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Ergodic theory of one dimensional Map

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Abstract

In this paper we study one dimensional linear and non-linear maps and its dynamical behavior. We study measure theoretical dynamical behavior of the maps. We study ergodic measure and Birkhoff ergodic theorem. Also, we study some problems using Birkhoff's ergodic theorem.

Key word: Mathematical space, Euclidean space, Probability, Dynamical system, Invariant

Introduction

We study dynamical systems of ergodic theory and the basic theory of measure theoretic dynamical systems, ergodic measure and ergodic theory.

A measure on a mathematical space is a way of assigning weights to different parts of the space, volume is a measure on ordinary three-dimensional Euclidean space. Probability distributions are measures, such that the largest measure of any set is 1 (and some other restrictions). We are interested in a dynamical system, a transformation that maps a space into itself. The set of points applying the transformation repeatedly to a point is called its trajectory or orbit. Some dynamical systems are measure preserving, meaning that the measure of a set is always the same as the measure of the set of points which map to it. Some sets may be invariant; they are the same as their images. An ergodic dynamical system is one in which, with respect to some probability distribution, all invariant sets either have measure 0 or measure 1.

Ergodic theory have been studied by many authors, notable amongst them are Pollicott and Yuri (1998), Billingseley (1965), Walters (2000), Parry (1981). In general the ergodic theorems of Birkhoff and Von Neumam are used in all aspects of dynamical systems and many problems in mathematical physics. Jakobson (2000) discussed ergodic theory of one-dimensional mappings. Jason Preszler (2003) applies ergodic theory in the study of the qualitative actions of a group on a space.

Central aspect of ergodic theory is the behavior of a dynamical system when it is allowed to run for a long period of time. This is expressed through ergodic theorems (Pollicott and Yuri 1998) which assert that, under certain conditions, the time average of a function along the trajectories exists almost everywhere and is related to the space average. If we take any well-behaved (integrable) function of our space, pick a point in the space at random (according to the ergodic distribution) and calculate the average of the function along the point's orbit, the time-average. Then, with probability 1, in the limit as the time goes to infinity (i) the time-average converges to a limit and (ii) that limit is equal to the weighted average of the value of the function at all points in the space (with the weights given by the same distribution), the space-average (Walters 2000).

We study the dynamics in a measure space is traditionally called ergodic theory (even when no ergodicity is involved), since the earliest work in this area countered around the problem of understanding the concept of ergodicity. Now we will give some of the basic definitions and easier results. The present analysis is shown that the measure of tent map is ergodic. Using this we solve some problems in this paper.

Basic Measure Theory

Definition 2.1. (σ-Algebra) A family β of subsets of X is called an **σ-algebra** (Royden 1987) if and only if

1) if $B_n \in \beta$ for $n = 1, 2, 3, \ldots$ then $\bigcap_{n=1}^{\infty} B_n \in \beta$,

2) for any $B \in \beta$ then $X / B \in \beta$,

3) the empty set ϕ belongs to β.

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The elements of β are usually referred to as measurable sets.

Definition 2.2. (Measure) A function $\mu : \beta \to \Re^+$ is called **measure** (Royden 1987) on β if and only if

1) $\mu(B) \geq 0 \ \forall B \in \beta$,

2) $\mu(\phi) = 0$,

3) for any sequence $\{B_n\}$ of disjoint measurable sets $B_n \in \beta, n = 1,2, \ldots\ldots$

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n).$$

Definition 2.3. (Measurable space) A measurable space is a set X with collection of subsets β of X such that

1) $X \in \beta$,

2) if $B \in \beta$ then $X - B \in \beta$,

3) $B_n \in \beta \Rightarrow \bigcup_{n=1}^{\infty} B_n \in \beta$.

The pair (X, β) is then called **a measurable space.**

Definition 2.4. The triple (X, β, μ) is then called a **finite measure space**. We will usually normalize a finite measure by assuming that $\mu(X) = 1$. With this normalization, μ is called **a probability measure** on (X, β) and (X, β, μ) is called a **probability space.** For a **probability measure,** note that $0 \leq \mu(B) \leq 1 \ \forall B \in \beta$.

