. Hence, $i^e(P_3 \Theta K_1) = \gamma^e(P_3 \Theta K_1) = 3$.

Case 3: $n = 4$

In $P_4 \Theta K_1$, vertices u_2, u_3 are equitable isolates and u_1, u_4 are pendant vertices. Then $D = \{v_1, u_2, u_3, v_4\}$ is an equi independent equitable dominating set of $P_4 \Theta K_1$ with minimum cardinality. Hence, $i^e(P_n \Theta K_1) = \gamma^e(P_4 \Theta K_1) = 4$

Case 4: $n \geq 5$

In this case $u_2, u_3, \ldots, u_{n-1}$ are equitable isolates of $P_n \Theta K_1$. Therefore the set $\{u_2, u_3, \ldots, u_{n-1}\}$ must be a subset of every equitable dominating set. While vertex u_1 is equitably adjacent to only v_1 and vertex u_n is equitably adjacent to only v_n . Therefore one of the pair from $\{v_1, v_n\}$ or $\{u_1, u_n\}$ must belongs to every equitable dominating set which implies that $i^e(P_n \Theta K_1) \geq i^e(P_{n-4}) + n$

Now depending upon the number of vertices of path P_n , consider the following subsets.

For $n \equiv 0(\bmod 3)$, $D = \{v_1 v_n v_{3i+1}, u_2, u_3, \ldots, u_{n-1}\}$ where $0 \leq i \leq \frac{n}{3} - 2$,

for, $n \equiv 1(\bmod 3)$ $D = \{v_1 v_, v_{3i+1}, u_2, u_3, \ldots, u_{n-1}\}$ where $0 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor - 2$,

for, $n \equiv 2(\bmod 3)$, $D = \{v_1 v_, v_{3i+1}, v_{n-2}, u_2, u_3, \ldots, u_{n-1}\}$ where $i = 0, 1, \ldots$ such that $3i + 1 < n - 1$.

In each of the above case $|D| = n + i^e(P_{n-4})$. Observe that $N^e(v_1) = \{u_1, v_2\}, N^e(v_n) = \{u_n, v_{n-1}\}$, $N^e(v_{3i+1}) = \{v_{3i}, v_{3i+2}\}$ and $N^e(v_{n-2}) = \{v_{n-3}, v_{n-1}\}$. Therefore $N^e[D] = V(P_n \Theta K_1)$ and D is an equitable dominating set of $P_n \Theta K_1$. Also D is an equitable independent set as pendant vertices $u_2, u_3, \ldots, u_{n-1}$ are not equitable adjacent to any other vertex of D and remaining vertices form $\gamma -$ set of P_n . Therefore D is an equi independent equitable dominating set of $P_n \Theta K_1$ with $|D| = n + i^e(P_{n-4})$. Hence, $i^e(P_n \Theta K_1) = n + i^e(P_{n-4})$.

Definition 2.4: The crown $C_n \Theta K_1$ is obtained by joining pendant edge to each vertex of cycle C_n .

Theorem 2.5: $i^e(C_n \Theta K_1) = \gamma^e(C_n \Theta K_1) = \gamma^e(C_n) + n$).

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of degree 3 and $u_1, u_2, \ldots, u_n$ vertices of degree 1 of crown $C_n \Theta K_1$. Observe that vertices $u_1, u_2, \ldots, u_n$ are equitable isolates of $C_n \Theta K_1$. This implies that they must belong to every equitable dominating set which implies that
$i^e(C_n \Theta K_1) \geq \gamma^e(C_n \Theta K_1) \geq \gamma^e(C_n) + n$).

Let S be the γ^e-set of C_n and $D = \{u_1, u_2, \ldots, u_n\} \cup S$ with $|D| = \gamma^e(C_n) + n$. We claim that D is an equi independent equitable dominating set of $C_n \Theta K_1$. Observe that D is an equitable dominating set of $C_n \Theta K_1$ with minimum cardinality as all the equitable isolates belongs to D and remaining all the vertices are dominated by set S. Also D is an equitable independent set as all $u_1, u_2, \ldots, u_n$ are equitable isolates and vertices of set S are not equitably adjacent to any other vertex of set D.
Hence, $i^e(C_n \Theta K_1) = \gamma^e(C_n \Theta K_1) = \gamma^e(C_n) + n$).

Definition 2.6: The armed crown ACr_n is a graph in which path P_2 is attached at each vertex of cycle C_n by an edge.

Theorem 2.7: $i^e(ACr_n) = \gamma^e(ACr_n) = n$.

