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(U, M)*-derivations in completely semiprime* Γ*-rings*

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Abstract

The objective of this paper is to extend and generalize some results of (Rahman and Paul, 2014) in completely semiprime Γ- rings. We prove that, if U is an admissible Lie ideal of a completely semiprime Γ-ring M and d is a (U, M)-derivation of M then $d(u\alpha v) = d(u)\alpha v + u\alpha d(v)$ for all $u, v \in U$ and $\alpha \in \Gamma$.

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Introduction

Herstein (1957) proved a well-known result in prime rings which states that 'every Jordan derivation on a 2-torsion free prime ring is a derivation'. Afterwards many Mathematicians studied extensively the derivations in prime rings. Awtar (1984) extended Herstein's result to Lie ideals. (U, R)-derivations in rings have been introduced by Faraj *et al* (2010) as a generalization of Jordan derivations on a Lie ideal U of a ring R. The notion of a (U, R)-derivation extends the concept given by Awtar (1984). Faraj *et al* (2010) proved that if R is a prime ring, char$(R) \neq 2$, U is a square closed Lie ideal of R and d is a (U, R)-derivation of R, then $d(ur) = d(u)r + ud(r)$ for all $u \in U, r \in R$. This result is a generalization of a result of Awtar (1984). Some extensive results of left derivation and Jordan left derivation of a Γ-ring were determined by Ceven (2002). Halder and Paul (2012) extended the results of Ceven (2002) to Lie ideals. In this article, we have generalized a result of (Rahman and Paul 2014) in completely semiprime Γ-rings by (U, M)-derivation.

Preliminaries

Barnes (1966) generalized the notion of a Γ-ring which was introduced by Nobusawa (1964). Let M and Γ be additive abelian groups. If there is a mapping $M \times \Gamma \times M \to M$ such that

(i) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)y = x\alpha y + x\beta y$, $x\alpha(y + z) = x\alpha y + x\alpha z$

(ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

then M is called a Γ-ring. This concept is more general than that of a ring. A Γ-ring M is *completely semiprime* if $a\Gamma a = 0$ (with $a \in M$) implies $a = 0$. A Γ-ring M is *2-torsion free* if $2a = 0$ implies $a = 0$ for all $a \in M$. For any $x, y \in M$ and $\alpha \in \Gamma$, the *Lie product* is defined by $[x, y]_\alpha = x\alpha y - y\alpha x$. An additive subgroup $U \subseteq M$ is said to be a *Lie ideal* if $u \in U, m \in M$ and $\alpha \in \Gamma$ implies $[u, m]_\alpha \in U$. A Lie ideal U is *square closed* if it satisfies $u\alpha u \in U$ for all $u \in U, \alpha \in \Gamma$ and a Lie ideal U is an *admissible Lie ideal* of M if U is square closed and $U \not\subseteq Z(M)$, where $Z(M)$ denotes the center of M.

We introduce the concept of (U, M)-derivation of a Γ-ring using the notion of (U, R)-derivation of a ring due to Faraj *et al* (2010) as follows:

Definition 2.1 Let M be a Γ-ring and U be a Lie ideal of M. An additive mapping $d: M \to M$ is said to be a (U, M)-derivation of M if for all $u \in U; m, s \in M$ and $\alpha \in \Gamma$

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$d(u\alpha m+s\alpha u)=d(u)\alpha m+u\alpha d(m)+d(s)\alpha u+s\alpha d(u)$ holds.

