Review article:

Mathematical Epidemic Model of HIV/AIDS in Pakistan

Syed Rizwan Ul Haq Tirmizi [1], Nasiruddin Khan[2], Syed Talha Tirmizi[3], Syeda Amara Tirmizi[4]

Abstract

In Pakistan the effect population mobility, specifically labor migration and refugees is also thought to have been important in explaining the rapid spread of HIV/AIDS. One of the effects labor migration is likely to have had increased the prevalence of the overlap of sexual partnership. A nonlinear fractional differential equation model is discussed for transmission and control of HIV/AIDS in Pakistan. We shall also discuss the disease free equilibrium and stability behavior of the model.

Keywords: HIV/AIDS; labor migration; Pakistan.

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Introduction

AIDS is the disease which is caused by the human immunodeficiency virus (HIV). HIV primarily targets CD4+ cells and without treatment this leads to the collapse of the host immune system and ultimately death. The clinical syndrome was called acquired immune deficiency syndrome (AIDS) in 1982 and four years later the causative virus was named HIV-I. A large number of modeling studies has been focused on HIV since its discovery. Presently there is no remedy for HIV so that once the AIDS stage of HIV infection is attained then ultimately death follows. Usually HIV is much less infectious than the short duration bacterial STI because the value of R_o is possibly minimum as compared to short duration bacterial STI. However, HIV is infectious for far longer than the short-duration STIs, increasing its R_o relative to short-duration bacterial STIs. Depending on the research questions, we may also need to consider that the infectiousness of HIV-infected individuals varies obviously with time since infection.

Preliminaries

In recent advanced research, fractional calculus field is developed as it has much application in engineering and medical sciences. Application of fractional derivative will be discussed to understand some more related definitions. These are given below:

Definition 1 Gamma function Γz is given by

$$\Gamma z = \int_0^{\infty} e^{-u} u^{z-1} du \quad \forall z \in R$$

We can also define function $\gamma(t)$ by using gamma function which will be more suitable for offering alternative form of the fractional integral $\sigma(t)$ is given by

$$\sigma_0(t) = \frac{t^{\alpha - 1}}{\Gamma(\alpha)}$$

Definition 2 In terms of the gamma function, Beta integral and its solution can be shown as

1. Syed Rizwan Ul Haq Tirmizi, Iqra University Main Campus, Karachi (Pakistan).
2. Nasiruddin Khan, Department of Mathematics, University of Karachi, Karachi-75270, (Pakistan).
3. Syed Talha Tirmizi, Petroleum Engineering Department, NED University of Engineering and Technology, Karachi (Pakistan).
4. Syeda Amara Tirmizi, Iqra University Main Campus, Karachi (Pakistan)

Correspondence to: Syed Rizwan Ul Haq Tirmizi, Mailing address: A-501 Diamond Residency, Defence view phase - III Karachi (Pakistan). Email: tirmizi50@yahoo.com

Definition 2 In terms of the gamma function, Beta integral and its solution can be shown as

$$\beta(p,q) = \int_0^1 (1-u)^{p-1}\, u^{q-1}\mathrm{du} = \frac{\lceil p \lceil q}{\lceil (p+q)} = \beta(p,q) \tag{3}$$

Beta function is significant association in fractional calculus.

Definition 3 The Mittage-Leffler can be defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\lceil(\alpha k+1)} \tag{4}$$

In two argument α and β

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\lceil(\alpha k+\beta)} \quad \alpha > 0, \quad \beta > 0$$

Mittage-Leffler shows the similar part in the solution of differential equation of non-integers.

Definition 4 let the function f (t)

$$\int_0^t f(t)dr = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)dr \tag{5}$$

For all $\alpha \epsilon R_+ J^\alpha f(t) = f_\alpha(t) = \frac{1}{\lceil(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau$

Definition 5 For the Grunwald-Letnikov formulation

As we know

$$f'(z) = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h}$$

Also

$$f''(z) = \lim_{h\to 0} \frac{f'(t+h)-f'(t)}{h}$$

$$f''(z) = \lim_{h_1\to 0}\lim_{h_2\to 0} \frac{\frac{f(t+h_1,h_2)-f(t+h_1)}{h_1} - \lim_{h_2\to 0}\frac{f(t+h_2)-f(t)}{h_2}}{h} \tag{1}$$

