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Continuous Dependence of Fixed Points on Parameters and Initial Conditions

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Abstract

The main purpose of the paper is to establish conditions for a continuous dependence of fixed points and an application to non-linear functional differential equation of neutral type have been made

I. Introduction

Functionalanalysis is a promising part of analysis. What is more functional analysis now provides a common language for all areas of mathematics involving the concept of continuity. No serious investigation in the theory of functions, differential equations, or mathematical physics, in numerical methods, mathematical economics or control theory, or innumerous other fields, take place or could take place without extensive use of the language and results of functional analysis. It is precisely this fact that explains on one hand, the rapid development of functional analysis as a mathematical discipline and on the other hand, the ever-increasing role played by its techniques in application.

We consider two types of convergence of mappings-point wise convergence and uniform convergence. The theorem regarding the continuity of fixed points of contraction mappings was proved by [3]. Subsequently [9] obtained results concerning sequence of contraction mappings and also gave an application suggested by [7], [8], [9] established conditions implying the strong convergence of the fixed points of a sequence of set-valued contractions. A similar result concerning the weak convergence of the fixed points of the set-valued non-expansive mappings in a Banach space was obtained by [7] who used it to obtain a stability result for generalized differential equations. In our writings we have tried to explain continuous dependence of fixed point sets. The main purpose of the present paper is to establish when the convergence of a sequence of ϕ-contractive mappings in a uniform space implies a convergence of the sequence of their fixed points. The notion of ϕ-contractive mapping in a uniform space has been introduced in [2]. In view of the applications given in [2] the problem of a continuous dependence of fixed points can be formulated as a continuous dependence of the solutions of a nonlinear functional differential equation.

To do that we have proved one important theorem regarding fixed points with the help of some lemmas which are also studied.

II. Main Theorem 1.

Let A be closed bounded subset of a Hilbert space H, d be the norm of H and D the Hausdorff metric on the closed subsets of A generated by d. If the family of set-valued maps F_k , k = 0, 1,… satisfy

(1) $F_k(x)$ is a non-empty closed convex subsets of A for each $x \in A$.

(2) Each F_k is a set-valued contraction, i.e., there is a $k \in [0,1]$ such that

$D(F_k(x), F_k(y)) \leq kd(x,y)$ for $x, y \in A$ and k = 0, 1,…,

(3) $\lim_{k\to\infty} D(F_k(x), F_0(x)) = 0$ uniformly for all $x \in A$.

Then the fixed point sets of the sequence $\{F_k\}$, k = 1, 2,…, converge to the fixed point set of F_0 in the Housdorff metric D.

Before giving the proof of the theorem some lemmas on the closest point projection map associated with F_k are required. For k = 0, 1,…, define the maps F_k by $f_k(x)$= {unique closest point $F_k(x)$ to x} for $x \in A$.

The iterates of each f_k are denoted by $f_k{}^n, n$ = 2,3,…,. The distance between any $x \in A$ and closed subset C of A will be $d(x,c) = \inf_{c\in C} d(x,c)$.

The following result was given in [5] for a finite dimensional space, but the statement and proof are valid for any Hilbert space.

Lemma 1. If E and F are closed convex subsets of H and e and f are the closest point in E and F to a point $v \in H$, then

$d(e,f) \leq \frac{1}{2}(h^2 + 4hl)$, where $h = D(E, F)$ and $l = d(v, E)$.

For s ≥ 0 we define the continuous monotone increasing function $g(s) = \frac{1}{2}(s^2 + 4sr)$, where r is the diameter of A.

Lemma 2. The maps f_k, k = 0, 1…, are equicontinuous on *A*.

Proof. The family of mappings f_k, k = 0, 1…, will be equicontinuous on *A* if for every $\varepsilon > 0$ $\exists \delta > 0$ such that

$$d(x,y) < \delta \Rightarrow d(f_k(x), f_k(y)) < \varepsilon .$$

Now for any k = 0, 1,…, and $x, y \in A$ let q denote the closest point in $F_k(x)$ to y.

