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The Condition of a Circle Containing other Circles in Determinant Form

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Abstract

This article examines the connection between circles and straight lines. A circle that contains another circle and is bounded to a fixed point is said to meet the containment requirement. The idea of putting a second circle outside the predetermined circle was previously discussed. For every possible combination of circles inside and outside the fixed circle, a set of equations is created. The graph is generated using "MATHEMATICA" software because they are produced in numbers. This is carried out for circles placed on any axis. The equation was demonstrated to represent the even and odd integers on a circle, however, this requirement had not previously been expressed explicitly. There are several uses and applications for formulating a circle containing another circle equation. It can be used to locate the spots where two circles intersect in geometry. The main objective of this essay is to offer a straightforward equation that describes how circles relate to their places. The usage of numbers is used to accomplish this. Every mathematical modeling and analysis tool can be implemented at runtime.

***Keywords*:** Circle, Determinant, Straight Line Equation, Series, Circle Containment, Voronoi Diagram.

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এ গবেষণায় কোন একটি বৃত্তে এর অভ্যন্তরে কেন্দ্রমুখী সঞ্চারিত ও বহিস্থে অসীম পর্যন্ত অবস্থান করলে উক্ত বৃত্ত তার অন্তস্থ ও বহিস্থ বৃত্তগুলি ধারন করার শর্ত নিরূপণ করা হয়েছে এবং শর্তদ্বয়কে নির্ণায়ক আকারে প্রকাশ করা হয়েছে। অক্ষের উপর যেকোনো স্থির বৃত্তকে মূল বৃত্ত ধরে নিয়ে তার থেকে সমান অক্ষ দূরত্ব বজায় রেখে এই গাণিতিক কাঠামোর ভিত্তি তৈরি হয়েছে। এটি জ্যামিতিক বৃত্তের ও যোগাশ্রয়ী বীজগণিতের মাধ্যমে একীভূত করে শর্তমূলক রাশিমালা গঠন করা হয়েছে। ইতঃপূর্বে কোন বৃত্ত আরেকটি বৃত্ত ধারণ করার নির্ণায়কের দ্বারা শর্ত প্রদান করা হয়নি। গবেষণার মডেলগুলো ম্যাথম্যাটিকা সফটওয়্যার দ্বারা প্রস্তুত করা হয়েছে। এই গবেষণার মাধ্যমে বৃত্ত সমীকরণ সম্বলিত একটি বৃত্ত প্রণয়নের জন্য বেশ কিছু ব্যবহার এবং প্রয়োগ রয়েছে। এটি দাগগুলি সনাক্ত করতে ব্যবহার করা যেতে পারে যেখানে দুটি বৃত্ত জ্যামিতিতে ছেদ করে। বৃত্ত সমীকরণগুলি চিত্রগুলির বৈশিষ্ট্যগুলি খুঁজে পেতে এবং সনাক্ত করতে কম্পিউটার গ্রাফিক্স এবং চিত্র প্রক্রিয়াকরণে ব্যবহার করা যেতে পারে। এই প্রবন্ধের মূল উদ্দেশ্য হল একটি সরল সমীকরণ দ্বারা শর্ত বর্ণনা করে কিভাবে বৃত্তগুলো একেকটির সাথে সম্পর্কিত থাকে। প্রতিটি গাণিতিক মডেলিং এবং বিশ্লেষণ ব্যবস্থায় প্রয়োগ করা যেতে পারে।