Example 2.1 Suppose R is an associative ring with 1, and U U is a Lie ideal of R. Let $M=M_{1,2}(R)$ and $\Gamma=\left\{\begin{pmatrix} n.1 \\ 0 \end{pmatrix}: n\in Z\right\}$, then M is a Γ-ring. Let $N=\{(x,x):x\in R\}\subseteq M$, then N is a sub Γ-ring. Let $U_1=\{(u,u):u\in U\}$, then for $una-anu\in U$

$$(u,u)\begin{pmatrix} n \\ 0 \end{pmatrix}(a,a)-(a,a)\begin{pmatrix} n \\ 0 \end{pmatrix}(u,u)$$

$$=(una,una)-(anu,anu)$$

$$=(una-anu,una-anu)\in U.$$

Thus, U_1 is a Lie ideal of N. Let $d:R\to R$ be a (U,R)-derivation. Now, we define a mapping $D:N\to N$ by $D((x,x))=(d(x),d(x))$. Then

$$D((u,u)\begin{pmatrix} n \\ 0 \end{pmatrix}(a,a)+(b,b)\begin{pmatrix} n \\ 0 \end{pmatrix}(u,u))$$
$$=D((una,una)+(bnu,bnu))$$
$$=D((una+bnu,una+bnu))$$
$$=(d(una+bnu),d(una+bnu))$$
$$=(d(u)na+und(a)+d(b)nu+bnd(u),d(u)na+und(a)+d(b)nu+bnd(u))$$
$$=(d(u)na+und(a),d(u)na+und(a))+(d(b)nu+bnd(u),d(b)nu+bnd(u))$$
$$=(d(u)na,d(u)na)+(und(a),und(a))+(d(b)nu,d(b)nu)+(bnd(u),bnd(u))$$
$$=(d(u),d(u))\begin{pmatrix} n \\ 0 \end{pmatrix}(a,a)+(u,u)\begin{pmatrix} n \\ 0 \end{pmatrix}(d(a),d(a))+(d(b),d(b))\begin{pmatrix} n \\ 0 \end{pmatrix}(u,u)+(b,b)\begin{pmatrix} n \\ 0 \end{pmatrix}(d(u),d(u))$$
$$=D((u,u))\begin{pmatrix} n \\ 0 \end{pmatrix}(a,a)+(u,u)\begin{pmatrix} n \\ 0 \end{pmatrix}(D((a,a))+D((b,b))\begin{pmatrix} n \\ 0 \end{pmatrix}(u,u)+(b,b)\begin{pmatrix} n \\ 0 \end{pmatrix}D((u,u))$$
$$=D(u_1)\alpha x+u_1\alpha D(x)+D(y)\alpha u_1+y\alpha D(u_1),$$

where $u_1=(u,u),\alpha=\begin{pmatrix} n \\ 0 \end{pmatrix},x=(a,a),y=(b,b)$.

Therefore,

$$D(u_1\alpha x+y\alpha u_1)=D(u_1)\alpha x+u_1\alpha D(x)+D(y)\alpha u_1+y\alpha D(u_1).$$

Hence D is a (U_1,N) – derivation of N.

(U, M) -Derivations in Semiprime Γ-Rings

In order to prove the main result, we have to establish some necessary results in the following way. All these results are due to the concept of (U,M)-derivations of a Γ-ring M.

We begin with the following lemma.

Lemma 3.1 Let d be a (U,M)-derivation of a Γ-ring M. Then for all $u\in U, m\in M$ and $\alpha,\beta\in\Gamma$, $d(u\alpha m\beta u)=d(u)\alpha m\beta u+u\alpha d(m)\beta u+u\alpha m\beta d(u)$.

Proof. By (U,M)-derivation of M, for all $u\in U, m,s\in M$ and $\alpha\in\Gamma$, we have

$$d(u\alpha m+s\alpha u)=d(u)\alpha m+u\alpha d(m)+d(s)\alpha u+s\alpha d(u)$$

If we replace both m and s by $(2u)\beta m+m\beta(2u)$ and suppose that

$$v=u\alpha((2u)\beta m+m\beta(2u))+((2u)\beta m+m\beta(2u))\alpha u$$

Then using (U,M)-derivation and the assumption (*).