Let $h_1 = h_2 = h$ then

$$f''(t) = \lim_{h\to 0} \frac{f(t+2h)-2f(t+h)+f(t)}{h^2}$$

For the nth derivative,

$$d^n f(t) = f'(z) = \lim_{h\to 0} \frac{1}{h^n}\sum_{m=0}^{n} (-1)^m \binom{n}{m} f(t-mh) \tag{2}$$

The general formula of the Grunwald-Letnikov fractional derivative

$$d^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^n}\sum_{m=0}^{\frac{t-n}{h}} (-1)^m \frac{\lceil(\alpha+1)}{m!\lceil\alpha-m+1} f(t-mh) \tag{6}$$

$$\binom{-\alpha}{m} = (-1)^m \frac{\lceil(\alpha+m)}{\lceil(\alpha)\ m!}$$

Then

$$d^{-\alpha} f(t) = \lim_{h\to 0} h^\alpha \sum_{m=0}^{\frac{t-\alpha}{h}} (-1)^m \frac{\lceil(\alpha+m)}{m!\lceil\alpha} f(t-mh) \tag{7}$$

Definition 6 Caputo defined the fractional derivative of a function $f(t)$ as

$$d^\alpha f(t) = \frac{1}{\lceil(n-\alpha)}\int_a^t (t-x)^{n-\alpha-1}\frac{d^n}{dt^n}\mathrm{f(x)dx} \quad ; \quad n-1 < \alpha < n$$

In our research, we will generalized HIV/AIDS model to a fractional order system of order α in the sense of caputo definition because it is equivalent to ordinary differential equation when $\alpha = 1$

Lemma 1: Let $x^* = (x_1^*, x_2^*, \dots\dots x_n^*)^T$ be an equilibrium point of the fractional differential equations:

$D^{\alpha}X(t) = F(x),\ \alpha \in [0,1]$ and X(0)=X_0

Where $X = (x_1^*, x_2^*, \dots\dots x_n^*)^T$ and $F = (f_1, f_2, \dots\dots f_n)^T$. Then $x^*$is locally asymptotically stable[8] if all the eigen values of the jacobian matrix $\beta(x^*)$ of the above system satisfies:

$|\arg(eigB(x^*)| > \frac{\alpha\pi}{2}$

Where $B(x^*) = \left[b_{i,j}\right]_{X=x^*}\ i,j = 1,2,3, \dots\dots\dots, n$ and $b_{i,j} = \frac{\partial f_i}{\partial X_j}$

Definition 7 The Riemann – Lioville fractional derivative of order $\alpha > 0, m-1 < \alpha < m, m\epsilon N$ is defined as

$$D_{t_o}^{\alpha} f(t) = \frac{1}{\lceil(m-\alpha)}\left(\frac{d}{dt}\right)^m \int_{t_o}^{t} (t-s)^{m-\alpha-1} f(s)\, ds\ ;\ m-1 < \alpha < m \qquad (9)$$

Definition 8 An invariant set M with respect to a system of O.D.E.

$z^{\cdot} = f(z)$ if $z(0)\epsilon M \Longrightarrow z(x) \in M\ ;\ \forall x \in R$

A positive invariant set with respect to $z^{\cdot} = f(z)$ if

$z(0) \in M \Longrightarrow z(x) \in M\ \forall\ x \geq 0$

Definition 9 Given a system of differential equation $x^{\cdot} = f(t)$ is an equilibrium $x^*$of this system is a point in the state space for which x (t)= x^* is a solution for f (t) = 0 $\forall\ t$

Definition 10 (a) Let $x = F(x)$ be an equation then its equilibrium solution x is said to be locally stable if $\forall\ \varepsilon > 0$, then $\exists$ a $\delta > 0$

s.t

$\|x_o - x\|_2 < \delta$With initial condition $x(t_0) = x_0$

and Satisfies the condition that

$\|x_t - x\|_2 < \varepsilon\ \forall t \geq t_0$

If the equilibrium is not locally stable it is said to unstable.