1. Introduction

This paper is aimed at promoting the use of such an analytical method for solving geometry problems relating to lines and circles in the Euclidean plane [1]. One, two, or no intersections between the circles are possible. Two circles cannot meet in the middle, but they can meet at one point, and they can meet at two locations [2]. However, if one circle lies within another, is smaller than it, and its centers are located at the same point, then the larger circle is referred to as an enclosure since it encompasses the smaller circles, also known as inner circles [3]. Straight lines and circles are the focus of the underlying geometry, and the fundamental constructions result in points of intersection, lines connecting points, and circles with one center and another going through it [4] In this paper, the ordinary Euclidean distance metric is used where the distance between a point and a circle is defined by the minimum distance from a point to the boundary of the circle [5]. In this model, we had already assumed that there is a set of true points that lie exactly along a straight [6]. The implicit equations of points, lines, and circles are discussed in this research using a homogeneous representation approach [7], and exemplify the established theorem [8,9]. Additionally, they claimed that these θ-FP results in metric spaces can be used to infer specific FP outcomes in incomplete metric spaces [10]. Consistency requires the use of the adjusted points, which meet this condition, rather than the observed points, which are dispersed about the fitted line because the equation implies that the exact data points would lie on a single parallel line. The generic equation of a circle is defined as $x^2 + y^2 + 2gx + 2fy + c = 0$ and it may be used to describe any circle by selecting the appropriate values for g, f, and c. A circle can be determined to meet three independent geometrical criteria and no more, which is represented by the three constants g, f, and c in the general equation. As a result, a circle can be identified when three of its points are specified or when it must touch three straight lines. Radii are not always the same and no smaller circle may be contained entirely within the larger one [11]. When the radii and distances between the centers are much lower than the distance of the centers from the origin, circle intersection methods frequently underachieve. If floating point arithmetic is used, this circumstance frequently leads to the computation of the difference between two very big values and the consequent loss of significance [12]. The two circles must meet if each point is both inside one circle and on the other. An envelope produced by a circle family is, by definition, a first-order linear differential equation solution with one constraint condition, just like an envelope produced by a hyper plane family [13]. The basic finding that any geometric data involving lines and circles may be easily packed in calculations using determinant theory is the basis for starting. Voronoi diagram can be helpful for a variety of geometric tasks, such as calculating the smallest diameter enclosing a set of inner circles with known radii that are free to move and that there are circles inside a larger circle on a plane. The radii of the circles don't have to be the same, and no circle inside the large circle completely fits inside another circle. When a line having one end at a fixed point makes a full rotation around that point, a circle is a planar figure that is left behind. The radius of the circle is the distance along the line from the fixed point, which is referred to as the center. In the conventional formulation, it is stated that a specific set must be covered by a specific number of equal circles [14]. The following rule reduces a third-order determinant to three determinants of the second order: Take the quantities that appear in the determinant's first row and multiply them one by one by the determinant that results from deleting the row and column. Where a nonlinear model of the issue is created, which is then converted into a linear issue with numerous variables, auxiliary variables, and constraints [15].

All things considered, it is clear that if a circle has circles both inside and outside of it, the relationship between the circle and the straight line has been established, and this can lead to the provision of a simple condition.

2. Methods

The quantitative approach is used to carry it out. Here, it is demonstrated that, if a circle's origin is fixed in the x and y axes, the circle's containment requirements for an infinite number of circles both inside and outside of the fixed circle are provided linearly and described as an infinite linear series. We may test the following mathematical formulations and concurrency conditions using the MATHEMATICA software. The particular methods would vary depending on the MATHEMATICA functions used, but in general, we enter equations and utilize the software's capabilities to evaluate and illustrate the results. Inside the circle surrounding the origin is a second circle B in Figure-2. At x_1 and x_2, circle A crosses the x-axis. Similar to circle A, circle B also crosses the x-axis at x_3 and x_4. The equations of straight lines for x_1 to x_3 and x_2 to x_4 are regarded as equivalent because the distances between them are also equal. Straight line equation across points (x_1, y_1) and (x_2, y_2) on circle A.

$$L_1 = (y-y_1)(x_1-x_2) - (x-x_1)(y_1-y_2) = 0$$

$$\text{or, } (y_2-y_1)x + (x_1-x_2)y + (x_2y_1-x_1y_2) = 0 \qquad (1)$$

The equation of the straight line connecting circle B and circle A through (x_4, y_4) and (x_2, y_2) is similar to that of the straight line connecting circle B and circle.

$$L_2 = (y-y_3)(x_3-x_4) - (x-x_3)(y_3-y_4) = 0$$

$$\text{or, } (y_3-y_4)x + (x_4-x_3)y + (x_4y_3-x_3y_4) = 0 \qquad (2)$$

$L_3 = (y-y_4)(x_4-x_2) - (x-x_4)(y_4-y_2) = 0$

or, $(y_4-y_2)x + (x_2-x_4)y + (x_4y_2-x_2y_4) = 0$ (3)