$$\begin{aligned}d(v)&=2(d(u)\alpha(u\beta m+m\beta u)+u\alpha d(u\beta m+m\beta u)\\&+d(u\beta m+m\beta u)\alpha u+(u\beta m+m\beta u)\alpha d(u))\\&=2(d(u)\alpha u\beta m+d(u)\alpha m\beta u+u\alpha d(u)\beta m\\&+u\alpha u\beta d(m)+u\alpha d(m)\beta u+u\alpha m\beta d(u)\\&+d(u)\beta m\alpha u+u\beta d(m)\alpha u+d(m)\beta u\alpha u\\&+m\beta d(u)\alpha u+u\beta m\alpha d(u)+m\beta u\alpha d(u))\\&=2(d(u)\alpha u\beta m+d(u)\alpha m\beta u+u\alpha d(u)\beta m\\&+u\alpha u\beta d(m)+u\alpha d(m)\beta u+u\alpha m\beta d(u)\\&+d(u)\alpha m\beta u+u\alpha d(m)\beta u+d(m)\alpha u\beta u\\&+m\alpha d(u)\beta u+u\alpha m\beta d(u)+m\alpha u\beta d(u))\end{aligned}$$

Again, we obtain

$$d(v) = d((2u\alpha u)\beta m + m\beta(2u\alpha u)) + 2d(u\alpha m\beta u) + 2d(u\beta m\alpha u)$$
$$= 2(d(u)\alpha u\beta m + u\alpha d(u)\beta m + u\alpha u\beta d(m) + d(m)\beta u\alpha u + m\beta d(u)\alpha u + m\beta u\alpha d(u) + 2d(u\alpha m\beta u) + 2d(u\alpha m\beta u)$$
$$= 2(d(u)\alpha u\beta m + u\alpha d(u)\beta m + u\alpha u\beta d(m) + d(m)\alpha u\beta u + m\alpha d(u)\beta u + m\alpha u\beta d(u)) + 4d(u\alpha m\beta u)$$

Equating the two expressions for $d(v)$ and cancelling the like terms from both sides, we get

$$4d(u\alpha m\beta u) = 4d(u)\alpha m\beta u + 4u\alpha d(m)\beta u + 4u\alpha m\beta d(u).$$

By the 2-torsion freeness of M, we obtain

$$d(u\alpha m\beta u) = d(u)\alpha m\beta u + u\alpha d(m)\beta u + u\alpha m\beta d(u)$$

for all $u \in U, m \in M$ and $\alpha, \beta \in \Gamma$.

Definition 3.1

Let M be a 2-torsion free completely semiprime Γ-ring satisfying the condition (*), and U be a Lie ideal of M. Let d be a (U, M)- derivation of M. Then for all $a, b \in U$ and $\alpha \in \Gamma$, we define $T_\alpha(a,b) = d(a\alpha b) - d(a)\alpha b - a\alpha d(b)$.

We get the following lemma as the consequece of the previous definition.

Lemma 3.2

Let M be 2-torsion free completely semiprime Γ-ring satisfying the condition (*) and U be a Lie ideal of M. Let d be a (U, M)- derivation of M. Then for all $a, b, c \in U$ and $\alpha, \beta \in \Gamma$, the following statements hold:

(i) $T_\alpha(a,b) + T_\alpha(b,a) = 0$;

(ii) $T_\alpha(a+b,c) = T_\alpha(a,c) + T_\alpha(b,c)$;

(iii) $T_\alpha(a,b+c) = T_\alpha(a,b) + T_\alpha(a,c)$;

(iv) $T_{\alpha+\beta}(a,b) = T_\alpha(a,b) + T_\beta(a,b)$.

To reach our goal we need an important result as below.

Lemma 3.3 Let M be a 2-torsion free completely semiprime Γ-ring satisfying the condition (*) and U be a Lie ideal of M. If $u \in U$ such that $[u,[u,x]_\alpha]_\alpha = 0$ for all $x \in M$ and $\alpha \in \Gamma$, then $[u,x]_\alpha = 0$.

Proof. Since $[u,[u,x]_\alpha]_\alpha = 0$ for all $x \in M$ and $\alpha \in \Gamma$. Let $\gamma \in \Gamma$ be any element.