Definition 2.5. (Invariant measures) Let (X, β, μ) be a measure space. Assume that μ is a probability measure, that is, $\mu(X) = 1$. A measurable map $T : X \to X$ (that is, $T^{-1}\beta \subset \beta$) is said to **preserve the measure** μ if for any $B \in \beta$ we have $\mu(B) = \mu(T^{-1}B)$. Alternatively, we say that μ is *T*-invariant.

Proposition 2.1 (Existence of invariant measures) Let X be a compact metric space and β be the Borel σ-algebra. Given any homeomorphism $T : X \to X$ (or more generally, a continuous map) there exists at least one probability measure μ preserving T.

Measure preserving transformation

The measure preserving transformations are functions on a measure space that preserve the given measure. Consider a measurable transformation T from (X, β) to itself. Also, T is a **measure preserving** if $T_*(\mu) = \mu$, or in other words, if $\mu(B) = \mu(T^{-1}(B))$ for every $B \in \beta$.

We say that T is an invertible **measure preserving transformation** if T is bijective and both T and T^{-1} are measure preserving.

We use the notation $T : (X, \beta, \mu) \to (X, \beta, \mu)$ to denote a measure preserving transformation of a probability space to itself. For instance, if X is a topological structure, then β is always the Borel σ-algebra (that is, the σ-algebra generated by open sets).

Definition 3.1. Suppose (X_1, β_1, μ_1) and (X_2, β_2, μ_2) are two probability spaces.

(i) A transformation $T : X_1 \to X_2$ is measurable if $T^{-1}(\beta_2) \subseteq \beta_1$ (i.e. T is surjective).

(ii) A transformation $T : X_1 \to X_2$ is measure-preserving if *T* is measurable and $\mu_1(T^{-1}(B_2)) = \mu_2(B_2) \quad \forall B_2 \in \beta_2$.

(iii) A transformation $T : X_1 \to X_2$ is an invertible measure-preserving transformation if T is measure-preserving, bijective, and T^{-1} is also measure-preserving.

Exercise 3.1. Verify that if $T_1 : X_1 \to X_2$ and $T_2 : X_2 \to X_3$ are measure preserving transformation then $T_2 \circ T_1 : X_1 \to X_3$ is also a measure preserving transformations.

In ergodic theory, we are interested in long term behavior, so we will focus on measure preserving transformations from a measure space onto itself, then $T : X_1 \to X_1$. Common examples of such measure preserving transformations are the identify transformation (which preserve any measure).

Theorem 3.1. Let (X, β, μ) be a normalized measure space and let $T : X \to X$ be measurable. Let P be a $\pi-$system (A family P of subsets of X is called a $\pi-$**system** if and only if for any A, B in P their intersection $A \cap B$ is also in P) that generates β. If $\mu(T^{-1}A) = \mu(A)$ for any $A \in P$, then T is measure preserving.

Example 3.1. Let $X = [0,1]$, $\beta =$ Borel $\sigma-$algebra of $[0,1]$ and $\lambda =$ Lebesgue measure on $[0,1]$. Let $T : X \to X$ be defined by $T(x) = rx(\text{mod } 1)$, where r is a positive integer greater than or equal to 2. Then T is measure preserving.

First, we would like to determine when two measure preserving transformations are isomorphic and other associated problems. The second type of problem is more external, how can we use results about measure preserving transformations to solve problems in other areas of mathematics or even outside of mathematics? The remainder of this paper will focus on the first type of problems, or the so called isomorphism problem.

Ergodic Measure

Definition 4.1. Given a probability space (X, β, μ), a transformation $T : X \to X$ is called **ergodic** if for every set $B \in \beta$ with $T^{-1}B = B$ then either $\mu(B) = 0$ or $\mu(B) = 1$. Alternatively we say that μ is $T-$**ergodic.**
The following lemma gives a simple characterization in terms of functions.

Lemma 4.1. T is ergodic with respect to μ if and only if whenever $f \in L^1(X, \beta, \mu)$ satisfies $f = foT$ then f is a constant function.