A condition that the three lines,

$A_1x+B_1y+C_1= 0$ (4)

$A_2x+B_2y+C_2= 0$ (5)

$A_3x+B_3y+C_3= 0$ (6)

To thoroughly prove or analyze the concurrency of the lines produced from those circles, we replaced the coefficients from these derived lines into the determinant and checked whether it equated to zero. If the determinant is zero, the three lines are confirmed to be concurrent. If it is more than zero, the lines are not coincident.

Should be concurrent or meet in a point is that the values of x and y which satisfy (1) and (2) simultaneously should satisfy also (3). The distances between the points x_1 to x_3 and x_2 to x_4 are equal.

The condition that the three equations should all be satisfied by the same values of x and y can be expressed at once in the form of a determinant:

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0 \qquad (7)$$

Tangent circles: We can use the knowledge that the point of tangency is on the line connecting the centers of the two circles to determine whether two circles are tangent to one another from the eq^n. (7). The point of tangency can be determined by writing the equations for the two circles, locating their centers, and then using the equation for the line that passes through the centers. This point will be on both circles, therefore we can verify that the larger circle contains the smaller one using the lemma proof of a circle enclosing another circle.

Concentric circles: When two circles share a center, we can exploit the fact from Figure 3 that the smaller circle's radius is smaller than the larger circle's radius to our advantage. We can formulate the equations for the two circles, determine their radii, and then verify that the larger circle encompasses the smaller one using the lemma proof of a circle including another circle and its deterministic series.

Overall, the theories and models we've created offer a strong tool for examining a variety of situations and arrangements of circles with inner and outer circles.

3. Results and Discussion

The deterministic equation of parallel lines is used to generate this linear equation. One can establish such an equation if two circles are contained within one another. Any circle will undoubtedly be anchored on the X and Y axes. Different circles outside or inside of this one at the same distance can have their X and Y axes fixed about that circle. Finding the straight-line equation along the diameter of any two circles. The equation of two straight lines going from one end to the other of their diameters between the first circle and its inner circle meet on the same line, as does the equation of two straight lines going from one end to the other of the diameter of another circle inside the first circle. A determinant format can be utilized to express these three concurrent points as x and y.

Given the determinant:

$$\Gamma_c = \begin{vmatrix} y_2 - y_1 & x_1 - x_2 & x_2y_1 - x_1y_2 \\ y_4 - y_3 & x_3 - x_4 & x_4y_3 - x_3y_4 \\ y_4 - y_2 & x_2 - x_4 & x_4y_2 - x_2y_4 \end{vmatrix}$$

Laplace's expansion will be used to enlarge the determinant. Using the first row as a starting point for expansion:

$$\Gamma_c = (y_2 - y_1) \times \begin{vmatrix} x_3 - x_4 & x_4y_3 - x_3y_4 \\ x_2 - x_4 & x_4y_2 - x_2y_4 \end{vmatrix}$$
$$\begin{vmatrix} y_4 - y_3 & x_4y_3 - x_3y_4 \\ y_4 - y_2 & x_4y_2 - x_2y_4 \end{vmatrix} \begin{vmatrix} y_4 - y_3 & x_3 - x_4 \\ y_4 - y_2 & x_2 - x_4 \end{vmatrix}$$

Solving each 2×2 determinants:

$$\begin{vmatrix} x_3 - x_4 & x_4y_3 - x_3y_4 \\ x_2 - x_4 & x_4y_2 - x_2y_4 \end{vmatrix} = (x_3 - x_4)\,(x_4y_2 - x_2y_4) - (x_4y_3 - x_3y_4)(x_2 - x_4)$$

And,

$$\begin{vmatrix} y_4 - y_3 & x_4y_3 - x_3y_4 \\ y_4 - y_2 & x_4y_2 - x_2y_4 \end{vmatrix} = (y_4 - y_3)\,(x_4y_2 - x_2y_4) - (y_4 - y_2)(x_4y_3 - x_3y_4)$$

Also,

$$\begin{vmatrix} y_4 - y_3 & x_3 - x_4 \\ y_4 - y_2 & x_2 - x_4 \end{vmatrix} = (y_4 - y_3)(x_2 - x_4) - (y_4 - y_2)(x_3 - x_4)$$

Plug the outcomes of these 2x2 determinants back into the 3x3 determinant expansion and simplify.