Replacing x by $x\gamma x$, we obtain

$$0 = [u,[u,x\gamma x]_\alpha]_\alpha$$
$$= [u, x\gamma[u,x]_\alpha + [u,x]_\alpha\gamma x]_\alpha$$
$$= [u, x\gamma[u,x]_\alpha]_\alpha + [u,[u,x]_\alpha\gamma x]_\alpha$$
$$= x\gamma[u,[u,x]_\alpha]_\alpha + [u,x]_\alpha\gamma[u,x]_\alpha + [u,[u,x]_\alpha]_\alpha\gamma x + [u,x]_\alpha\gamma[u,x]_\alpha$$
$$= 2[u,x]_\alpha\gamma[u,x]_\alpha.$$

By the 2-torsion freeness of M, $[u,x]_\alpha\gamma[u,x]_\alpha = 0$. Since M is completely semiprime, so $[u,x]_\alpha = 0$ for all $x \in M$ and $\alpha \in \Gamma$. This completes the proof.

It follows the following lemma.

Lemma 3.4 Let M be a 2-torsion free completely emiprime Γ-ring satisfying the condition (*), and U be a commutative Lie ideal of M, then $U \subseteq Z(M)$.

Proof. Since U is a commutative Lie ideal of a completely semiprime Γ-ring M, so we have $[u,[u,x]_\alpha]_\alpha = 0$ for all $u \in U, x \in M$ and $\alpha \in \Gamma$. In view of Lemma 3.3, we get $[u,x]_\alpha = 0$, which implies $U \subseteq Z(M)$.

Then in view of Lemma 3.4 we can state the following:

Lemma 3.5 If U is a non-zero sub-Γ-ring and a Lie ideal of a 2-torsion free completely semiprime Γ-ring M, then either $U \subset Z(M)$ or U contains a non-zero ideal of M.

Proof. If we consider U is a commutative Lie ideal of M, then by Lemma 3.4, $U \subseteq Z(M)$. So let U be non-commutative, then for some $u, v \in M$ and $\alpha \in \Gamma$, we have $[u,v]_\alpha \in U$. Hence there exists an ideal S of M generated by $[u,v]_\alpha (\neq 0)$ and $S \subseteq U$.

This leads us to state the following:

Lemma 3.6 : Let M be a 2-torsion free completely semiprime Γ-ring satisfying the condition (*). If $U \subseteq Z(M)$, then $Z(U) = Z(M)$.

Proof. Since $Z(U)$ is both a sub-Γ-ring and a Lie ideal of M such that $Z(U)$ does not contain non-zero ideal of

M. Therefore, by Lemma 3.5, we obtain that $Z(U)\subseteq Z(M)$. Hence $Z(U)=Z(M)$.

In view of Lemma 3.4 and Lemma 3.6 we can conclude the following :

Lemma 3.7 Let M be a 2-torsion free completely semiprime Γ-ring satisfying the condition (*) and U be a Lie ideal of M, then $Z([U,U]_\Gamma)=Z(U)$.

Proof. Suppose that a is any element of M. If $[a,[U,U]_\Gamma]_\Gamma=0$, then we have $[a,U]_\Gamma=0$. Thus we get $Z([U,U]_\Gamma)=Z(U)$. If $[U,U]_\Gamma\subseteq Z(M)$ then by Lemma 3.6, $a\in Z(U)$. So a centralizes U. Now, let $[U,U]_\Gamma\subseteq Z(M)$. Then we have $[u,[u,a]_\alpha]_\alpha=0$ for all $u\in U, a\in M$ and $\alpha\in\Gamma$. Using Lemma 3.4, we get $[u,a]_\alpha=0$ for all $u\in U, a\in M$ and $\alpha\in\Gamma$. Therefore, $a\in Z(U)$. Thus the proof of the lemma is completed.

In order to prove our main result, we need to costruct the following important result.

