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Haar Wavelet Representation of Continuous and Discrete Functions

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Abstract

Wavelets are mathematical functions that cut up data into different frequency components and then study each component with a resolution matched to its scale. They have advantages over traditional Fourier methods in analyzing physical situations where the signal contains discontinuities and sharp spikes. Wavelets were developed independently in the fields of mathematics, quantum physics, electrical engineering, seismic geology etc. Interchanges between these fields during the last twenty years have led to many new wavelet applications such as image compression, turbulence, human vision, radar and earthquake prediction. The wavelet representation of a function is a new technique and it does not loss time information. In this study we try to represent how continuous and discrete functions represent in wavelet form, especially in the Haar wavelet representation.

Key word: Wavelet, Wavelet Coefficients, Haar Function, Haar Basis Function.

Introduction

Wavelets are functions that satisfy certain mathematical requirements and are used in representing data or functions. Approximation using superposition of functions has existed since the early 1800's, when Joseph Fourier discovered that he could superpose sines and cosines to represent other functions. However, in wavelet analysis, the scale that we use to look at data plays a special role. Wavelet algorithms process a data at different scales or resolutions. This makes wavelets interesting and useful. For many decades, scientists have wanted more appropriate functions than the sines and cosines which comprise the bases of Fourier analysis, details in O Christensen (2004). By their definition, these functions are non-local and stretch out to infinity for details we can be seen Kaiser, G. (1994). They therefore do a very poor job in approximating sharp spikes. But with wavelet analysis, we use approximating functions that are contained neatly in finite domains, for details M. Nielsen (2004). Wavelets are well suited for approximating data with sharp discontinuities, details in M. R. Islam (2005). In this paper we have tried to represent continuous and discrete function into Haar wavelets.

Basic Definitions

Wavelets

Wavelets are functions that are confined in finite domains and are used to represent data or a function. In an analogous way to Fourier analysis, which analyzes the frequency content in a function using sines and cosines, wavelet analysis analyzes the scale of a function's content with special basis functions called wavelets. For details we refer Debnath, L. (2002). Equivalent mathematical conditions for wavelet are:

$$(i) \int_{-\infty}^{\infty} |\psi(x)|^2 dx < \infty,$$

$$(ii) \int_{-\infty}^{\infty} |\psi(x)| dx = 0,$$

$$(iii) \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty$$

where $\hat{\psi}(\omega)$ is the Fourier Transform of $\psi(x)$, (iii) is called the admissibility condition.

Wavelet Transform

Wavelet transform analysis uses little wave like functions known as wavelets. Wavelets are used to transform the signal under investigation into another representation, which presents the signal information in a more useful form. This transformation of signal is known as the wavelet transform. Mathematically speaking, the wavelet transform is a

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convolution of the wavelet function with the signals. Jean Morlet in 1982, first considered wavelets as a family of functions constructed from translations and dilations of a single function called the "mother wavelet", ψ(x). They are defined by

$$\psi_{j,k}(x) = \frac{1}{\sqrt{|j|}}\psi\left(\frac{x-k}{j}\right) j,k \in \mathrm{R}, j \neq 0$$

where j and k represent the scaling parameter which measures the degree of compression and the translation parameter which determines the time location of the wavelet respectively. The wavelet transform of f can be defined as

$$W_{\psi}f(j,k) = \frac{1}{\sqrt{|j|}}\int_{-\infty}^{\infty}\overline{\psi\left(\frac{x-k}{j}\right)}f(x)dx \qquad (2.2.1)$$

where $\overline{\psi\left(\frac{x-k}{j}\right)}$ is the complex conjugate of $\psi\left(\frac{x-k}{j}\right)$. There are many kinds of wavelet transforms such as continuous, discrete, fast, complex transforms as well as wavelet packet transforms.

Wavelet Series & Wavelet Coefficients

Now a day's wavelet representation of a function is very popular. Because Fourier transformation loss time information but wavelet representation does not loss time information. Thus, the series $\sum_{j\in Z}\sum_{k\in Z}\langle f,\psi_{j,k}\rangle\psi_{j,k}(x)$ is called the wavelet series of f if the function $f \in L_2(R)$, and then $\langle f,\psi_{j,k}\rangle = d_{j,k} = \int_{-\infty}^{\infty} f(x)\psi_{j,k}(x)dx$ are called the wavelet coefficients of f.

The Inverse Wavelet Transform

There is an inverse wavelet transform, defined as

$$f(x) = \frac{1}{c_g}\int_{-\infty}^{\infty}\int_{0}^{\infty} f(j,k)\psi_{j,k}(x)\frac{djdk}{j^2}$$

which allows the original signal to be recovered from its wavelet transform by integrating over all scales and location j and k. For the inverse transform, the original wavelet function is used, rather than its conjugate, which is used in the forward transformation and c_g is the admissibility constant.

Continuous Wavelet Transform

The continuous wavelet transform of $f \in L_2(R)$ can be defined as

$$T_{\psi}f(j,k) = |j|^{-\frac{1}{2}}\int_{-\infty}^{\infty} f(x)\overline{\psi}\left(\frac{x-k}{j}\right)dx = \langle f,\psi_{j,k}\rangle. \qquad (2.5.1)$$

where $\overline{\psi}_{j,k}(x)$ is the complex conjugate of $\psi_{j,k}(x)$, $\psi_{j,k}(x) = |j|^{-\frac{1}{2}}\psi\left(\frac{x-k}{j}\right)$ is translated by k and dilated by j of ψ and $T_{\psi}f(j,k)$ is called the wavelet transform of $f(x)$ in $L_2(\mathrm{R})$.

Discrete Wavelet Transform

The foundations of the DWT go back to 1976 when Cruiser, Esteban and Galand devised a technique to decompose discrete time signals. Our discrete wavelets are not time-discrete, only the translation and the scale step are discrete. For details we refer to Addition, P. S. (2002). It turns out that it is better to discretize it in a different way, first we fix two positive constants a_0 and b_0 and define

$$\psi_{j,k}(x) = a_0^{-j/2}\psi(a_0^{-j}x - kb_0) \qquad (2.6.1)$$

where both $j,k \in Z$ the discrete wavelet transform of a given function $f(x)$ is defined by

$$W_{\psi}f(j,k) = \langle f,\psi_{j,k}\rangle = \int_{-\infty}^{\infty} f(x)\overline{\psi}_{j,k}(x)dx$$

$$= a_0^{-\frac{j}{2}}\int_{-\infty}^{\infty} f(x)\overline{\psi}(a_0^{-j}x - kb_0)dx \qquad (2.6.2)$$

where both f and ψ are continuous, $\overline{\psi}_{j,k}(x)$ is the complex conjugate of $\psi_{j,k}(x)$. For computational efficiency, $a_0 = 2$ and $b_0 = 1$ are commonly used so that results lead to a binary dilation of 2^j and a dyadic translation of $k2^j$. From (2.6.1) we get, $\psi_{j,k}(x) = 2^{-\frac{j}{2}}\psi(2^{-j}x - k)$.