Proof: Let $w_1, w_2, \ldots, w_n$ be the pendant vertices, $u_1, u_2, \ldots, u_n$ be the vertices of degree 2 and $v_1, v_2, \ldots, v_n$ be the vertices of degree 3 of ACr_n. To dominate pendant vertices $w_1, w_2, \ldots, w_n$ equitably at least one of the u_i or w_i must belong to any equitable dominating set which implies that,
$i^e(ACr_n) \geq n$

Let $D = \{u_1, u_2, \ldots, u_n\}$ with $|D| = n$. Note that $N^e[D] = V(ACr_n)$ and $|d(u_i) - d(v_i)| = |d(u_i) - d(w_i)| = 1$. Therefore D is an equitable dominating set with minimum cardinality. Also D is an equitable independent set as vertices of D are non adjacent to each other. Therefore D is an equi independent equitable dominating set with $|D| = n$. Hence, $i^e(ACr_n) = \gamma^e(ACr_n) = n$.

Definition 2.8: The gear graph G_n is obtained from the wheel W_n by subdividing each of its rim edge.

Theorem 2.9: $i^e(G_n) = \gamma^e(G_n) = \begin{cases} 2 & \text{for } n = 3 \\ 3 & \text{for } n = 4 \\ i^e(C_{2n}) + 1 & \text{for } n \geq 5 \end{cases}$

Proof: Let v be the apex vertex, $v_1, v_2, \ldots, v_n$ be rim vertices, each vertex u_i that subdividing an edge $v_i v_{i+1}$ for $1 \leq i \leq n-1$ and vertex u_n subdividing an edge $v_n v_1$ of gear graph G_n.

Case 1: $n=3$

Gear graph G_3 is not complete graph which implies that $i^e(G_3)>1$. Consider $D=\{v_1,u_2\}$ with $|D|=2$. Note that D is an equi independent equitable dominating set. Hence, $i^e(G_3)=2$.

Case 2: $n=4$

Observe that $|d(v)-d(v_i)|=1=|d(v_i)-d(u_j)|$ where $v_iu_j\in E(G_4)$. Consider $D=\{v_1,u_2,u_3\}$ with $|D|=3$. Note that $N^e[v_1]\cup N^e[u_2]\cup N^e[u_3]=V(G_4)$ and vertices v_1,u_2,u_3 are non adjacent vertices. Therefore D is an equi independent equitable dominating set of G_4. Hence, $i^e(G_4)=3$.

Case 3: $n\geq 5$

In this case apex vertex v is an equitable isolates and set $V(G_n)-\{v\}$ form C_{2n}. Let S be the i^e-set of C_{2n}. Therefore $D=\{v\}\bigcup S$ is an equi independent equitable dominating set with minimum cardinality. Hence, $i^e(G_n)=i^e(C_{2n})+1$.

Definition 2.10: The fan f_n is graph on $n+1$ vertices obtained by joining all the vertices of P_n to a new vertex called apex vertex.

Theorem 2.11: $i^e(f_n)=\gamma^e(f_n)=\begin{cases}1 & \text{for } n=3\\ 2 & \text{for } n=4\\ i^e(P_n)+1 & \text{for } n\geq 5\end{cases}$

Proof: Let v be a apex vertex and $v_1,v_2,\ldots,v_n$ be the rim vertices of fan f_n.

Case 1: $n=3$

In f_3, $D=\{v_2\}$ equitably dominate all the vertices and D is an equitable independent set. Hence, $i^e(f_3)=1$.

Case 2: $n=4$

Observe that vertices v_2 and v_4 are non adjacent to each other. While $N^e[v_2]\bigcup N^e[v_4]=V(f_4)$. Hence, $D=\{v_2,v_4\}$ is an equi independent equitable dominating set of f_n and $i^e(f_3)=2$.

Case 3: $n\geq 5$

In this case, apex vertex v is equitable isolates and $V(f_n)-\{v\}$ form path P_n. Let S be the i^e-set of P_n. Therefore $D=\{v\}\bigcup S$ is an equi independent equitable dominating set with minimum cardinality. Hence, $i^e(f_n)=i^e(P_n)+1$.

Definition 2.12: The double fan Df_n is given by P_n+2K_1.