Lemma 3.8 : Let M be 2-torsion free completely semiprime Γ-ring satisfying the condition (*) and U be a Lie ideal of M. If d is a (*U, M*)- derivation of M. Then for all $a,b\in U$ and $\alpha,\beta\in\Gamma$: $T_\alpha(a,b)\beta[a,b]_\alpha+[a,b]_\alpha\beta T_\alpha(a,b)=0$.

Proof. Let $a,b\in U$ and $\alpha,\beta\in\Gamma$ be any elements. Suppose $x=2(a\alpha b\beta b\alpha a+b\alpha a\beta a\alpha b)$.

Using Definition 2.1, we get

$$\begin{aligned} d(x) &= d((2a\alpha b)\beta(b\alpha a)+(b\alpha a)\beta(2a\alpha b)) \\ &= 2d(a\alpha b)\beta(b\alpha a)+2(a\alpha b)\beta d(b\alpha a) \\ &\quad + 2d(b\alpha a)\beta(a\alpha b)+2(b\alpha a)\beta d(a\alpha b). \end{aligned}$$

Using Lemma 3.1, we obtain

$$\begin{aligned} d(x) &= d(2(a\alpha(b\beta b)\alpha a)+2(b\alpha(a\beta a)\alpha b)) \\ &= 2d(a)\alpha(b\beta b)\alpha a+2a\alpha d(b\beta b)\alpha a \\ &\quad + 2a\alpha(b\beta b)\alpha d(a)+2d(b)\alpha(a\beta a)\alpha b \\ &\quad + 2b\alpha d(a\beta a)\alpha b+2b\alpha(a\beta a)\alpha d(b) \\ &= 2d(a)\alpha b\beta b\alpha a+2a\alpha d(b)\beta b\alpha a \\ &\quad + 2a\alpha b\beta d(b)\alpha a+2a\alpha b\beta b\alpha d(a) \\ &\quad + 2d(b)\alpha a\beta a\alpha b+2b\alpha d(a)\beta a\alpha b \\ &\quad + 2b\alpha a\beta d(a)\alpha b+2b\alpha a\beta a\alpha d(b). \end{aligned}$$

Comparing the two expressions for $d(x)$

$$\begin{aligned} & 2(d(a\alpha b)-d(a)\alpha b-a\alpha d(b))\beta b\alpha a \\ & + 2(d(b\alpha a)-d(b)\alpha a-b\alpha d(a))\beta a\alpha b+ \\ & 2a\alpha b\beta(d(b\alpha a)-d(b)\alpha a-b\alpha d(a)) \\ & + 2b\alpha a\beta(d(a\alpha b)-d(a)\alpha b-a\alpha d(b))=0. \end{aligned}$$

Using Definition 3.1, we obtain

$$\begin{aligned} & 2T_\alpha(a,b)\beta b\alpha a+2T_\alpha(b,a)\beta a\alpha b+2a\alpha b\beta T_\alpha(b,a) \\ & +2b\alpha a\beta T_\alpha(a,b)=0. \end{aligned}$$

Using Lemma 3.2 (i) and applying 2-torsion freeness of M, we get

$$\begin{aligned} & 2T_\alpha(a,b)\beta b\alpha a-2T_\alpha(a,b)\beta a\alpha b-2a\alpha b\beta T_\alpha(a,b) \\ & +2b\alpha a\beta T_\alpha(a,b)=0. \\ & \Rightarrow T_\alpha(a,b)\beta[a,b]_\alpha+[a,b]_\alpha\beta T_\alpha(a,b)=0. \end{aligned}$$

To prove Corollary 3.1 we need the following lemma.

Lemma 3.9 Let M be a 2-torsion free completely semiprime Γ-ring, U be a Lie ideal of M and let $a,b\in U$ and $\alpha\in\Gamma$. If $a\alpha b+b\alpha a=0$ then $a\alpha b=0=b\alpha a$.