Now the equation (2.6.2) can be written as

$$W_\psi f(j,k) = \langle f, \psi_{j,k} \rangle = 2^{-\frac{j}{2}} \int_{-\infty}^{\infty} f(x)\bar{\psi}(2^{-j}x-k)dx \quad (2.6.3)$$

where j and k are integers that scale dilute the mother function $\psi(x)$ to generate wavelets. The scale index j indicates the wavelet's width and the location index k gives its position. The discrete wavelet transform of a given function $f(x)$ can be defined in another way, which is given by $f(x)=\frac{1}{\sqrt{M}}\sum_{k=0}^{2^{j_0}-1} c_{j_0,k}\varphi_{j_0,k}(x)+\frac{1}{\sqrt{M}}\sum_{j=j_0}^{\infty}\sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k}(x)$

for $j \geq j_0$ (2.6.4)

$$c_{j_0,k} = W_\varphi(j_0,k) = \frac{1}{\sqrt{M}}\sum_{x=0}^{2^J-1} f(x)\varphi_{j_0,k}(x) \quad (2.6.5)$$

$$d_{j,k} = W_\psi(j,k) = \frac{1}{\sqrt{M}}\sum_{x=0}^{2^J-1} f(x)\psi_{j,k}(x) \quad (2.6.6)$$

here, $f(x), \varphi_{j_0,k}(x)$, and $\psi_{j,k}(x)$ are functions of the discrete variable $x = 0,1,2,\ldots, M-1$ and we consider $j = 0,1,2,\ldots\ldots J-1$ & $M = 2^J$.

Haar Function

A function defined on the real line R as

$$\psi(x) = \begin{cases} 1 & \text{for } x \in [0,1/2) \\ -1 & \text{for } x \in [1/2,1) \\ 0 & \text{otherwise} \end{cases} \quad (2.6.7)$$

is known as the Haar function. The Haar function $\psi(x)$ is the simplest example of a wavelet. The Haar function $\psi(x)$ is a wavelet because it satisfies all the conditions of wavelet. Haar function is discontinuous at $x = 0, \frac{1}{2}, 1$ and it is very well localized in the time domain. Haar function is known as Haar wavelet.

Translation and Dilation of a wavelet function and its representation

In signal analysis it is common to consider functions belonging to the vector space

$$L_2(\mathrm{R}) = \left\{ f : R \to C \,/ \int_{-\infty}^{\infty} |f(x)dx|^2 < \infty \right\}$$

except for $f = 0$, functions in $L_2(R)$ are never periodic. Yet, it is possible to obtain series expansions like

$$f(x) = \sum_{n=0}^{\infty} a_n f_n(x) \quad (3.1)$$

of functions in $L_2(\mathrm{R})$. Let us consider a function $\psi(x) = e^{-x^2}$, for ψ associate a family of functions defined by

$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$; $x \in \mathrm{R}$ & $j,k \in \mathrm{Z}$.

For the relationship between these functions and the given function $\psi(x)$ we first consider $j = 0$ and observe when $k = 0$, then $\psi_{0,0}(x) = 2^0\psi(2^0 x - 0) = \psi(x)$ which indicate $\psi_{0,0}(x)$ is ψ itself. Again

$$\psi_{0,k}(x) = 2^0\psi(2^0 x - k) = \psi(x-k) \quad \text{for} \quad k \in \mathrm{Z}.$$

Thus, the graph of the function $\psi_{0,k}$ appears by translating the graph of ψ by k units to the right in Fig. 2 to Fig. 3.

In order to understand the role of the parameter j, putting $k = 0$ then $\psi_{j,0}(x) = 2^{j/2}\psi(2^j x)$. As from Fig. 4 to Fig. 7 demonstrate, the functions $\psi_{j,0}$ are scaled versions of ψ, when j is a large positive number, the graph of $\psi_{j,0}$ is a compressed version of the graph of ψ, while negative values of j lead to less localized versions of ψ. Putting everything together, we see that the functions $\psi_{j,k}$ are scaled and translated versions of ψ. We say that the functions $\psi_{j,k}, \forall j,k \in \mathrm{Z}$, form the wavelet system associated to the function ψ.

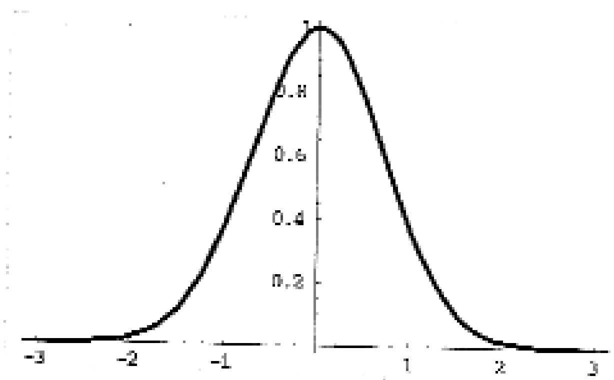

Fig. 1: The function $\psi_{0,0}(x) = e^{-x^2} = \psi(x)$.

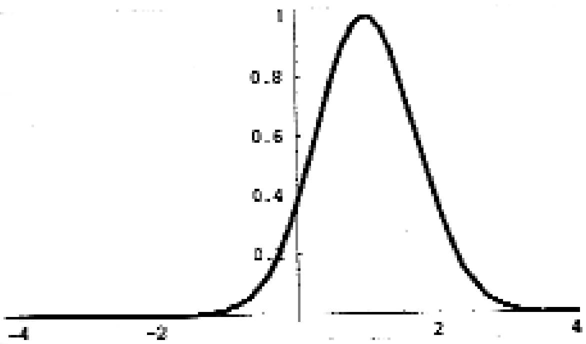

Fig. 2: The function $\psi_{0,1}(x)=e^{-(x-1)^2}$.

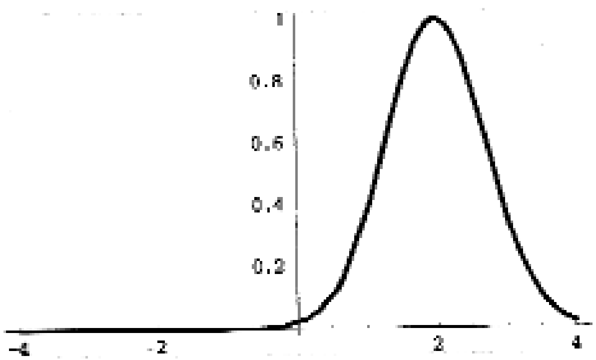

Fig. 3: The function $\psi_{0,2}(x)=e^{-(x-2)^2}$.

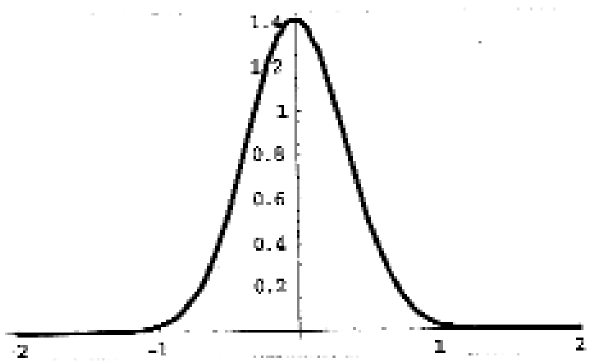

Fig. 4: The function $\psi_{1,0}(x)=\sqrt{2}\,e^{-4x^2}$.

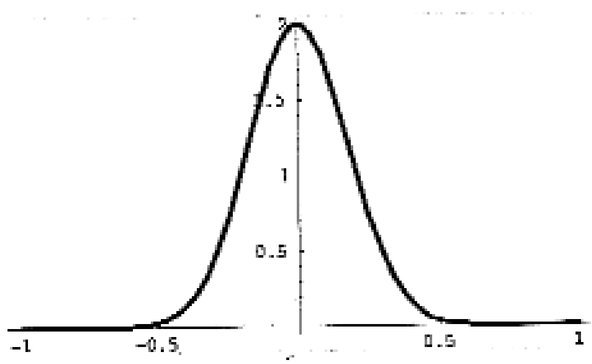

Fig. 5: The function $\psi_{2,0}(x)=2\,e^{-16x^2}$.