(b) An equilibrium solution x is said to be locally asymptomatic stable if it is locally stable and if there exist $r > 0$

s.t

$$\|x_o - x\|_2 < r \Longrightarrow \lim_{t\to\infty}\|x(t) - x\|_2 = 0$$

Definition 11 (Routh – Hurwitz Criteria)

Consider the characteristic equation

$\mathcal{F}^n + a_1\mathcal{F}^{n-1} + a_2\mathcal{F}^{n-1} + a_3\mathcal{F}^{n-2} + \cdots\dots\dots\dots + a_{n-1}\mathcal{F} + a_n = 0$

determine the n eign values $\mathcal{F}$ of a real nxn square matrix A. Then the Eign values $\mathcal{F}$ all have positive real parts if $H_1, H_2, , \dots\dots\dots\dots H_n < 0$

Where

$$H_n = \begin{vmatrix} a_1 & 1\dots\dots. & 0 \\ a_3 & a_2\dots\dots. & 0 \\ \vdots & \vdots\dots\dots. & \vdots \\ 0 & 0\dots\dots. & a_n \end{vmatrix}$$

The steady state is stable (i.e.$R_e\mathcal{F} < 0$) $\forall\ \lambda$

Iff $H_j \geq 0\ \forall\ j = 1,2, \dots\dots\dots n$

Description of the Model

To construct the model, all parameters are supposed to be non-negative. We divide the population into four subclasses, the susceptible class s(t), the infective class I(t) that do not know that they have HIV and the known infective class J(t) and the AIDS class A(t). Thus we formulated the fractional order system model as:

$D^{\alpha}s(t) = \mu(k-s) - c\beta_1(Is + \beta_2 Js)$

$D^{\alpha}I(t) = \gamma J - c\beta_1(I + \beta_2 J)s - (\mu + t_1)I$

$$D^{\alpha}J(t) = t_1 I - (\mu + t_2 - \gamma)J \quad (10)$$
$$D^{\alpha}A(t) = t_2 J - \mu A - dA$$

Table 1 . Parameters with their description used in the above model.

Parameters	Description
μ	Constant death rate
k	Average value of connections of an individual per unit time
β_1	Probability of transmission of disease as per connection by an infected individual in the primary stage
β_2	Probability of transmission of disease as per connection by an infected individual in the advanced stag
t_1	Transfer rate constant from asymptomatic phase to symptomatic phase
t_2	Transfer rate constant from symptomatic phase to asymptomatic phase
A	Individual with AIDS
γ	Treatment rate from symptomatic phase to asymptomatic phase
D	disease related death

Basic Reproduction Ratio.

The basic reproduction rate (R_o) is considered to be the estimation of the transmission intensity of a disease. It is defined as the value of secondary infections produced by a distinctive case of an infection in an entirely susceptible population. Mathematical epidemiology is very helpful to identify the threshold conditions of an infectious disease and the threshold conditions are generally described by the reproductive numberR_0.

This basic reproductive rate is affected by several factors including:

- The time period of infectivity
- The infectiousness of the organism
- It can estimate the susceptible people by which the infected patient comes in contact

In general,

if R_0> 1 then the disease will stay to spread within a population.

If R_0< 1 then the disease will ultimately vanish from a population.

Now we investigate the dynamic behavior of system. First we understand the basic reproduction number R_0 by the method of next generation matrix then we obtain the equilibrium of the system (10).Setting the time derivatives in the equations (10) to be equal to zero.

The equilibrium point at $s_c = N$ and $I_c = A_c = 0$ which represents the disease free equilibrium.

Then $S_c = (\mu + \alpha)\frac{N}{\beta}$

$$\boldsymbol{I_c} = \frac{\mu N}{\beta}(R_0 - 1)$$

If model has one infected compartment, we obtain R_0

$$R_0 = \beta.\frac{1}{\mu + \alpha} = \frac{\beta}{\mu + \alpha}$$

Where β is the rate of transmission and the infectious period?

The Next Generation Matrix

The Next Generation Method introduced by the Van den Driessche and Vatmough. This is the method of finding R_0. In a case where we have more than one infected compartment. Suppose we have n compartment in an infected class and m compartment in non-infected compartment.

Let $x \in R^n$

$y \in Rm$

If we denote rate of secondary infection by $\mathcal{F}$ and $\mathcal{V}$ the rate of disease progression, death and recovery.