Proof. Let $a,b\in U$ and $\alpha\in\Gamma$ such that $a\alpha b+b\alpha a=0$.

Suppose $\beta\in\Gamma$ be any element. Then applying $a\alpha b=-b\alpha a$ and 2-torsion freeness of M :

$$\begin{aligned} 4(a\alpha b)\beta(a\alpha b) &= -4(b\alpha a)\beta(a\alpha b) \\ &= -4(b(\alpha a\beta)a)\alpha b \\ &= 4(a(\alpha a\beta)b)\alpha b \\ &= 2a\alpha(2a\beta b)\alpha b \\ &= -2a\alpha(2b\beta a)\alpha b \\ &= -4(a\alpha b)\beta(a\alpha b. \\ &\Rightarrow 8((a\alpha b)\beta(a\alpha b))=0. \\ &\Rightarrow (a\alpha b)\beta(a\alpha b)=0, \quad \text{for all } \beta\in\Gamma. \\ &\Rightarrow (a\alpha b)\Gamma(a\alpha b)=0. \end{aligned}$$

Since M is completely semiprime, so $a\alpha b=0$. Proceeding in the similar way $b\alpha a=0$.

As an immediate consequence, we have

Corollary 3.1 Let M be a 2-torsion free completely semiprime Γ-ring satisfying the condition (*), U be a Lie ideal of M and let d be a (*U, M*)- derivation of M. Then for all $a,b \in U$ and $\alpha,\beta \in \Gamma$: (i) $T_\alpha(a,b)\beta[a,b]_\alpha = 0$;

(ii) $[a,b]_\alpha \beta T_\alpha(a,b) = 0$.

Proof. Using the result of Lemma 3.9 in that of Lemma 3.8, we get these results.

Next, we go through the following results.

Lemma 3.10 Let M be a 2-torsion free completely semiprime Γ-ring satisfying the condition(*), U be a Lie ideal of M and d be a (U, M)- derivation of M. Then for all $a,b,x,y \in U$ and $\alpha,\beta,\gamma \in \Gamma$:

(i) $T_\alpha(a,b)\beta[x,y]_\alpha = 0$; (ii) $[x,y]_\alpha \beta T_\alpha(a,b) = 0$

(iii) $T_\alpha(a,b)\beta[x,y]_\gamma = 0$; (iv) $[x,y]_\gamma \beta T_\alpha(a,b) = 0$.

Proof. (i) Replacing a by $a+x$ in Corollary 3.1 (i) and using Lemma 3.2(ii), we get

$T_\alpha(a+x,b)\beta[a+x,b]_\alpha = 0.$

$\Rightarrow T_\alpha(a,b)\beta[a,b]_\alpha + T_\alpha(a,b)\beta[x,b]_\alpha$
$+ T_\alpha(x,b)\beta[a,b]_\alpha + T_\alpha(x,b)\beta[x,b]_\alpha = 0.$

Using Corollary 3.1(i)

$$T_\alpha(a,b)\beta[x,b]_\alpha + T_\alpha(x,b)\beta[a,b]_\alpha = 0.$$

$$\Rightarrow T_\alpha(a,b)\beta[x,b]_\alpha = -T_\alpha(x,b)\beta[a,b]_\alpha$$

Since

$(T_\alpha(a,b)\beta[x,b]_\alpha)\beta(T_\alpha(a,b)\beta[x,b]_\alpha)$
$= -T_\alpha(a,b)\beta[x,b]_\alpha \beta T_\alpha(x,b)\beta[a,b]_\alpha = 0.$

By the complete semiprimeness of M, we have

$T_\alpha(a,b)\beta[x,b]_\alpha = 0.$

If we replace b by $b+y$ in this result, we get $T_\alpha(a,b)\beta[x,y]_\alpha = 0.$

(ii) By the similar replacements successively in Corollary 3.1 (ii), we get

$[x,y]_\alpha \beta T_\alpha(a,b) = 0$ for all $a,b,x,y \in U$ and $\alpha,\beta \in \Gamma$.