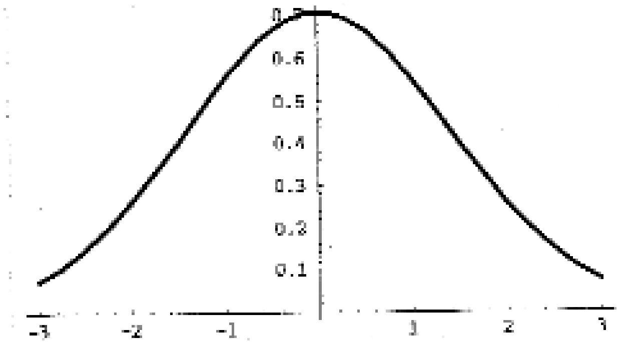

Fig. 6: The function $\psi_{-1,0}(x)=\frac{1}{\sqrt{2}}e^{-\frac{x^2}{4}}$

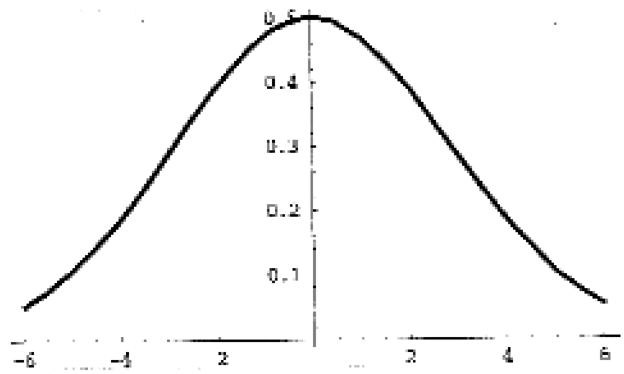

Fig. 7: The function $\psi_{-2,0}(x)=\frac{1}{2}e^{-\frac{x^2}{16}}$.

Our goal is to expand functions in $L_2(\mathrm{R})$ in terms of functions of the type $\psi_{j,k}$ *i.e.* we want to obtain expansions like (3.1) with f_n replaced by $\psi_{j,k}$. So our first problem is to determine a function ψ such that every function f in $L_2(\mathrm{R})$ has a representation of the form

$$f(x)=\sum_{j\in Z}\sum_{k\in Z}d_{j,k}\,\psi_{j,k}(x). \tag{3.2}$$

for a certain choice of the coefficients $\{d_{j,k}\}$ which depend on the given function f. Now put the additional condition on ψ.

$$\int_{-\infty}^{\infty}\psi_{j,k}(x)\overline{\psi}_{j',k'}(x)dx=\begin{cases}1 & \text{if } j=j',k=k',\\ 0 & \text{otherwise}\end{cases} \tag{3.3}$$

This extra condition implies that if (3.2) is possible for all $f\in L_2(\mathrm{R})$, then the coefficients appearing in the expansion of an arbitrary f are unique and we have a convenient expression for them, in fact

$$d_{j,k} = \int_{-\infty}^{\infty} f(x)\overline{\psi}_{j,k}(x)dx. \quad (3.4)$$

where $\overline{\psi}_{j,k}(x)$ is the complex conjugate of $\psi_{j,k}(x)$. The condition (3.3) puts extra restrictions on the choice of ψ. We might ask for choices of ψ satisfying some extra conditions, for details D.F. Walnut (2001). Some of the conditions which appear often are:

(i) that ψ is smooth, may be even infinitely often differentiable.

(ii) that ψ has a computationally convenient form; for example that ψ is a piecewise polynomial, *i.e.* a spline.

(iii) that ψ has compact support, i.e. all its function values are zero outside a certain bounded interval.

(iv) $\hat{\psi}$ has compact support.

Haar Wavelet Representation of functions

Haar Scaling Function

The Haar scaling function can be defined as

$$\varphi(x) = \chi_{[0,1)}(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Haar Wavelet Function

Haar wavelet function $\psi(x)$ in terms of scaling function can be written as

$$\psi(x) = \chi_{\left[0,\frac{1}{2}\right)}(x) - \chi_{\left[\frac{1}{2},1\right)}(x)$$

$$= \begin{cases} 1, & \text{if } 0 \le x \le \frac{1}{2} \\ -1, & \text{if } \frac{1}{2} \le x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Haar Wavelet Series and Wavelet Coefficients

Let f be defined on [0, 1], then it has an expansion in terms of Haar functions as follows. For any integer $j_0 \ge 0$,

$$f(x) = \sum_{k=0}^{2^{j_0}-1} \langle f, \varphi_{j_0,k}\rangle \varphi_{j_0,k}(x) + \sum_{j=j_0}^{\infty}\sum_{k=0}^{2^j-1} \langle f, \psi_{j,k}\rangle \psi_{j,k}(x)$$

$$= \sum_{k=0}^{2^{j_0}-1} c_{j_0,k}\varphi_{j_0,k}(x) + \sum_{j=j_0}^{\infty}\sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k}(x) \quad (4.3.1)$$

The series (4.3.1) is known as the Haar wavelet series for the given function f. $d_{j,k}$ and $c_{j_0,k}$ are known as the Haar wavelet co-efficient and the Haar scaling co efficient respectively. Where,

$$d_{j,k} = \int_{-\infty}^{\infty} f(x)\psi_{j,k}(x)dx \;\;\&\;\; c_{j_0,k} = \int_{-\infty}^{\infty} f(x)\varphi_{j_0,k}dx$$

Haar Representation of Continuous Function

<u>**Example 4.4.1**</u>

Consider a simple function

$$f(x) = \begin{cases} x^2 + x & \text{if } 0 \le x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

We can represent $f(x)$ as (4.3.1). Let the starting scale be $j_0 = 0$, so the scaling coefficient

$$c_{0,0} = \int_0^1 (x^2 + x)\varphi_{0,0}(x)dx = \frac{5}{6}$$

And the wavelet coefficients are

$$d_{0,0} = \int_0^1 (x^2 + x)\psi_{0,0}(x)dx = -\frac{1}{2}, \text{ Similarly}$$

$$d_{1,0} = -\frac{3\sqrt{2}}{32},\; d_{1,1} = -\frac{5\sqrt{2}}{32},\; d_{2,0} = -\frac{5}{128},$$
$$d_{2,1} = -\frac{7}{128},\; d_{2,2} = -\frac{9}{128},\; d_{2,3} = -\frac{11}{128}$$

and so on. Then

$$f(x) = x^2 + x = \frac{5}{6}\varphi_{0,0}(x) + \left[-\frac{1}{2}\psi_{0,0}(x)\right]$$
$$+\left[-\frac{3\sqrt{2}}{32}\psi_{1,0}(x) - \frac{5\sqrt{2}}{32}\psi_{1,1}(x)\right] +$$
$$\begin{bmatrix} -\frac{5}{128}\psi_{2,0}(x) - \frac{7}{128}\psi_{2,1}(x) - \\ \frac{9}{128}\psi_{2,2}(x) - \frac{11}{128}\psi_{2,3}(x) \end{bmatrix} + \ldots\ldots\ldots\ldots$$