We have

$$\mathcal{F} = \begin{bmatrix} \omega\beta k & \omega\beta\theta_1 k & \omega\beta\theta_2 k(1-q) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{V} = \begin{bmatrix} (\mu+\rho) & 0 & 0 \\ -\sigma & (\mu+\sigma) & 0 \\ 0 & -\rho & (\mu+\sigma)(\mu+\rho) \end{bmatrix}$$

Using adjoint method

$$\mathcal{V}^{-1} = \frac{1}{(\mu+\rho)(\mu+\sigma)} \begin{bmatrix} (\mu+\rho)(\mu+\sigma) & 0 & 0 \\ \sigma(\mu+\delta) & (\mu+\sigma)(\mu+\delta) & 0 \\ \sigma\rho & 0 & (\mu+\sigma)(\mu+\rho) \end{bmatrix}$$

Leading Eigen values are denoted by the equation given below:

$$\rho(\mathcal{F}\mathcal{V}^{-1}) = \frac{\omega\beta k}{\mu+\sigma}[1 + \frac{\theta_1\sigma}{\mu+\rho} + \rho\frac{\theta_2\rho\sigma(1-q)}{(\mu+\rho)(\mu+\delta)}$$

Where

$$R_1 = \frac{\omega\beta k}{\mu+\sigma}$$

$$R_2 = \frac{\omega\beta k\theta_1}{(\mu+\rho)(\mu+\sigma)} \qquad (11)$$

$$R_3 = \frac{\omega\beta k\theta_2}{(\mu+\rho)(\mu+\sigma)(\mu+\delta)}$$

If $R_0 > 1$, a positive endemic equilibrium $\Lambda^* = (S^*,\ I^*, J^*, A^*)$ is given by the following:

$$S^* = \frac{(\sigma+\mu)(\alpha\mu+\beta\omega\Gamma+\alpha\mu(R_0-1))}{\beta\omega\Gamma(\alpha\mu+\beta\omega\Gamma)}$$

$$I^* = \frac{\mu(R_0-1)}{\alpha\mu+\beta\omega\Gamma}$$

$$J^* = \frac{\alpha\mu(R_0-1)}{(\sigma+\mu)(\alpha\mu+\beta\omega\Gamma)}$$

$$A^* = \frac{\rho\alpha\mu(R_0-1)}{(\sigma+\mu)(\alpha\mu+\beta\omega\Gamma)(\delta+\mu)} \qquad (12)$$

Where

$$R_0 = \frac{c\beta k(\mu+t_2+\gamma+bt_1)}{(\mu+t_1)(\mu+t_2)+\mu\gamma}$$

(P.Driessche-2002)

Stability and Equilibrium

Theorem1 A model A is asymptotically stable in an invariant region if $R_o < 1$. If $R_o > 1$, then the system is unstable.

Proof. Consider the function $\mathcal{T} = \mathcal{T}(I(t), J(t), A(t))$, and we have to prove that a Lyapunov function at the point $(I, J, A) = (0, 0, 0)$.

$$\mathcal{T} = a_1 I + a_2 J + a_3 A$$

Formerly $\mathcal{T}' = a_1 I' + a_2 J' + a_3 A'$

Now mentioning that S (t) < K ∀ t

While λ <$\omega\beta(I + \theta_1 J + \xi A)$,

It gives that we can mark:
$\mathcal{T}' < \mathcal{T}_1 I + \mathcal{T}_2 J + \mathcal{T}_3 A$
Where the coefficients values of $\mathcal{T}_i$:

$$\mathcal{T}_1 = a_1[\omega K\beta - (\mu+\sigma)] + a_2\,\sigma$$
$$\mathcal{T}_2 = a_1\theta_1\omega K\beta - a_2(\mu+\rho) + a_3\,\rho$$
$$\mathcal{T}_3 = a_1\xi\omega K\beta - a_3(\mu+\delta) \qquad (13)$$

Now substituting the values of a_1, a_2, a_3 and ξ, we get,
$\mathcal{T}_2 = a_1\theta_1\omega K\beta - \omega k\beta[\theta_1 \ \ (\mu+\delta) + a_3\,\rho$
$\mathcal{T}_2 = a_1\theta_1\omega K\beta - a_1\theta_1\omega K\beta - \omega K\beta\rho_\xi(\mu\rho) + \omega k\beta\xi(\mu+\rho)\rho$=0
Similarly, for $\mathcal{T}_3$ we have

$$\mathcal{T}_3 = \xi\omega K\beta(m\mu+\rho)(\mu+\delta) - \xi\omega K\beta(m\mu+\rho)(\mu+\delta) = 0$$