Theorem 2.13: $i^e(Df_n) = \gamma^e(Df_n) = \begin{cases} 1 & for\, n = 2 \\ 1 & for\, n = 3 \\ 2 & for\, n = 4 \\ 2 & for\, n = 5 \\ i^e(P_n) + 2 & for\, n \geq 6 \end{cases}$

Proof: Let v_1, v_n be the vertices of degree 3, u, v be the vertices of degree n and the vertices $v_2, v_3, \ldots, v_{n-1}$ are of degree 4 of double fan Df_n.

Case 1: $n = 2$
In Df_2, $D = \{v_1\}$ is an equi independent equitable dominating set of Df_2. Hence, $i^e(Df_2) = \gamma^e(Df_2) = 1$.

Case 2: $n = 3$
Observe that Df_3 is wheel W_4. Hence, $i^e(Df_3) = i^e(W_3) = 1$.

Case 3: $n = 4$
Observe that $N^e(v_1) \bigcup N^e(v_4) = V(Df_4)$ and v_1, v_4 are not adjacent vertices. Also none of the vertex is equitably adjacent to every vertex of Df_4. Therefore $D = \{v_1, v_4\}$ is an equi independent equitable dominating set of Df_4. Hence, $i^e(Df_4) = \gamma^e(Df_4) = 2$.

Case 4: $n = 5$
In Df_5, no vertex is adjacent to all the vertices of Df_5. Consider $D = \{v_2, v_4\}$. Observer that D is an equi independent equitable dominating set and D is an independent set of Df_5 with $|D| = 2$. $i^e(Df_5) = \gamma^e(Df_5) = 2$.

Case 5: $n \geq 6$
In this case u and v are equitable isolates and $V(Df_n) - \{u, v\}$ form path P_n. Let S be the i^e – set of P_n. Therefore $D = S \bigcup \{u, v\}$ is an equi independent equitable dominating set of Df_n with minimum cardinality. Hence, $i^e(Df_n) = i^e(P_n) + 2$.

Definition 2.14: The flower Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the apex of the helm.

Theorem 2.15: $i^e(Fl_n) = \gamma^e(C_n) + n + 1$.

Proof: Let v be the apex vertex, $u_1, u_2, \ldots, u_n$ be the vertices of degree 2 and $v_1, v_2, \ldots, v_n$ be the vertices of degree 4 of Fl_n. Observe that vertices $v, u_1, u_2, \ldots, u_n$ are equitable isolate of Fl_n. Therefore they must belong to every equitable dominating set of F_n which implies that $i^e(Fl_n) \geq \gamma^e(C_n) + n + 1$.

Let S be the i^e – set of C_n and $D=\{v,u_1,u_2,\ldots,u_n\}\cup S$ with $|D|=i^e(C_n)+n+1=\gamma^e(C_n)+n+1$. Observe that $N^e[D]=V(Fl_n)$. Therefore D is an equitable dominating set of Fl_n. Also D is an equitable independent set of Fl_n as $v,u_1,u_2,\ldots,u_n$ are equitable isolates and vertices of S are non adjacent to each other. Hence, D is an equi independent equitable dominating set of Fl_n and $i^e(Fl_n)=\gamma^e(C_n)+n+1$.

Definition 2.16: A one point union $C_n^{(k)}$ of k copies of cycle C_n is the graph obtained by taking v as a common vertex such that any two cycles $C_n^{(i)}$ and $C_n^{(i)}$ $(i\neq j)$ are edge disjoint and do not have any vertex in common except v.

Definition 2.17: The friendship graph F_n is a one-point union of n copies of cycles C_3.

Theorem 2.18: $i^e(F_n)=\gamma^e(F_n)=n+1$

Proof: Let v be the apex vertex and $v_1,v_2,\ldots,v_{2n}$ be other vertices of F_n. Here v is an equitable isolates of F_n. Therefore v must belong to every equitable dominating set of F_n. Also vertex v_i is equitably adjacent to only v_{i+1} and vice versa, where $i=1,3,\ldots,2n-1$ which implies that at least one vertex from each pair $\{v_i,v_{i+1}\}$ must belong to every equitable dominating set of F_n. Therefore $D=\{v,v_1,v_3,\ldots,v_{2n-1}\}$ is an equitable dominating set as well as it is an independent set of F_n. Hence, D is an equi independent equitable dominating set of F_n and $i^e(F_n)=\gamma^e(F_n)=n+1$.

3. Concluding Remarks

Some fundamental results on the concept of equi independent equitable domination number are established by the authors but for the sake of brevity they are not reported here. Here we investigated equi independent equitable domination number for some special graphs. To establish the bounds in terms of various graph theoretic parameters in the context of equi independent equitable domination number is an open area of research.

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