Here, $v_0 = \frac{5}{6}\varphi_{0,0}(x)$, $w_0 = -\frac{1}{2}\psi_{0,0}(x)$,

$$w_1 = -\frac{3\sqrt{2}}{32}\psi_{1,0}(x) - \frac{5\sqrt{2}}{32}\psi_{1,1}(x)$$

$$w_2 = -\frac{5}{128}\psi_{2,0}(x) - \frac{7}{128}\psi_{2,1}(x) - \frac{9}{128}\psi_{2,2}(x) - \frac{11}{128}\psi_{2,3}(x)$$

$$v_1 = v_0 \oplus w_0 = \frac{5}{6}\varphi_{0,0}(x) + \left[-\frac{1}{2}\psi_{0,0}(x)\right]$$

$$v_2 = v_1 \oplus w_1 = v_0 \oplus w_0 \oplus w_1 = \frac{5}{6}\varphi_{0,0}(x) + \left[-\frac{1}{2}\psi_{0,0}(x)\right] + \left[-\frac{3\sqrt{2}}{32}\psi_{1,0}(x) - \frac{5\sqrt{2}}{32}\psi_{1,1}(x)\right]$$

$$v_3 = v_2 \oplus w_2 = v_1 \oplus w_1 \oplus w_2$$
$$= \frac{5}{6}\varphi_{0,0}(x) + \left[-\frac{1}{2}\psi_{0,0}(x)\right] + \left[-\frac{3\sqrt{2}}{32}\psi_{1,0}(x) - \frac{5\sqrt{2}}{32}\psi_{1,1}(x)\right] + \left[\begin{array}{l} -\frac{5}{128}\psi_{2,0}(x) - \frac{7}{128}\psi_{2,1}(x) \\ -\frac{9}{128}\psi_{2,2}(x) - \frac{11}{128}\psi_{2,3}(x) \end{array}\right]$$

where, v_j and w_j, $j \geq 0$ are the orthogonal subspaces of $L_2[0, 1]$ and $\oplus$ is the direct sum.

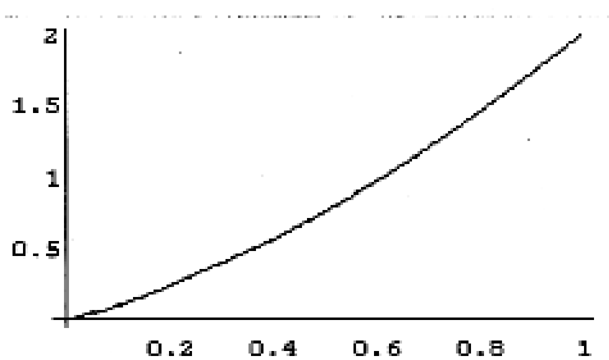

Fig. 8: Graph of $f(x)$

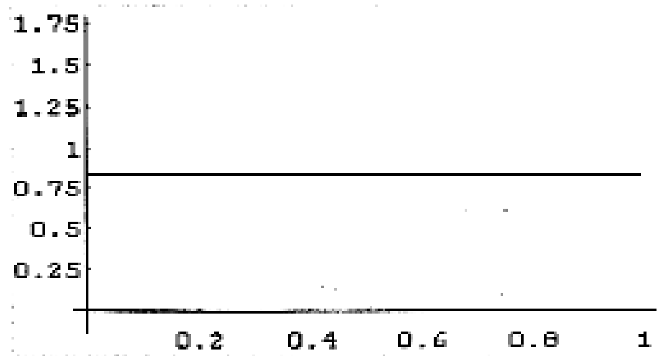

Fig. 9: Graph of V_1

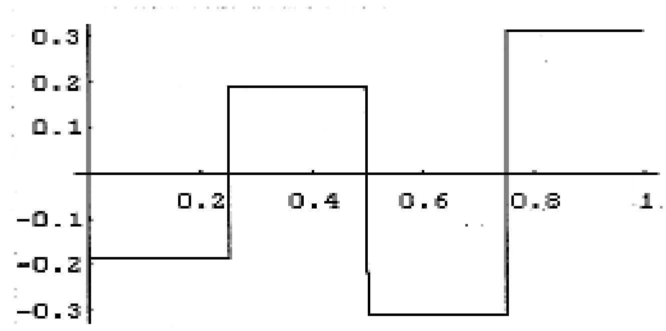

Fig. 10: Graph of W_1

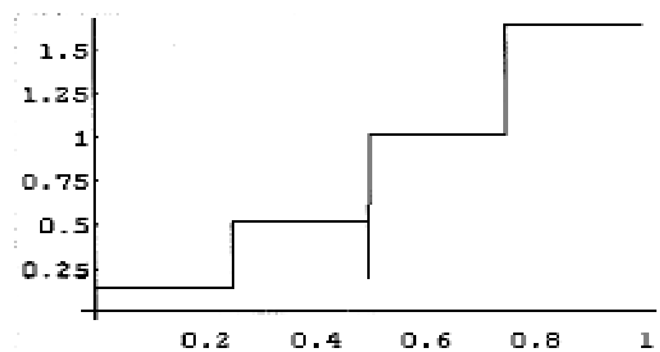

Fig. 11: Graph of V_2

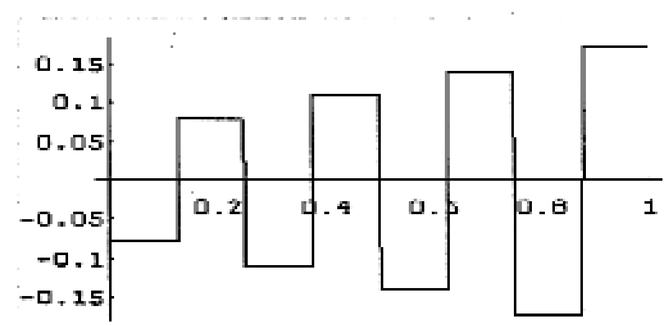

Fig. 12: Graph of W_2

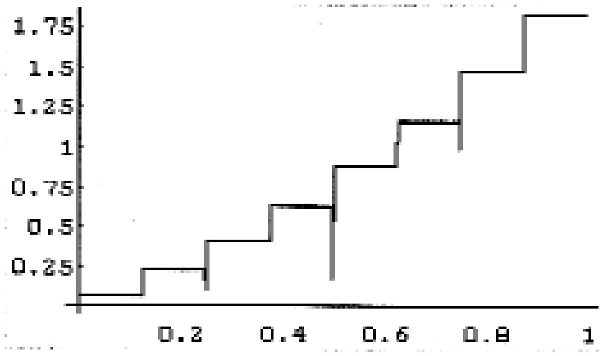

Fig. 13: Graph of V_3

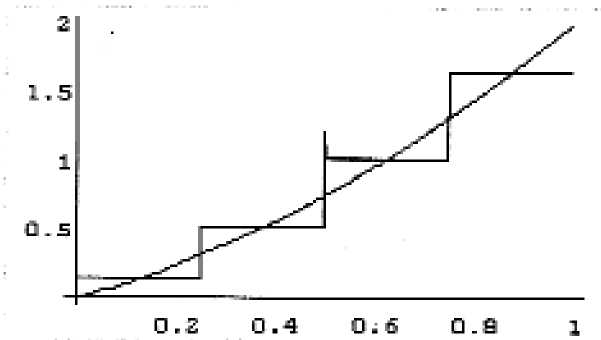

Fig. 14: Graph of $f(x)$ and V_2 jointly

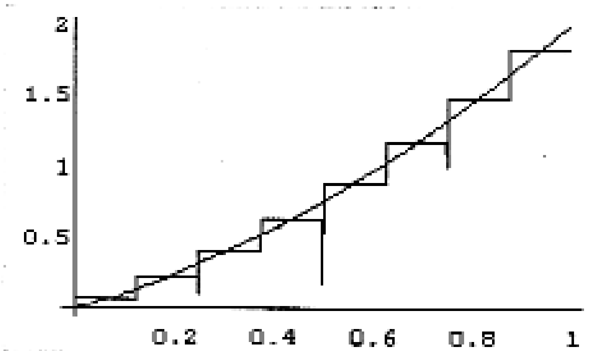

Fig. 15: Graph of $f(x)$ and V_3 jointly

Example 4.4.2.