Then for $\mathcal{T}_1$

$$\mathcal{T}_1 = a_1[\omega K\beta - (\mu+\sigma)] + a_2\,\sigma$$
$$= (\mu+\sigma)[\frac{\omega k\beta}{(\mu+\sigma)} + \frac{a_2\sigma}{a_1(\mu+\sigma)} - 1]$$
$$= (\mu+\sigma)[\frac{\omega k\beta}{(\mu+\sigma)} + \frac{\theta_1\omega K\beta(\mu+\delta)+\xi}{(\mu+\rho)(\beta+\delta)(\mu+\sigma)} - 1]$$
$$= a_1(\mu+\sigma)[\frac{\omega k\beta}{(\mu+\sigma)}\{1 + \frac{\theta_1\sigma}{(\mu+\rho)} + \frac{\rho\xi\sigma}{(\mu+rho)(\mu+\delta)}\} - 1]$$
$$= (R_0 - 1)\,((\mu+\rho)(\mu+\delta)(\mu+\sigma)) \leq 0$$

Since $R_0 \leq 0$It follows that $\mathcal{T}$ is a Lyapunov function as asserted.

■

The characteristic equation of the Jacobian matrix for the system (10) is
$p(\lambda) = -(\lambda+\mu+d)(\lambda^3 + q_1\lambda^2 + q_2\lambda + q_3 = 0)$
The coefficientsq_1,q_2, and q_3 are calculated in (S. Samko-1993)
The Eigen values of above equation $\lambda = -\mu - d$, and the roots of the above equation

$$\lambda^3 + q_1\lambda^2 + q_2\lambda + q_3 = 0$$

Let $P(D)$ be the discriminant of a polynomial$P(\lambda)$. Then

$$P(D) = 18q_1q_2q_3 + (q_1q_2)^2 - 4q_3q_1^3 - 4q_2^3 - 27q_3^3 \qquad (14)$$

Where q_1, q_2, q_3 are clearly defined in (V. Lakshmikantham-2008).
The system is asymptotically stable if it satisfied the conditions given below:
For $P(D) > 0$ and $\alpha\epsilon[0,1]$s.t $q_1, q_3 > 0$ and $q_1q_2 > q_3$ (Routh-Hurwitz conditions)
If $P(D) < 0$ and $\alpha\epsilon[0, {}^{2}/_{3}]$ s.t $q_1, q_2 \geq 0$ then $q_3 > 0$
And $P(D) < 0$ s.t $q_1, q_2 < 0$
Now examined the consequences of incubation period on the stability performance of system (10). Let τ denotes the incubation period from the start of treatment in the stage (J) which is symptomatic. By using different techniques until and unless the influence of this treatment happens. Therefore, we modified system (10):

$$D^\alpha s(t) = \mu(k-s) - c\beta_1(Is + \beta_2 Js)$$
$$D^\alpha I(t) = \gamma J(t-\tau) - c\beta_1(I + \beta_2 J)s - (\mu + t_1)I \qquad (15)$$
$$D^\alpha J(t) = t_1 I - (\mu + t_2 - \gamma)J(t-\tau)$$

Where $0 \leq t \leq \tau$and$s(0) = s_0$, $I(0) = I_0$, and $J(t) = J_0$
We are going to study the activities of the disease free equilibrium.
Theorem 2 The disease free equilibrium point Λ^* of system (10) with α ∈ (0, 1] is asymptotically stable when*ever* $R_0 < 1$ for any incubation period $\tau \geq 0$ if:

$$\gamma = \delta^2 - \min\{Q_1, Q_2\} < 0 \qquad (16)$$

Proof:
The Jacobin matrix of the fractional differential equations (10) at $\Lambda^*$is

$$B(\Lambda^*) = \begin{bmatrix} -\mu & -c\beta k & -c\beta k \\ 0 & c\beta k - (\mu + k_1) & c\,\beta\,k + \delta e^{-\lambda\tau} \\ 0 & k_1 & -(\mu + k_2 + \delta e^{-\lambda\tau}) \end{bmatrix}$$

The characteristic equation of $B(\Lambda^*)$ is

$$(\lambda + \mu)[\lambda^2 + (m_1 + n_1 e^{-\lambda\tau})\lambda + (m_2 + n_2 e^{-\lambda\tau})] = 0 \quad (17)$$

Its coefficients are

$m_1 = 2\mu + t_1 + t_2 - n\beta t$,

$n_1 = \delta, m_2 = (\mu + t_2)(\mu + t_1 - c\beta t) - n\beta t\, t_1$

$n_2 = \delta(\mu - c\beta t)$.