If $\Gamma_c = 0$ then the lines represented by those equations L_1, L_2 and L_3 are concurrent.

Hence,

$$\Gamma_c = \begin{vmatrix} y_2 - y_1 & x_1 - x_2 & x_2y_1 - x_1y_2 \\ y_4 - y_3 & x_3 - x_4 & x_4y_3 - x_3y_4 \\ y_4 - y_2 & x_2 - x_4 & x_4y_2 - x_2y_4 \end{vmatrix} = 0 \qquad (8)$$

For instance, consider a fixed circle with the center at the origin and coordinates (3,0) and (-3,0), and another circle inside it with coordinates (5,0) and (-5,0) centered on the x and y axes.

$$\Gamma_c = \begin{vmatrix} 0 & 5-(-5) & \{(-5).0 - 5.0\} \\ 0 & 3-(-3) & \{(-3).0 - 3.0\} \\ 0 & -5-(-3) & \{(-3).0 - (-5).0\} \end{vmatrix}$$

$$\text{or, } \Gamma_c = \begin{vmatrix} 0 & 10 & 0 \\ 0 & 6 & 0 \\ 0 & -2 & 0 \end{vmatrix}$$

or, $\Gamma_c = 0$

where Γ_c *is denoted as a condition of a circle containing another circle.*

Lemma 1:

The determinant created by the coefficients of three lines is zero if they cross at a single point.

Proof: Let the equations of the three lines be ax + by + c = 0, dx + ey + f = 0, and gx + hy + i = 0. We can write these equations in matrix form:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If the three lines intersect at a single point, then there exists a unique solution to this system of equations. This means that the matrix is singular, i.e., its determinant is zero.

Therefore, if the equations of three lines in a circle intersect at a single point (i.e., they intersect at the center of the circle), then the determinant Γ_c formed by their coefficients is zero. Figure 3 shows the three lines in a circle intersecting at a single point.

By Figure 1, the series equation of n numbers of circles from the fixed circle to the origin or infinity is given by the preceding equation, which is the determinant equation of any one circle containing another circle.

$$\Gamma_{c_1} + \Gamma_{c_2} + \Gamma_{c_3} + \Gamma_{c_3} + \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots + \Gamma_{c_n} = 0 \qquad (9)$$

The determinant equation Γ_{c_i} represents a specific geometric condition concerning another process or reference for each circle. The series equation represents the cumulative influence of these conditions in the system $\Gamma_{c_1} + \Gamma_{c_2} + \Gamma_{c_3} + \Gamma_{c_3} + \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots + \Gamma_{c_n} = 0$

Comprehending the individual impact versus. The combined effect of the geometric configurations is the key to understanding the link between the two. Although not shown, the representation in figure 1 contextualizes these mathematical relationships in a geometric environment.

Lemma 2:

If the sum of the coefficients of any two equations of three lines in a circle is equal to zero, then the summation series of the determinant Γ_c is zero.

Proof: Let the equations of the three lines be ax + by + c = 0, dx + ey + f = 0, and gx + hy + i = 0. We can write these equations in matrix form:

1. $ax + by + c = 0$
2. $dx + ey + f = 0$
3. $gx + hy + i = 0$

These can be represented in matrix form as:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Suppose that the sum of the coefficients of the first two equations is zero,

i.e., a + d = b + e = c + f = 0. Then we can write the matrix in terms of the first two rows:

1. $\mathrm{a} + \mathrm{d} = 0 \implies \mathrm{d} = -\mathrm{a}$
2. $\mathrm{b} + \mathrm{e} = 0 \implies \mathrm{e} = -\mathrm{b}$
3. $\mathrm{c} + \mathrm{f} = 0 \implies \mathrm{f} = -\mathrm{c}$

Substituting the following values for the coefficients in the second row:

$$\begin{bmatrix} a & b & c \\ -a & -b & -c \\ g & h & i \end{bmatrix} \qquad (10)$$

However, the matrix representation for three lines stays unaltered, thus splitting the matrix into two matrices, the sum of which yields the original matrix. One of these matrices contains a row of zeros, representing the zero-sum of coefficients.