Consider a simple function

$$f(x)=\begin{cases} e^x & \text{if} \quad 0 \le x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

We can represent $f(x)$ as (4.3.1). Let the starting scale be $j_0=0$, so the scaling coefficient,

$$c_{0,0}=\int_0^1 e^x \varphi_{0,0}(x)dx = 1.7182$$

and the wavelet coefficients are

$d_{0,0} = -0.42083,$ $\quad d_{1,0}=\sqrt{2}\,(0.080670),$
$d_{1,1}=\sqrt{2}\,(0.13300),$ $\quad d_{2,0} = -0.03545,$
$d_{2,2}=-0.05845$, $d_{2,3}=-0.07506$, and so on. Then

$$f(x)=1.71828\varphi_{0,0}(x)+\left[-0.42083\psi_{0,0}(x)\right]+\begin{bmatrix}\sqrt{2}\,(-0.080670)\psi_{1,0}(x)+ \\ \sqrt{2}\,(-0.13300)\psi_{1,1}(x)\end{bmatrix}+\begin{bmatrix}(-0.03545)\psi_{2,0}(x)-(0.04552)\psi_{2,1}(x)- \\ (0.05845)\psi_{2,2}(x)-(0.07506)\psi_{2,3}(x)\end{bmatrix}+\dots$$

$$v_0=1.71828\varphi_{0,0}(x),\ w_0=-0.42083\psi_{0,0}(x),$$

$$w_1=-\begin{bmatrix}\sqrt{2}\,(-0.080670)\psi_{1,0}(x) \\ +\sqrt{2}\,(-0.13300)\psi_{1,1}(x)\end{bmatrix}$$

$$w_2=(-0.03545)\psi_{2,0}(x)-(0.04552)\psi_{2,1}(x) \\ -(0.05845)\psi_{2,2}(x)-(0.07506)\psi_{2,3}(x)$$

$$v_1=v_0\oplus w_0$$
$$=1.71828\varphi_{0,0}(x)+\left[-0.42083\psi_{0,0}(x)\right]$$

$$v_2=v_1\oplus w_1=v_0\oplus w_0\oplus w_1$$
$$=1.71828\varphi_{0,0}(x)+\left[-0.42083\psi_{0,0}(x)\right]+\begin{bmatrix}\sqrt{2}\,(-0.080670)\psi_{1,0}(x) \\ +\sqrt{2}\,(-0.13300)\psi_{1,1}(x)\end{bmatrix}$$

$$v_3=v_2\oplus w_2=v_1\oplus w_1\oplus w_2$$
$$=1.71828\ \varphi_{0,0}(x)+\left[-0.42083\ \psi_{0,0}(x)\right]+\begin{bmatrix}\sqrt{2}\,(-0.080670)\psi_{1,0}(x)+ \\ \sqrt{2}\,(-0.13300)\psi_{1,1}(x)\end{bmatrix}+\begin{bmatrix}(-0.03545)\psi_{2,0}(x)-(0.04552)\psi_{2,1}(x) \\ -(0.05845)\psi_{2,2}(x)-(0.07506)\psi_{2,3}(x)\end{bmatrix}$$

where v_j and w_j, $j\ge 0$ are the orthogonal subspaces of $L_2[0,\ 1]$ and $\oplus$ is the direct sum.