From Eq. (17) the eigenvalues are $\lambda_1 = -\mu$ and $\lambda_{2,3}$ are the roots of the equation:

$$\lambda^2 + (m_1 + n_1 e^{-\lambda\tau})\lambda + (m_2 + n_2 e^{-\lambda\tau}) = 0 \quad (18)$$

Since

$$Re\lambda_i < 0, \quad i = 1,2,3\ for\ \tau = 0.$$

By increasing τ

We try to find about it if $R_e\lambda_{2,3}$ change sign to be positive.

If we get pure imaginary Eigen values ($\lambda_{2,3} = \pm i\omega$).

Then Putting the value of $\lambda = \pm i\omega$ in Eq. (18), then we get:

$$\omega n_1 sin\omega\tau + n_2 cos\omega\tau = \omega^2 - n_2 \quad (19)$$

$$\omega n_1 cos\omega\tau - n_2 sin\omega\tau = -\omega m_1$$

Removing τ from the above equations (19), we get

$$y^2 + (m_1^2 - 2m_2 - n_1^2)y + (m_2^2 - n_2^2) = 0 \quad (20)$$

Where y = ω^2 , hence there is no positive roots for Eq. (18)

If $m_1^2 - 2m_2 - n_1^2 > 0$ and $m_2^2 - n_2^2 > 0$.

In this case the values of $Re\lambda_{2,3}$ will remain positive. Since

$$m_1^2 - 2m_2 - n_1^2 = (2\mu + t_1 + t_2 - c\hat{a}t)^2 - 2[(\mu + t_2)(\mu + t_1 - n\hat{a}t) - c\hat{a}t\square t_1] - \ddot{a}^2 > 0 \quad (21)$$

By applying the condition (22) we have:

$m_2^2 - n_2^2 = (m_2 + m_2)(m_2 - n_2)$

Since we can write:

$m_2 + n_2 = [(\mu + t_1)(\mu + t_2) + \ddot{a}\mu](1 - R_0) > 0$

Where $R_0 < 1$, and

$$m_2 - n_2 = [(\mu + t_1 - n\hat{a}t) - n\,\hat{a}mt\square_1 - \ddot{a}(\mu - n\,\hat{a}t)](\mu + t_2)$$
$$= [\ddot{a}t_1(1 + R_0)(t_2 + \mu(1 - m)) + (1 - R_0)(\mu + t_2 + mt_1)(\mu + t_1)(\mu + t_2)$$
$$- \mu\ddot{a}_2]/(\mu + t_2 + \ddot{a} + mt_1) > 0$$

By using (16) proof is completed.

Numerical Analysis, Simulation and Discussion.

The local stability of the model for fractional order time derivative is evaluated using fractional Routh-Hurwitz stability criterion. The fractional derivative is described in Caputo sense. The results obtained through numerical procedure show that the technique is effective and reliable. To study the behavior of system numerically a fourth order Range- Kutta method is needed for this we used a computer simulation software Berkeley Madonna and data was collected from Pakistan Demographic Health Survey for this purpose. We intend the numerical simulations that shows the stability and equilibrium state of the disease and effectiveness of the model. In this study, we have proposed reported data of HIV/AIDS and Sentinel surveillance centers' reports on HIV/AIDS. These parameter values is to be included into the entire theme for development of appropriate models to predict the spread of HIV/AIDS in Pakistan.

Table 2: Values of parameters of the model

Parameters	Description	values
ì	Death rate constant	0.085
k	Average number of contacts of an individual per unit time	1.4
$\hat{a}_1$	In the first stage, Probability of disease transmission per contact by an infective person	0.75
$\hat{a}_2$	Probability of disease transmission per contact by an infective person in the second stage	0.34
I	Asymptomatic phase	1.95
S	Susceptible class	0.01
J	Symptomatic phase	0.058
t_1	Transfer rate constant from asymptomatic phase to symptomatic phase	0.31
t_2	Transfer rate constant from symptomatic phase to asymptomatic phase	0.014
A	Individual with AIDS	0.02
ã	Treatment rate from symptomatic phase to asymptomatic phase	0.01
D	disease related death	0.073

We examined the prediction, incident rate and intensity of HIV and secondary infection through this model .It has been seen that (Figure 1), the incidence of HIV/AIDS is predicted to rise steadily that shows recruitment rate of new higher risk individuals. It will be increasing with raised in susceptible individuals. While earlier prevalence rate is increased then attained constant trend. This is because the higher risk individual exterminates through the population and new higher risk partners are not inducted at the similar rate. Therefore, partner rate change will be dropped with the passage of time. If nothing else changes, HIV prevalence will become steady with the stability in the new HIV infections and per year number of HIV deaths, (Figure 2). Equilibria and corresponding stability of the system (Figure 4) are analyzed and obtain some significant results. These shows that HIV infection control efficiently if we increase the incubation time period and minimize the intensity of secondary infections with the proper treatment. However, it is very difficult to explain HIV trend because variation in HIV prevalence and incidence are also due to the natural dynamics of infection and result of intervention.