Definition 4.2. (Ergodicity and transitivity) Let μ be a measure on (X, β). A measurable transformation $T : (X, \beta) \to (X, \beta)$ is said to be **ergodic**, with respect to the measure class of μ, if it is not possible to express X as the union of two disjoint set of positive measure, $X = S \cup S_1$ with $S \cap S_1 = \phi$, $\mu(S) > 0$, and $\mu(S_1) > 0$, where $T^{-1}(S) = S$ or equivalently $T^{-1}(S_1) = S_1$, so that S and S_1 are $T-$invariant closely related is the concept of measure transitivity. By definition, T is measure transitive if for any $S, S_1 \in \beta$ with $\mu(S) > 0$ and $\mu(S_1) > 0$ there exists $n > 0$ such that $f^{on}(S) \cap S_1 \neq \phi$, or equivalently $S \cap T^{-n}(S_1) \neq \phi$.
A completely equivalent formulation would be that if $\mu(S_1) > 0$ then the union

$$T^{-1}(S_1) \cup T^{-2}(S_1) \cup T^{-3}(S_1) \cup \Lambda \; \Lambda \; \Lambda$$

is a set of **full measure**, so that it must intersect every set of positive measure.

Corollary 4.1. A measure preserving transformation on a finite measure space is **ergodic** if and only if it is **transitive**.

Theorem 4.1. (Poincare Recurrence Theorem) Let T be a measure-preserving transformation on a normalized measure space (X, β, μ). Let $E \in \beta$ such that $\mu(E) > 0$. Then almost all points of E return infinitely often to E under iterations of T.

Definition 4.3. We call a measure preserving transformation $T : (X, \beta, \mu) \to (X, \beta, \mu)$ **ergodic** if for any $B \in \beta$, such that $T^{-1}B = B$, $\mu(B) = 0$ or $\mu(X \setminus B) = 0$. Since ergodicity (Pollicott and Yuri 1998) is a property of the pair (T, μ) we often say that (T, μ) is ergodic.
As for example Tent map $T(x) = 2x\,(\text{mod } 1), x \in [0, 1]$ is ergodic.

Lemma 4.2. The extremal points in the convex set M are ergodic measures (that is, $\mu \in M$ is ergodic if whenever $\exists\, \mu_1, \mu_2 \in M$ and $0 < \alpha < 1$ with $M = \alpha\,\mu_1 + (1-\alpha)\mu_2$ then $\mu_1 = \mu_2$).
The symbol Δ denotes the **symmetric difference** of sets: $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Definition 4.4. Let (X, β, μ, T) be a **dynamical system**. A set $B \in \beta$ is called $T-$**invariant** if $T^{-1}(B) = B$ and almost $T-$invariant if

$\mu(T^{-1}(B)\Delta B)=0$. Similarly, a measurable function is called $T-$invariant if $f \circ T = f$ and almost $T-$invariant if $f \circ T = f$ is $\mu-$almost everywhere.

Theorem 4.2. The following statements are equivalent for the transformation $T:(X,\beta,\mu)\to(X,\beta,\mu)$ preserving a normalized measure μ:

(i) T is ergodic.
(ii) $\mu(T^{-1}B\,\Delta B)=0,\ B\in\beta \Rightarrow \mu(B)=0$ or 1.
(iii) For any $A,B\in\beta$ with $\mu(A)>0,\quad \mu(B)>0$, there exists $n>0$ such that $\mu(T^{-n}A\cap B)>0$.

Now we write some important lemmas which are related with invariant and ergodic measure.

Lemma 4.3. If a normalized measure μ is $T-$invariant (Billingseley 1965) and $T^{-1}B\subset B$, then there exists a set $B_1\subset B$, $\mu(B\setminus B_1)=0$ and $T^{-1}(B_1)=B_1$.

Lemma 4.4. If a normalized measure μ is $T-$invariant and $\mu(T^{-1}(B)\Delta B)=0$, then there exists a set B_1 such that $\mu(B\,\Delta B_1)=0$, and $T^{-1}(B_1)=B_1$.

Lemma 4.5. If a normalized $T-$invariant measure μ is ergodic, then for any set B such that $T^{-1}(B)\subset B$, we have $\mu(B)$ equal to 0 or 1.