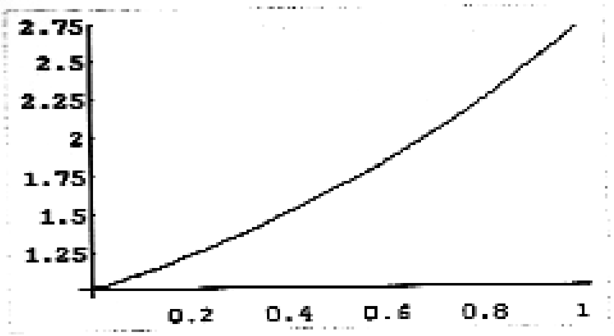

Fig. 16: Graph of $f(x)$

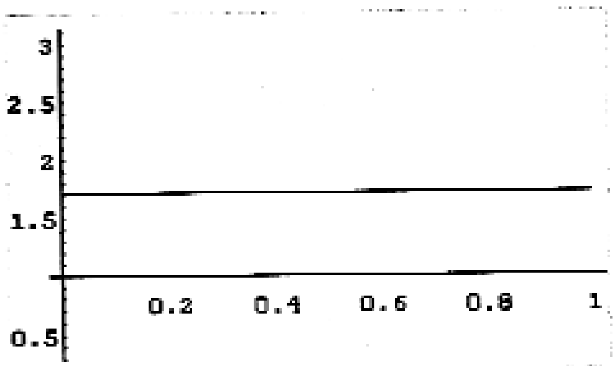

Fig. 17: Graph of V_0

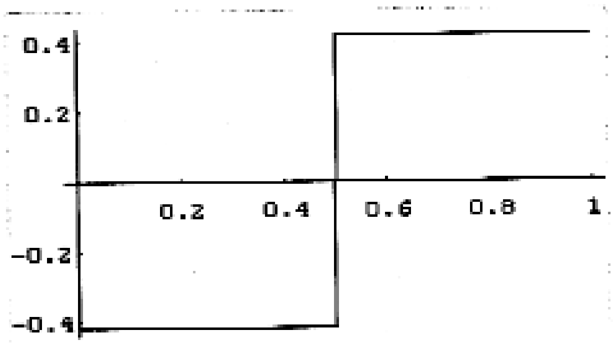

Fig. 18: Graph of W_0

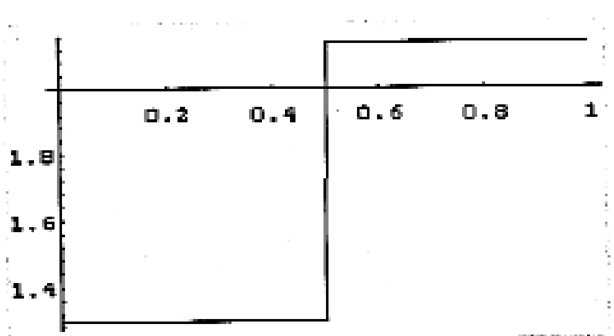

Fig. 19: Graph of V_1

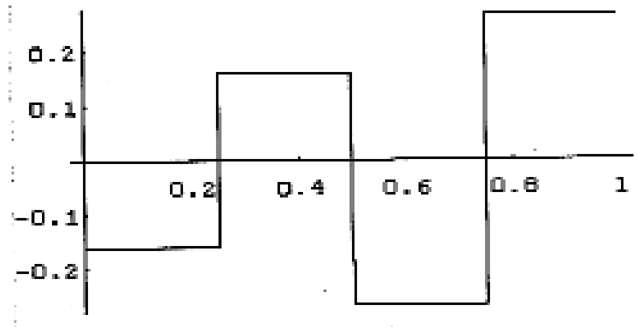

Fig. 20: Graph of W_1

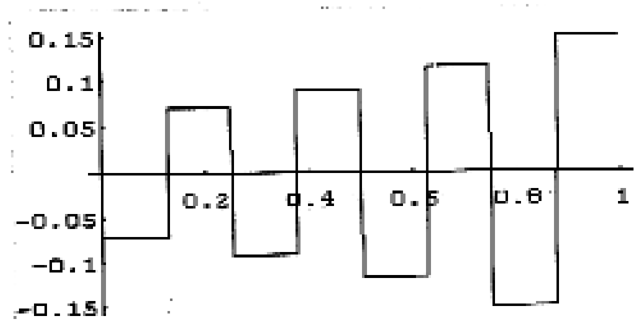

Fig. 21: Graph of V_2

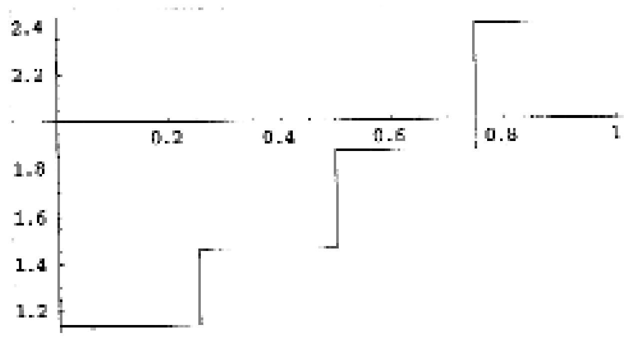

Fig. 22: Graph of W_2

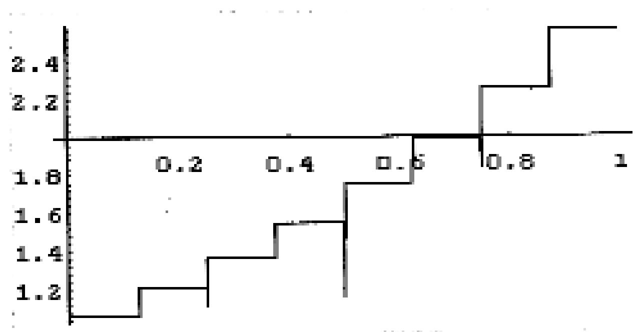

Fig. 23: Graph of V_3

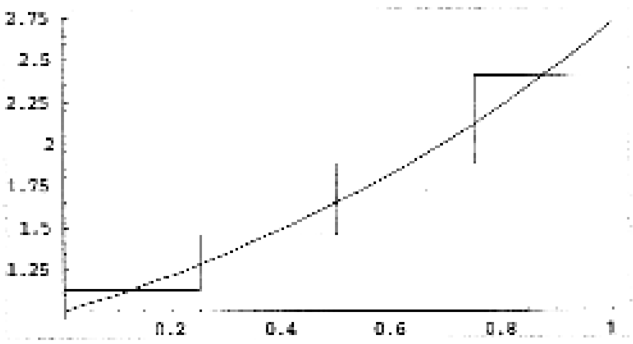

Fig. 24: Graph of $f(x)$ and V_2 jointly

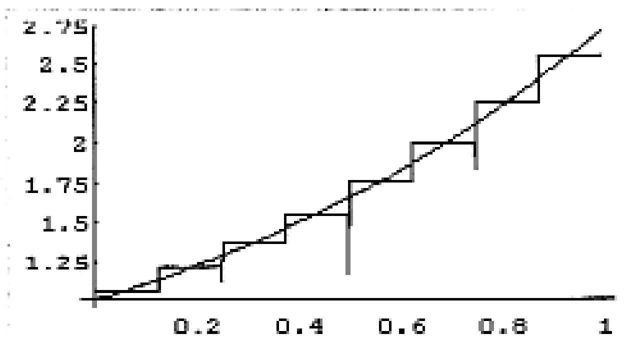

Fig. 25: Graph of $f(x)$ and V_3 jointly

Haar Basis Function

The Haar basis function $h_x(z)$ are defined over the continuous and closed interval $z \in [0, 1]$ for $x = 0, 1, 2, \ldots\ldots, M-1$, where $M = 2^J$. These functions are contained in the $M \times M$ transformation matrix H. To generate H, we define integer K such as $K = 2^j + k - 1$ where $0 \le j \le J-1, k = 0$ or 1 for $j = 0$ and $1 \le k \le 2^j$ for $j \ne 0$. Then the Haar basis functions are

$$h_0(z) = h_{00}(z) = \frac{1}{\sqrt{M}},\ z \in [0,1] \quad (4.5.1)$$

$$h_x(z) = h_{jk}(z)$$

$$= \frac{1}{\sqrt{M}} \begin{cases} 2^{\frac{j}{2}} \text{ if } \dfrac{(k-1)}{2^j} \le z < \dfrac{(k-0.5)}{2^j} \\ -2^{\frac{j}{2}} \text{ if } \dfrac{(k-0.5)}{2^j} \le z < \dfrac{k}{2^j} \\ 0 \quad \text{otherwise, } z \in [0,1] \end{cases} \quad (4.5.2)$$

The i^{th} row of a $M \times M$ Haar transformation matrix contains the elements of $h_i(z)$ for

$z = 0/M, 1/M, 2/M, \ldots\ldots\ldots, \frac{M-1}{M}$ given that,

$$K = 2^j + k - 1 \quad (4.5.3)$$

$$0 \le j \le J-1 \quad (4.5.4)$$

$$0 \le j \le J-1,\ k = 0 \text{ or } 1 \text{ for } j = 0 \text{ and } 1 \le k \le 2^j \text{ for } j \ne 0 \quad (4.5.5)$$

Example 4.5.1

Consider a signal {1, 4,-3, 0, 2,-1, 5, 3} which can be represent by the discrete functions $f(0) = 1$, $f(1) = 4$, $f(2) = -3$, $f(3) = 0$, $f(4) = 2$, $f(5) = -1$, $f(6) = 5$, $f(7) = 3$, We have $M = 2^J - 8 - 2^3$; $\therefore J = 3$. From (4.5.4), we get $0 \le j \le 2$ when j=0, then k=0 or 1 [From (4.5.5)] and $K = 2^0 + 0 - 1 = 0$ or $K = 2^0 + 1 - 1 = 1$ [From the equation (4.5.3)] when j=1, then $1 \le k \le 2$ [From equation (4.5.5)]. Thus for j=1 and k=1, then K=2, & for j=1 and k=2, then K=3. Again for j=2, then $1 \le k \le 4$. So for j=2 and k=1, then $K = 4$ & for j=2 and k=2, then K=5. Also for j=2 and k=3 then $K = 6$ & for j=2 and k=4 then K=7.

Table I : The values for *K*, *j* and *k*.

K	j	k
0	0	0
1	0	1
2	1	1
3	1	2
4	2	1
5	2	2
6	2	3
7	2	4

We get, $h_0(0) = h_{00}(0) = \frac{1}{\sqrt{M}} = \frac{1}{\sqrt{8}}$

For, $h_1(z) = h_{01}(z) = \frac{1}{\sqrt{8}}\begin{cases} 1 & \text{if } 0 \le z < 0.5 \\ -1 & \text{if } 0.5 \le z < 1 \\ 0 & \text{otherwise}, z \in [0,1] \end{cases}$

$h_1\left(\frac{0}{8}\right) = h_{01}(0) = \frac{1}{\sqrt{8}}$, $h_1\left(\frac{1}{8}\right) = h_{01}(0.125) = \frac{1}{\sqrt{8}}$,

$h_1\left(\frac{2}{8}\right) = h_{01}(0.25) = \frac{1}{\sqrt{8}}$, $h_1\left(\frac{3}{8}\right) = h_{01}(0.375) = \frac{1}{\sqrt{8}}$,

$h_1\left(\frac{4}{8}\right) = h_{01}(0.5) = -\frac{1}{\sqrt{8}}$, $h_1\left(\frac{5}{8}\right) = h_{01}(0.625) = -\frac{1}{\sqrt{8}}$,

$h_1\left(\frac{6}{8}\right) = h_{01}(0.75) = -\frac{1}{\sqrt{8}}$, $h_1\left(\frac{7}{8}\right) = h_{01}(0.875) = -\frac{1}{\sqrt{8}}$

For $h_2(z) = h_{11}(z) = \frac{1}{\sqrt{8}}\begin{cases} \sqrt{2} & \text{if } 0 \le z < 0.25 \\ -\sqrt{2} & \text{if } 0.25 \le z < 0.5 \\ 0 & \text{otherwise}, z \in [0,1] \end{cases}$