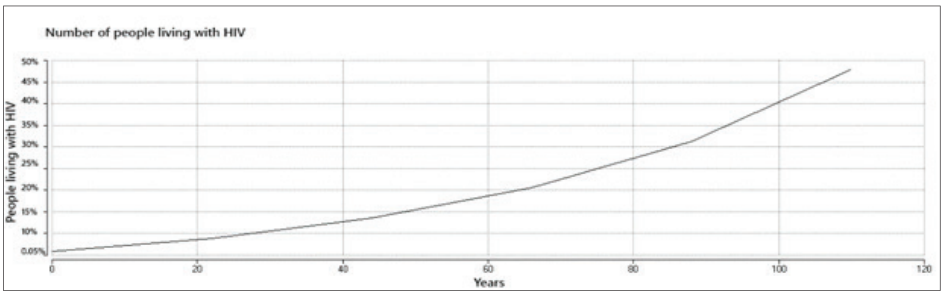

Figure. 1 Prediction of people living with HIV/AIDS.

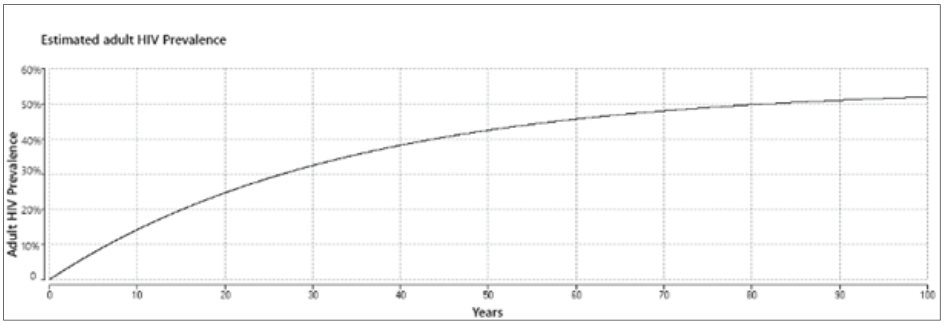

Figure. 2 Prediction of prevalence of HIV/AIDS

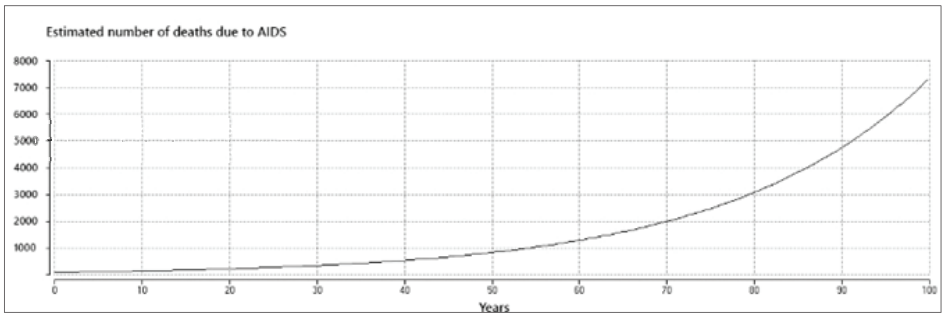

Figure. 2 Prediction of number of deaths due to AIDS

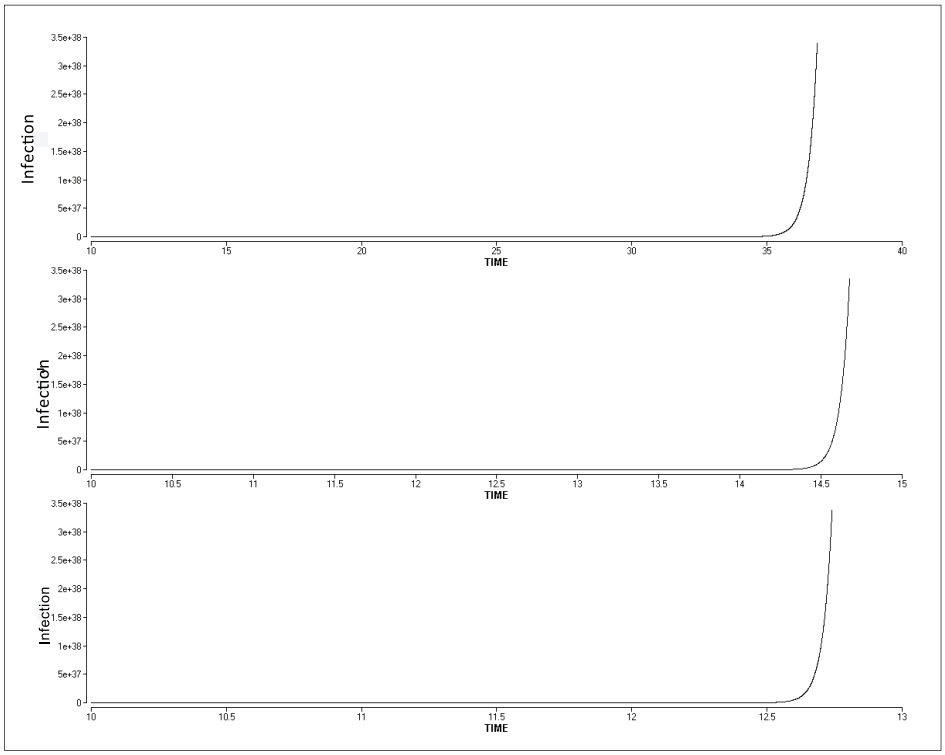

Figure. 4 Incubation Time Rate Of J (T) For Ô= 100,300,500

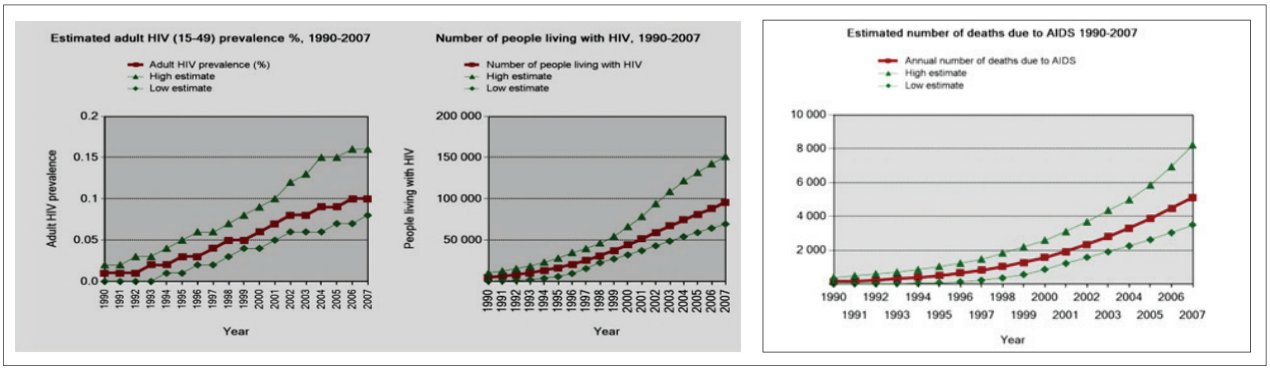

Figure.5 Estimated trend of HIV in Pakistan.
Source: UNAIDS/WHO, 2008

Conclusion

In our study, a nonlinear mathematical model having equilibrium and corresponding stability of the system are analyzed and obtain some significant results i.e. HIV infection control efficiently, if we increase the incubation time period and minimize the intensity of secondary infections with the proper regimen fractional order á is presented. Otherwise leads to the collapse of the host immune system and ultimately death.

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Author contributions: Data gathering and idea owner of this study:
Study design:
Syed Rizwan Ul Haq Tirmizi
Data gathering:
Syed Rizwan Ul Haq Tirmizi, Nasiruddin Khan
Writing and submitting manuscript:
Syed Rizwan Ul Haq Tirmizi
Syed Talha Tirmizi
Editing and approval of final draft:
Syed Rizwan Ul Haq Tirmizi , Nasiruddin Khan, Syed Talha Tirmizi, Syeda Amara Tirmizi

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