(iii) Replacing $\alpha+\gamma$ for α in (i), we obtain

$T_{\alpha+\gamma}(a,b)\beta[x,y]_{\alpha+\gamma} = 0.$

Using Lemma 3.2(iv), we get

$(T_\alpha(a,b)+T_\gamma(a,b))\beta([x,y]_\alpha + [x,y]_\gamma) = 0.$

$\Rightarrow T_\alpha(a,b)\beta[x,y]_\alpha + T_\alpha(a,b)\beta[x,y]_\gamma$
$+ T_\gamma(a,b)\beta[x,y]_\alpha + T_\gamma(a,b)\beta[x,y]_\gamma = 0.$

Using (i), we get

$T_\alpha(a,b)\beta[x,y]_\gamma + T_\gamma(a,b)\beta[x,y]_\alpha = 0.$

$\Rightarrow T_\alpha(a,b)\beta[x,y]_\gamma = -T_\gamma(a,b)\beta[x,y]_\alpha.$

Therefore,

$(T_\alpha(a,b)\beta[x,y]_\gamma)\beta(T_\alpha(a,b)\beta[x,y]_\gamma)$
$= -T_\alpha(a,b)\beta[x,y]_\gamma \beta T_\gamma(a,b)\beta[x,y]_\alpha = 0.$

Since M is completely semiprime, thus

$T_\alpha(a,b)\beta[x,y]_\gamma = 0.$

(iv) By the similar replacement in (ii), we obtain this.

Now, we are ready to prove our main result as follows.

Remark 3.2: If U is a commutative Lie ideal of a Γ-ring M, then $U \subseteq Z(M)$. So, by the Definition 2.1 and using 2-torsion freeness of M, we get $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$ for all $a,b \in U$ and $\alpha \in \Gamma$.

Therefore, for the final result we consider $U \not\subseteq Z(M)$.

Theorem 3.1 Let M be a 2-torsion free completely semiprime Γ-ring satisfying the condition(*), U be an admissible Lie ideal of M and d be a (U, M)- derivation of M, then $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$ for all $a,b \in U$ and $\alpha \in \Gamma$.

Proof. In view of Lemma 3.10 (iii), we have $T_\alpha(a,b)\beta[x,y]_\gamma = 0$ for all $a,b,x,y \in U$ and $\alpha,\beta,\gamma \in \Gamma$. By Lemma 3.10(iv), $[x,y]_\gamma \beta T_\alpha(a,b) = 0$ for all $a,b,x,y \in U$ and $\alpha,\beta,\gamma \in \Gamma$.

Since

$[T_\alpha(a,b),[x,y]_\gamma]_\beta = T_\alpha(a,b)\beta[x,y]_\gamma$
$-[x,y]_\gamma \beta T_\alpha(a,b) = 0.$

So $T_\alpha(a,b) \subseteq Z([U,U]_\Gamma) = Z(U) = Z(M)$, by Lemma 3.6 and 3.7.

Therefore, $T_\alpha(a,b)\in Z(M)$. Now, we obtain

$$\begin{aligned}2T_\alpha(a,b)\beta T_\alpha(a,b) &= T_\alpha(a,b)\beta(T_\alpha(a,b)+T_\alpha(a,b))\\ &= T_\alpha(a,b)\beta(T_\alpha(a,b)-T_\alpha(b,a))\\ &= T_\alpha(a,b)\beta(d(a\alpha b)-d(a)\alpha b\\ &\quad -a\alpha d(b)-d(b\alpha a)+d(b)\alpha a\\ &\quad +b\alpha d(a))\\ &= T_\alpha(a,b)\beta(d(a\alpha b-b\alpha a)\\ &\quad +(b\alpha d(a)-d(a)\alpha b)\\ &\quad +(d(b)\alpha a-a\alpha d(b)))\\ &= T_\alpha(a,b)\beta(d([a,b]_\alpha)\\ &\quad +[b,d(a)]_\alpha+[d(b),a]_\alpha)\\ &= T_\alpha(a,b)\beta d([a,b]_\alpha)\\ &\quad +T_\alpha(a,b)\beta[b,d(a)]_\alpha\\ &\quad +T_\alpha(a,b)\beta[d(b),a]_\alpha.\end{aligned}$$

Since $d(a),d(b)\in M$ and $a,b\in U$ implies that $[b,d(a)]_\alpha,[d(b),a]_\alpha\in U$.