$h_2\left(\frac{0}{8}\right) = h_{11}(0) = \frac{\sqrt{2}}{\sqrt{8}}$, $h_2\left(\frac{1}{8}\right) = h_{11}(0.125) = \frac{\sqrt{2}}{\sqrt{8}}$,

$h_2\left(\frac{2}{8}\right) = h_{11}(0.25) = -\frac{\sqrt{2}}{\sqrt{8}}$, $h_2\left(\frac{3}{8}\right) = h_{11}(0.375) = -\frac{\sqrt{2}}{\sqrt{8}}$,

$h_2\left(\frac{4}{8}\right) = h_{11}(0.5) = 0$, $h_2\left(\frac{5}{8}\right) = h_{11}(0.625) = 0$,

$h_2\left(\frac{6}{8}\right) = h_{11}(0.75) = 0$, $h_2\left(\frac{7}{8}\right) = h_{11}(0.875) = 0$

For, $h_3(z) = h_{12}(z) = \frac{1}{\sqrt{8}}\begin{cases} \sqrt{2} & \text{if } 0.5 \le z < 0.75 \\ -\sqrt{2} & \text{if } 0.75 \le z < 1 \\ 0 & \text{otherwise}, z \in [0,1] \end{cases}$

$h_3\left(\frac{0}{8}\right) = h_{12}(0) = 0$, $h_3\left(\frac{1}{8}\right) = h_{12}(0.125) = 0$,

$h_3\left(\frac{2}{8}\right) = h_{12}(0.25) = 0$, $h_3\left(\frac{3}{8}\right) = h_{12}(0.375) = 0$,

$h_3\left(\frac{4}{8}\right) = h_{12}(0.5) = \frac{\sqrt{2}}{\sqrt{8}}$, $h_3\left(\frac{5}{8}\right) = h_{12}(0.625) = \frac{\sqrt{2}}{\sqrt{8}}$,

$h_3\left(\frac{6}{8}\right) = h_{12}(0.75) = -\frac{\sqrt{2}}{\sqrt{8}}$, $h_3\left(\frac{7}{8}\right) = h_{12}(0.875) = -\frac{\sqrt{2}}{\sqrt{8}}$

For, $h_4(z) = h_{21}(z) = \frac{1}{\sqrt{8}}\begin{cases} 2 & \text{if } 0 \le z < 0.125 \\ -2 & \text{if } 0.125 \le z < 0.25 \\ 0 & \text{otherwise}, z \in [0,1] \end{cases}$

$h_4\left(\frac{0}{8}\right) = h_{21}(0) = \frac{2}{\sqrt{8}}$, $h_4\left(\frac{1}{8}\right) = h_{21}(0.125) = -\frac{2}{\sqrt{8}}$,

$h_4\left(\frac{2}{8}\right) = h_{21}(0.25) = 0$, $h_4\left(\frac{3}{8}\right) = h_{21}(0.375) = 0$,

$h_4\left(\frac{4}{8}\right) = h_{21}(0.5) = 0$, $h_4\left(\frac{5}{8}\right) = h_{21}(0.625) = 0$,

$h_4\left(\frac{6}{8}\right) = h_{21}(0.75) = 0$, $h_4\left(\frac{7}{8}\right) = h_{21}(0.875) = 0$

For, $h_5(z) = h_{22}(z) = \frac{1}{\sqrt{8}}\begin{cases} 2 & \text{if } 0.25 \le z < 0.375 \\ -2 & \text{if } 0.375 \le z < 0.5 \\ 0 & \text{otherwise}, z \in [0,1] \end{cases}$

$h_5\left(\frac{0}{8}\right) = h_{22}(0) = 0$, $h_5\left(\frac{1}{8}\right) = h_{22}(0.125) = 0$,

$h_5\left(\frac{2}{8}\right) = h_{22}(0.25) = \frac{2}{\sqrt{8}}$, $h_5\left(\frac{3}{8}\right) = h_{22}(0.375) = -\frac{2}{\sqrt{8}}$,

$h_5\left(\frac{4}{8}\right) = h_{22}(0.5) = 0$, $h_5\left(\frac{5}{8}\right) = h_{22}(0.625) = 0$,

$h_5\left(\frac{6}{8}\right) = h_{22}(0.75) = 0$, $h_5\left(\frac{7}{8}\right) = h_{22}(0.875) = 0$

For, $h_6(z) = h_{23}(z) = \frac{1}{\sqrt{8}}\begin{cases} 2 & \text{if } 0.5 \le z < 0.625 \\ -2 & \text{if } 0.625 \le z < 0.75 \\ 0 & \text{otherwise}, z \subset [0,1] \end{cases}$

$h_6\left(\frac{0}{8}\right) = h_{23}(0) = 0$, $h_6\left(\frac{1}{8}\right) = h_{23}(0.125) = 0$,

$h_6\left(\frac{2}{8}\right) = h_{23}(0.25) = 0$, $h_6\left(\frac{3}{8}\right) = h_{23}(0.375) = 0$,

$h_6\left(\frac{4}{8}\right) = h_{23}(0.5) = \frac{2}{\sqrt{8}}$, $h_6\left(\frac{5}{8}\right) = h_{23}(0.625) = -\frac{2}{\sqrt{8}}$,

$h_6\left(\frac{6}{8}\right) = h_{23}(0.75) = 0$, $h_6\left(\frac{7}{8}\right) = h_{23}(0.875) = 0$

For, $h_7(z) = h_{24}(z) = \frac{1}{\sqrt{8}}\begin{cases} 2 & \text{if } 0.75 \le z < 0.875 \\ -2 & \text{if } 0.875 \le z < 1 \\ 0 & \text{otherwise}, z \in [0,1] \end{cases}$

$h_7\left(\frac{0}{8}\right) = h_{24}(0) = 0$, $h_7\left(\frac{1}{8}\right) = h_{24}(0.125) = 0$,

$h_7\left(\frac{2}{8}\right) = h_{24}(0.25) = 0$, $h_7\left(\frac{3}{8}\right) = h_{24}(0.375) = 0$,

$h_7\left(\frac{4}{8}\right) = h_{24}(0.5) = 0$, $h_7\left(\frac{5}{8}\right) = h_{24}(0.625) = 0$,

$h_7\left(\frac{6}{8}\right) = h_{24}(0.75) = \frac{2}{\sqrt{8}}$, $h_7\left(\frac{7}{8}\right) = h_{24}(0.875) = -\frac{2}{\sqrt{8}}$

Now, we construct the 8×8 transformation matrix, H_8 is

$$H_8 = \frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix} \quad (4.5.6)$$

We have the discrete wavelet transform of a given function $f(x)$ is given by

$$f(x) = \frac{1}{\sqrt{M}}\sum_{k=0}^{2^{j_0}-1} c_{j_0,k}\varphi_{j_0,k}(x) + \frac{1}{\sqrt{M}}\sum_{j=j_0}^{\infty}\sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k}(x), \forall j \geq j_0 \quad (4.5.7)$$

$$c_{j_0,k} = \frac{1}{\sqrt{M}}\sum_{k=0}^{2^j-1} f(x)\varphi_{j_0,k}(x) \quad (4.5.8)$$

$$d_{j,k} = \frac{1}{\sqrt{M}}\sum_{k=0}^{2^j-1} f(x)\psi_{j,k}(x) \quad (4.5.9)$$

Here $f(x)$, $\varphi_{j_0,k}(x)$ and $\psi_{j,k}(x)$ are functions of the discrete variable $x = 0,1,2,\ldots,M-1$. we let, $j_0 = 0$ and $M = 2^J$ which are performed over $x = 0,1,2,\ldots,M-1$, $j = 0,1,2,\ldots,J-1$.