Lemma 4.6. If a normalized $T-$invariant measure μ is ergodic and $\mu(A)>0$, then $\mu\left(\bigcup_{k=1}^{\infty}T^{-k}(A)\right)=1$.

Theorem 4.3. Let $T:(X,\beta,\mu)\to(X,\beta,\mu)$ be measure preserving. Then the following statements are equivalent:

(i) T is ergodic.
(ii) If f is measurable and $(f\circ T)(x)=f(x)$ almost everywhere, then f is constant almost everywhere.
(iii) If $f\in L^2(\mu)$ and $(f\circ T)(x)=f(x)$ almost everywhere, then f is constant almost everywhere.

Proposition 4.1. Let X be a compact metric space and let μ be a Borel normalized measure on X, which gives positive measure to every non-empty open sets. If $T:X\to X$ is continuous and ergodic with respect to μ, then $\mu\{x:\{T^n x : n\ge 0\}$ is dense in $X\}=1$.

Definition 4.5. A system $T:(X,\beta)\to(X,\beta)$ is chaotic if and only if it has an ergodic measure and exhibits sensitive dependence on initial conditions with respect to the measure.

Theorem 4.4. If the topological entropy of a map T is positive, then there exists an ergodic measure such that the measurable entropy is positive.

Ergodic Theorem

Let f be a function which is an observable for a physical quantity. One of the main themes in ergodic theory is to study the asymptotic behavior of their time evolution $\{f\circ T^k\}_{k\in Z^+}$. Under the ergodic hypothesis, their averages $\frac{1}{N}\sum_{k=0}^{N-1} f\circ T^k$ converge to the space average $\int f\,d\mu$. This property also implies the well-known law of large numbers, which is a key concept in statistics (that is the distribution of the long term average converges to the Dirac measure supported on $\int f\,d\mu$).

Let (X,β,μ) be a probability space, and assume that the transformation $T:X\to X$ preserves μ. The Birkhofff "individual" ergodic theorem gives a strong type of ergodic theorem in that it describes the average of functions along individual "typical" orbits. We prove the theorem under the additional assumption that μ is ergodic.

Theorem 5.1. (Birkhoff's Theorem (Ergodic Version)) Consider $f\in L^1(X,\beta,\mu)$. If the measure μ is ergodic then for almost all $x\in X$ we have that the averages

$$\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x)\to\int f\,d\mu \text{ as } N\to+\infty$$

that is, $\mu\left\{x\in X:\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x)\ne\int f\,d\mu\right\}=0$.

Clearly if T is ergodic then f^* is constant almost everywhere and if $\mu(X)<\infty$ then $f^*=\left(\frac{1}{\mu(X)}\right)\int f\,dm$. Furthermore if (X,β,μ) is a probability space and T is ergodic $\forall f \in L^1(\mu)$ then $\lim_{N\to\infty}\left(\frac{1}{N}\right)\sum_{n=0}^{N-1} f(T^n x) = \int f\, dm \quad a.e.$

The Birkhoff ergodic theorems are uses in statistical mechanics, but also to number theory and dynamical systems.

The following corollary is due to Von Neumann.

Corollary 5.1. (L^p Ergodic Theorem of Von Neumann) Let $1 \le p < \infty$ and let T be a measure-preserving transformation of the probability space (X, β, μ). If $f \in L^P(\mu)$ there exist $f^* \in L^P(\mu)$ with $f^* oT = f^*$ a.e. and $\left\| \left(\frac{1}{N}\right)\sum_{n=0}^{N-1} f\left(T^n(x) - f^*(x)\right) \right\|_p \to 0$.

Interestingly enough the theorem of Von Neumann was published a year before Birkhoff's result.

Next corollary provides another criteria for ergodicity.

Corollary 5.2. Let (X, β, μ) be a probability space and let $T : X \to X$ be a measure preserving transformation. Then T is ergodic if and only if $\forall\ A, B \in \beta$ then

$$\frac{1}{N}\sum_{n=0}^{N-1} \mu\left(T^{-n} A \cap B\right) \to \mu(A)\,\mu(B).$$

Problem 5.1. Let $f(x) = ax(1-x)$ be the Logistic map. Let $T : [0,1] \to [0,1]$ be the doubling map. Use Birkhoff's ergodic theorem to show that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f\left(T^i x\right) = \frac{a}{6}$$

for Lebesgue almost every $x \in [0,1]$.