Thus by Lemma 3.10 , we have

$$T_\alpha(a,b)\beta[b,d(a)]_\alpha=T_\alpha(a,b)\beta[d(b),a]_\alpha=0.$$

Therefore, we get

$$2T_\alpha(a,b)\beta T_\alpha(a,b)=T_\alpha(a,b)\beta d([a,b]_\alpha). \qquad (1)$$

Now, we obtain

$$\begin{aligned}0 &= d(T_\alpha(a,b)\beta[x,y]_\gamma+[x,y]_\gamma\beta T_\alpha(a,b))\\ &= d(T_\alpha(a,b))\beta[x,y]_\gamma+T_\alpha(a,b)\beta d([x,y]_\gamma)\\ &\quad +d([x,y]_\gamma)\beta T_\alpha(a,b)+[x,y]_\gamma\beta d(T_\alpha(a,b))\\ &= d(T_\alpha(a,b))\beta[x,y]_\gamma+2T_\alpha(a,b)\beta d([x,y]_\gamma)\\ &\quad +[x,y]_\gamma\beta d(T_\alpha(a,b)).\end{aligned}$$

$T_\alpha(a,b)\in Z(M)$ implies that

$$d([x,y]_\gamma)\beta T_\alpha(a,b)=T_\alpha(a,b)\beta d([x,y]_\gamma).$$

Therefore, we get

$$2T_\alpha(a,b)\beta d([x,y]_\gamma)=-d(T_\alpha(a,b))\beta[x,y]_\gamma -[x,y]_\gamma\beta d(T_\alpha(a,b)). \qquad (2)$$

From (1) and (2), we have

$$\begin{aligned}4T_\alpha(a,b)\beta T_\alpha(a,b) &= 2T_\alpha(a,b)\beta d([a,b]_\alpha)\\ &= -d(T_\alpha(a,b))\beta[a,b]_\alpha\\ &\quad -[a,b]_\alpha\beta d(T_\alpha(a,b)).\end{aligned}$$

Therefore,

$$4T_\alpha(a,b)\beta T_\alpha(a,b)\beta T_\alpha(a,b)=-d(T_\alpha(a,b))\beta[a,b]_\alpha\beta T_\alpha(a,b)-[a,b]_\alpha\beta d(T_\alpha(a,b))\beta T_\alpha(a,b).$$

Since $[a,b]_\alpha\beta T_\alpha(a,b)=0$ and $d(T_\alpha(a,b))\in M$, so we have $[a,b]_\alpha\beta d(T_\alpha(a,b))\beta T_\alpha(a,b)=0$.

Therefore, we obtain

$$\begin{aligned}&4T_\alpha(a,b)\beta T_\alpha(a,b)\beta T_\alpha(a,b)=0\\ &\Rightarrow T_\alpha(a,b)\beta T_\alpha(a,b)\beta T_\alpha(a,b)=0.\end{aligned}$$

This shows that $T_\alpha(a,b)$ is a nilpotent element of the completely semiprime Γ-ring M, where $T_\alpha(a,b)\in Z(M)$. Since the centre of a completely semiprime Γ-ring does not contain any nonzero nilpotent elements, so we get $T_\alpha(a,b)=0$ for all $a,b\in U$ and $\alpha\in\Gamma$. Hence $d(a\alpha b)=d(a)\alpha b+a\alpha d(b)$ for all $a,b\in U$ and $\alpha\in\Gamma$. Which is the required result.

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