Using determinant properties, if a determinant's two rows (or columns) are identical or proportional, its value is zero.

$$\begin{bmatrix} a & b & c \\ -a & -b & -c \\ g & h & i \end{bmatrix}$$

The determinant is 0 since the first and second rows are proportional with a factor of -1. Furthermore, the determinant of any matrix with a row (or column) of zeros is also zero.

Thus:

$$\left|\begin{bmatrix} 0 & 0 & 0 \\ d & e & f \\ 0 & 0 & 0 \end{bmatrix}\right| = 0 \qquad (11)$$

$Adding\ up\ the\ determinants, we\ have$:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ -a & -b & -c \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & 0 & 0 \\ d & e & f \\ 0 & 0 & 0 \end{vmatrix} = 0 + 0 = 0$$

Similarly, if the sum of the coefficients of the second and third equations, or the first and third equations, is zero, we can use the same method to show that the summation series of the determinant Γ_c is zero, which is showing at figure 4.

Therefore, if the sum of the coefficients of any two equations of three lines in a circle is equal to zero, then the summation series of the determinant Γ_c is zero.

4. Conclusions

To conclude, this study uses determinants to demonstrate how the linear condition of a circle that contains circles both inside and outside its perimeter can be proven. the study investigates the conditions under which a circle might include other circles in the form of a determinant. The characteristics that describe this condition are determined by the study using mathematical ideas and methods. The results of this study will be helpful to mathematicians, physicists, and engineers who study circles and their properties, as well as to designers and engineers who have to build circular structures that can hold additional circles. In the end, this study adds to the body of information about circles and their behavior and has applications in a variety of industries that use circular shapes. Circle containment conditions provide a potent modeling and analysis tool for circular objects and systems, and have a wide range of applications in a variety of disciplines. Insights, techniques, and applications in the fields of geometry, physics, engineering, computer science, and other fields may be discovered as a result of further study and development in this area.

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Figures:

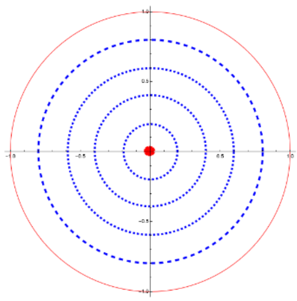

Figure 1 . A circle containing small circles.

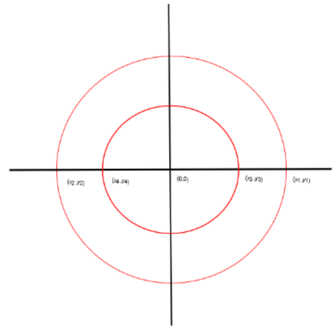

Figure 2. Fixed circle containing another circle and make three equations.

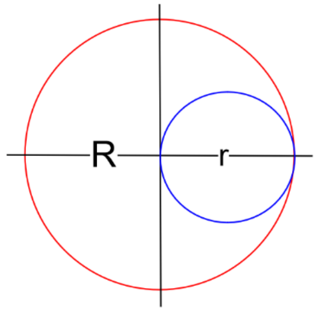

*Figure 3.*The circle containing other circle then its radius bigger than small circle.

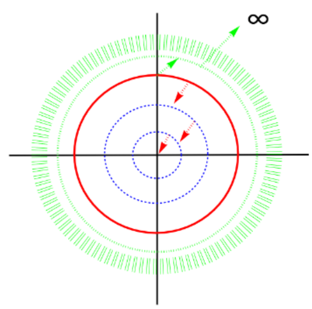

Figure 4. n numbers of circles from the fixed circle to the origin or infinity.