The given discrete functions are

$f(0) = 1$, $f(1) = 4$, $f(2) = -3$, $f(3) = 0$, $f(4) = 2$, $f(5) = -1$, $f(6) = 5$, $f(7) = 3$.

Here, $M = 2^J = 8 = 2^3 \therefore J = 3$ and with $j_0 = 0$, the summation are performed over $x = 0, 1, \ldots\ldots, 7$; $j = 0,1,2$ We will use the Haar scaling & wavelet functions and assumes that eight samples of $f(x)$ are distributed over the support of the basis functions. So, from the first row of the matrix H_4 [from (4.5.6)]

$\varphi_{0,0}(0) = \varphi_{0,0}(1) = \varphi_{0,0}(2) = \varphi_{0,0}(3) =$

$\varphi_{0,0}(4) = \varphi_{0,0}(5) = \varphi_{0,0}(6) = \varphi_{0,0}(7) = 1$.

From (4.5.9), we find

$$c_{0,0} = W_\varphi(0,0) = \frac{1}{\sqrt{8}}\sum_{x=0}^{7} f(x)\varphi_{0,0}(x) = \frac{11}{\sqrt{8}},$$

similarly we get, $\psi_{j,k}(x)$ corresponding to rows of 2, 3, 4, 5, 6, 7 and 8 of H_8.

$\psi_{0,0}(0) = 1$, $\psi_{0,0}(1) = 1$, $\psi_{0,0}(2) = 1$, $\psi_{0,0}(3) = 1$,
$\psi_{0,0}(4) = -1$, $\psi_{0,0}(5) = -1$, $\psi_{0,0}(6) = -1$,
$\psi_{0,0}(7) = -1$, $\psi_{1,0}(0) = \sqrt{2}$, $\psi_{1,0}(1) = \sqrt{2}$,
$\psi_{1,0}(2) = -\sqrt{2}$, $\psi_{1,0}(3) = -\sqrt{2}$, $\psi_{1,0}(4) = 0$,
$\psi_{1,0}(5) = 0$, $\psi_{1,0}(6) = 0$, $\psi_{1,0}(7) = 0$, $\psi_{1,1}(0) = 0$,
$\psi_{1,1}(1) = 0$, $\psi_{1,1}(2) = 0$, $\psi_{1,1}(3) = 0$, $\psi_{1,1}(4) = \sqrt{2}$,
$\psi_{1,1}(5) = \sqrt{2}$, $\psi_{1,1}(6) = -\sqrt{2}$, $\psi_{1,1}(7) = -\sqrt{2}$,
$\psi_{2,1}(0) = 2$, $\psi_{2,1}(1) = -2$, $\psi_{2,1}(2) = 0$, $\psi_{2,1}(3) = 0$,
$\psi_{2,1}(4) = 0$, $\psi_{2,1}(5) = 0$, $\psi_{2,1}(6) = 0$, $\psi_{2,1}(7) = 0$,
$\psi_{2,2}(0) = 0$, $\psi_{2,2}(1) = 0$, $\psi_{2,2}(2) = 2$, $\psi_{2,2}(3) = -2$,
$\psi_{2,2}(4) = 0$, $\psi_{2,2}(5) = 0$, $\psi_{2,2}(6) = 0$, $\psi_{2,2}(7) = 0$,
$\psi_{2,3}(0) = 0$, $\psi_{2,3}(1) = 0$, $\psi_{2,3}(2) = 0$, $\psi_{2,3}(3) = 0$,
$\psi_{2,3}(4) = 2$, $\psi_{2,3}(5) = -2$, $\psi_{2,3}(6) = 0$, $\psi_{2,3}(7) = 0$,
$\psi_{2,4}(0) = 0$, $\psi_{2,4}(1) = 0$, $\psi_{2,4}(2) = 0$, $\psi_{2,4}(3) = 0$,
$\psi_{2,4}(4) = 0$, $\psi_{2,4}(5) = 0$, $\psi_{2,4}(6) = 2$, $\psi_{2,4}(7) = -2$

Again from (4.5.9), we get

$$d_{0,0} = W_\psi(0,0) = \frac{1}{\sqrt{8}}\sum_{x=0}^{7} f(x)\psi_{0,0}(x) = -\frac{7}{\sqrt{8}},$$

$$d_{1,0} = W_\psi(1,0) = \frac{1}{\sqrt{8}}\sum_{x=0}^{7} f(x)\psi_{1,0}(x) = \frac{8\sqrt{2}}{\sqrt{8}},$$

$$d_{1,1} = W_\psi(1,1) = \frac{1}{\sqrt{8}}\sum_{x=0}^{7} f(x)\psi_{1,1}(x) = -\frac{7\sqrt{2}}{\sqrt{8}},$$

$$d_{2,1} = W_\psi(2,1) = \frac{1}{\sqrt{8}}\sum_{x=0}^{7} f(x)\psi_{2,1}(x) = -\frac{6}{\sqrt{8}},$$

$$d_{2,2} = W_\psi(2,2) = \frac{1}{\sqrt{8}}\sum_{x=0}^{7} f(x)\psi_{2,2}(x) = -\frac{6}{\sqrt{8}},$$

$$d_{2,3} = W_\psi(2,3) = \frac{1}{\sqrt{8}}\sum_{x=0}^{7} f(x)\psi_{2,3}(x) = \frac{6}{\sqrt{8}},$$

$$d_{2,4} = W_\psi(2,4) = \frac{1}{\sqrt{8}}\sum_{x=0}^{7} f(x)\psi_{2,4}(x) = \frac{4}{\sqrt{8}}$$

Thus the discrete wavelet transform of given eight sample functions relative to the Haar scaling wavelets are

$$\left\{\frac{11}{\sqrt{8}}, -\frac{7}{\sqrt{8}}, \frac{8\sqrt{2}}{\sqrt{8}}, -\frac{7\sqrt{2}}{\sqrt{8}}, -\frac{6}{\sqrt{8}}, -\frac{6}{\sqrt{8}}, \frac{6}{\sqrt{8}}, \frac{4}{\sqrt{8}}\right\}.$$

Using (4.5.8), $f(0)=1$, $f(1)=4$, $f(2)=-3$, $f(3)=0$, $f(4)=2$, $f(5)=-1$, $f(6)=5$, $f(7)=3$, Thus the graph of the given discrete signal in wavelet transform is

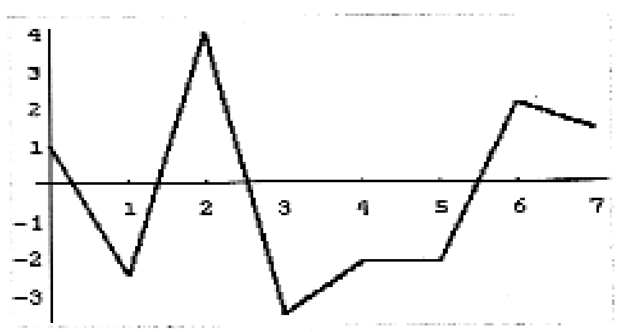

Fig. 26: Graph of Wavelet Transform of the given Function

Conclusion

In our study we discussed about wavelets, wavelet transforms, represent of a function in terms of Haar wavelet. Wavelet transform is a new tool to approximate a function. By using Haar scaling coefficients and Haar wavelet coefficients we approximate continuous functions such as algebraic and exponential functions. We also represent here discrete functions by Haar discrete wavelet. We represent continuous and discrete functions in terms of Haar wavelet because wavelets are well localized in both time and frequency domain, by using wavelet transform we can scale and translate the function and approximate the function by using only a few coefficients.

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