SECTIONALLY PSEUDOCOMPLEMENTED RESIDUAL LATTICE

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Abstract : *At first, we recall the basic concept, By a residual lattice is meant an algebra $L=(L,\vee,\wedge,*,\circ,0,1)$ such that*

(i) $L=(L,\vee,\wedge,0,1)$ is a bounded lattice,

(ii) $L=(L,*,1)$ is a commutative monoid,

(iii) it satisfies the so-called adjoin ness property: $(x\vee y)*z=y$ if and only if $y\le z\le x\circ y$

Let us note [7] that $x\vee y$ is the greatest element of the set $(x\vee y)*z=y$

Moreover, if we consider $x*y=x\wedge y$, then $x\circ y$ is the relative pseudo-complement of x with respect to y, i. e., for $*=\wedge$ residuated lattices are just relatively pseudo-complemented lattices. The identities characterizing sectionally pseudo-complemented lattices are presented in [3] i.e. the class of these lattices is a variety in the signature $\{\vee,\wedge,\circ,1\}$. We are going to apply a similar approach for the adjointness property:

Key words: *Residuated lattice, non Distributive, Residuated Abeliean, commutative monoid:*

1. Introduction

Residuated lattices were introduced by Ward and Dilworth [5] and studied by several authors. Two monographs contain a compendium on residuated lattices. They are that by Blyth and Janowitz [1] (where it is renamed as a residuated Abelian semi-group with a unit) and the book by R. Belohavek [7].

In this short note we will compare a certain modification of a residuated lattice with already introduced [2], [3]. At first, we recall the basic concept:

Definition 1. A lattice $L=(L,\vee,\wedge,1)$ with the greatest element 1 is sectionally pseudo-complemented if each interval [y, 1] is a pseudo-complemented lattice.

From now on, denote by $x\vee y$ the pseudo-complement of $x\vee y$ in the interval [y, 1].

Naturally, $x\vee y\in[y,1]$ thus $L=(L;\vee,\wedge,1)$ is sectionally pseudo-complemented if and only if "$\circ$" is an (everywhere defined) operation on L.

Definition 2. An algebra $L=(L;\vee,\wedge,*,\circ,1)$ is called a sectionally residuated *lattice if*

(i) $L=(L,\vee,\wedge,0,1)$ is a lattice with the greatest element 1;

(ii) $L=(L,*,1)$ is a commutative monoid ;

(iii) it satisfies the sectional adjointness property: $(x\vee y)*z=y$ if and only if $y\le z\le x\circ y$

Lemma 1.1 Let $L=(L;\vee,\wedge,*,\circ,1)$ be a sectionally residuated lattice. Then $x*y$ *is the greatest element of the set* $\{z;(x\vee y)*z=y\}$

This immediately yields the following facts:

$(x\vee y)*(x\circ y)=y$, (1)

$(x\vee y)*y=y,$ (2)

$y\le x\circ y,$ (3)

Lemma 1.2 Let $L=(L;\vee,\wedge,*,\circ,1)$ be a sectionally residuated lattice. Then $x\le y,$ *if and only If* $x\circ y=1$

Proof: Suppose $x\le y,$ Then $x\vee y=y,$ and by Lemma 1.1, $x\circ y$ is the greatest element of the set $\{z;y*z=y\}$ By Definition 2, $y*1=1$ thus $x\circ y=1$. Conversely,

Suppose $x\circ y=1$ .Then, by [1], we have $y=(x\vee y)*(x\circ y)=(x\vee y)*1=x\vee y$

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whence $x \le y$

Lemma 1.3 In a sectionally residuated lattice, the following identities are satisfied:

and $1 \circ x = x$

Proof: The first three identities follow directly by Lemma 1.2. Further, by Lemma 1.1,

$1 \circ x$ is the greatest element of the set $\{z; 1 * z = x\} = \{x\}$ thus $1 \circ x = x$

Lemma 1.4 In a sectionally residuated lattice, $a * b = a$ if and only if $a = b$

Proof: Putting $x = y = a$ and $z = b$ in the sectional adjointness property, the assumption $a * b = a$ yields $(a \vee a) * b$ *iff* $a \le b \le a \circ a = 1$ thus $a \le b$

Conversely, $a \le b$ implies by Lemma 3 $a \le b \le 1 = a \circ a$ and, by sectional adjointness, $a * b = (a \vee a) * b = a$

Applying Lemma 1.2 and Lemma 1.4, we get

Corollary 1.5 In a sectionally residuated lattice,

(a) $x * y = x$ if and only if $x \circ y = 1$;

(b) $x * x = x$

Lemma 1.6 In a sectionally residuated lattice, $x \wedge y \le x * y$.

Proof: By [3] we have $x \wedge y \le x \circ (x \wedge y)$. Applying sectional adjointness, we infer $x * (x \wedge y) = (x \vee (x \wedge y)) * (x \wedge y)$ and, analogously, $y * (x \wedge y) = x \wedge y$. Hence, by Corollary 1.5 (b),

$x * y * (x \wedge y) = x * (x \wedge y) * y * (x \wedge y)$

$= (x \wedge y) * (x \wedge y) = x \wedge x$

and by Lemma 1.4, $x \wedge y \le x * y$.

Theorem 1.7 Let $L = (L; \vee, \wedge, *, \circ, 1)$ be a sectionally residuated lattice. Then it is a sectionally pseudo-complemented lattice.

Proof: Replacing y by $x \wedge y$ in the sectional adjointness property, we obtain $x * z = x \wedge y$ *iff* $x \wedge y \le z \le x \circ (x \wedge y)$.

However, $x \circ (x \wedge y)$ is the greatest element of the set $\{t; (x \vee (x \wedge y)) * t = x \wedge y\} = \{t; x * t = x \wedge y\}$.

By Lemma 1.4, $x \wedge t \le x * t = x \wedge y$, thus the greatest t of this property satisfies $t \ge y$.

Thus $y \le x \circ (x \wedge y)$, i.e., $x \wedge y \le y \le x \circ (x \wedge y)$ and by the sectional adjointness, $x * y = (x \wedge (x \vee y)) * y = x \wedge y$.

Hence, $x \circ y$ is the pseudo-complement of $x \vee y$ in the interval $[y, 1]$

2. Conclusion

It is well known that every relatively pseudo-complemented lattice is distributive.

An extension of relative pseudo-complementation for the non-distributive case was already involved in [3], [4]:

Reffrences

[1] Blyth, T.S. and Janowttz, M.F. : Residuation Theory, Pergamon Press, Oxford, 1972.

[2] Chajda, I. and Halas, R: Sectionally pseudocomplemented lattice and semilattice, Advances in algebra, Springer Verlag, 2003, 282-290.

[3] Chajda, I.: An extension of relative pseudocomplemention to non distributive lattices, Acta Sci. Math(Szeged), 69 (2003), 491-496.

[4] Chajda, I. and Radeleczki, S.: On varities defined by pseudo-complemented nondistributive lattices, Publ. Math. (Debrecen). 63 (2003), 737-750.

[5] Ward, M. and Dilworth, R.P.: Residuated lattices, Trans.Amer.Math. Soc., 45(1939), 335-354.

[6] Chajda, I. : Sectionally Residuated lattices, Miskolc Mathmatical notes, Vol. 6 (2005), no. 1,pp. 27-30

[7] R. Belohavek; Fuzzy relational systems, Kluwer Academic/ Plemun Publ., New York, Boston, Dordrecht, London, Moscow, 2002