Solution: We know that, by Birkhoff's ergodic theorem,

$$\frac{1}{n}\sum_{i=0}^{n-1} f\left(T^i x\right) \to \int f\, d\mu \text{ as } n \to +\infty$$

that is, $\mu\left\{x \in X : \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f(T^n x) \ne \int f\, dx\right\} = 0.$

Here, $f(x) = ax(1-x)$, $0 < a \le 4$.. Now, $\int_0^1 ax(1-x)dx = \frac{a}{6}$.

Therefore, $\frac{1}{n}\sum_{i=0}^{n-1} f\left(T^i x\right) \to \frac{a}{6}$. So we write $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f\left(T^i x\right) \to \frac{a}{6}$.

Hence $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f\left(T^i x\right) = \frac{a}{6}$.

When $a = 4$ then this map is chaotic and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f\left(T^i x\right) = \frac{2}{3}.$$

Problem 5.2. Let $f : [0,1] \to R$ be defined by $f(x) = x^2$. Let $T : [0,1] \to [0,1]$ be the doubling map. Use Birkhoff's ergodic theorem to show that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f\left(T^i x\right) = \frac{1}{3}$$

for Lebesgue almost every $x \in [0,1]$.

Solution: We know that, by Birkhoff's ergodic theorem,

$\frac{1}{n}\sum_{i=0}^{n-1} f\left(T^i x\right) \to \int f\, d\mu$ as $n \to +\infty$

that is, $\mu\left\{x \in X : \lim_{n\to+\infty}\frac{1}{n}\sum_{i=0}^{n-1} f(T^i x) \ne \int f\, d\mu\right\} = 0.$

Here $f(x) = x^2$. Now, $\int_0^1 x^2 dx = \frac{1}{3}$.

Therefore, $\frac{1}{n}\sum_{i=0}^{n-1} f(T^i x) \to \frac{1}{3}$.

So, we write $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f\left(T^i x\right) \to \frac{1}{3}$. Hence,

$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f\left(T^i x\right) = \frac{1}{3}$.

Corollary 5.3. Let (X, β, μ) be a probability space, and assume that the transformation $T : X \to X$ preserves μ. The proportion of time spent by almost all points in a subset $B \in \beta$ is given by it measure $\mu(B)$, that is,

$$\lim_{N\to+\infty}\frac{1}{N} Card\{0 \le n \le N-1 : T^n x \in B\} = \mu(B)$$

for almost all points $x \in X$.

Example 5.1. (Lack of convergence on a set of zero measure) Consider the map $T : \Re/Z \to \Re/Z$ defined by $T(x) = 2x \bmod(1)$ and the usual Haar-Lebesgue measure μ.

Consider any continuous function $f : \Re/Z \to \Re$ such that $f(0) \neq \int f \, d\mu$. Clearly,

$$\frac{1}{N}\sum_{n=0}^{N-1} f(T^n 0) = f(0).$$

Since $T^n 0 = 0$, for all $n \geq 0$. But since $f(0) \neq \int f \, d\mu$ we see that at the point 0 the sequence does not converge to the integral.

Conclusion

Ergodic measures are closely related to invariant measure. The collection of invariant probability measures for a given map form a convex subset of the set of all probability measures on the space X. The ergodic probability measures are precisely the extremal points of the set of invariant probability measures. In this paper, we discuss Birkhoff theorem for ergodic version. We try to solve some problems using this theorem. We explain some of the important examples of measure preserving transformation.

Applications of ergodic theory to other parts of mathematics usually involve establishing ergodicity properties for systems of special kind. In geometry, methods of ergodic theory have been used to study the geodesic flow on Riemannian manifolds, starting with the results of Eberhard Hopf for Riemann surfaces of negative curvature. Ergodic theory has fruitful connections with, harmonic analysis, Lie theory (representation theory, lattices in algebraic groups), and number theory.

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