Then,

$$d(f_k(x), f_k(y)) \le d(f_k(x), q) + d(q, f_k(y)) \quad (1)$$

The term $d(f_k(x), q)$ is bounded by $d(x,y)$, since projection onto a closed convex set is non-expansive. Since $F_k(x)$, $F_k(y)$ are non-empty closed convex subsets of *A* for $x, y \in A$, Lemma 1 implies that $d(q, f_k(y))$ is bounded by

$$g(D(F_k(x), F_k(y)) .$$

Since F_k is a set-valued contraction and *g* is monotone, we have,

$$g(D(F_k(x), F_k(y)) \le g(kd(x,y)).$$

The inequality (1) can then be written as

$$d(f_k(x), f_k(y)) \le d(x,y) + g(kd(x,y)) .$$

The map *g* is continuous and by definition of *g*, $g(s) \to 0$ as $s \to 0$.

Therefore we have,

$$d(f_k(x), f_k(y)) \le d(x,y)$$

But $d(x,y) < \delta \Rightarrow d(f_k(x), f_k(y)) < \varepsilon$ whenever $\delta = \varepsilon$. This completes the proof of the Lemma. □

Lemma 3. The sequence of maps $\{f_k^{\,n}\}$, k = 0. 1, …, converges uniformly on *A* to $f_0^{\,n}$ for each *n*.

Proof. For $n = 1$ the uniform convergence follows from,

$$d(f_k(x), f_0(x)) \le g(D(F_k(x), F_0(x)))$$

and the uniform convergence of the maps F_k to F_0.

Now we make the induction assumption that $f_k^{\,n-1}$, k = 1, 2, …, converges uniformly on *A* to $f_0^{\,n-1}$.

From the definition of equi-continuity of f_k, we have, for $\varepsilon > 0$, there exists a $\delta > 0$ such that for $u, v \in A$ and $d(u,v) < \delta$ implies

$$d(f_k(u), f_k(v)) < \varepsilon/2 \text{ for all } k.$$

The uniform convergence of the sequence $\{f_k\}$ and $\{f_k^{\,n-1}\}$ to f_0 and $f_0^{\,n-1}$ permits the choice of an integer *N* such that $k \ge N$ implies

$$d(f_k^{\,n-1}(x), g_0^{\,n-1}(x)) < \delta \quad \text{and}$$

$$d(f_k(x), f_0(x)) < \varepsilon/2 \qquad \forall \quad x \in A .$$

Considering the inequality

$$d(f_k^{\,n}(x), f_0(x)) = d(f_k(f_k^{\,n-1}(x), f_0^{\,n-1}(x))$$
$$\le d(f_k(f_k^{\,n-1}(x), f_k(f_0^{\,n-1}(x)) + d(f_k(f_0^{\,n-1}(x), f_0(f_0^{\,n-1}(x))$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \forall k \ge N$$

i. e., $d(f_k^{\,n}(x), f_0^{\,n}(x) < \varepsilon \qquad \forall\, k \ge N$.

Therefore, the sequence of maps $\{f_k^{\,n}\}$, k = 0. 1,.., converges uniformly on *A* to $f_0^{\,n}$ for each *n*.

III. Proof of the Main Theorem 1

The sequence of iterates $\{f_k^{\,n}(x)\}$ converges to a fixed point of F_k for k = 0, 1,.., and for all $x \in A$. If p_k denotes the fixed point set of f_k, then from the theorem it follows that

$$d(f_k^{\,n}(x), p_k) \le \sum_{i=1}^{\infty} (r+i)k^i \quad (2)$$

where *k* is the Lipschitz constant and $r = d(f_k(x), p_k)$.

Each p_k is a closed subset and can be written as

$$p_k = \{y \in A : \lim_{n\to\infty} f_k^{\,n}(x) = y, x \in A\}$$

Given $\varepsilon > 0$, choose any $x \in A$ and let

$p_k(x) = \lim_{n\to\infty} f_k^{\,n}(x)$, k = 0. 1, …

Consider the inequality

$$d(p_k(x), p_0(x)) \le d(p_k(x), f_k^{\,n}(x)) + d(f_k^{\,n}(x), f^{\,n}{}_0(x)) + d(f_0^{\,n}(x), p_0(x)) \quad (3)$$

By the estimate (2) we can choose an integer *N* such that for all $x \in A$, $d(f_k^{\,n}(x), p_k(x)) < \varepsilon/3$ for k = 0, 1, …, The uniform convergence of $\{f_k{}^N\}$ to $f_0{}^N$ permits the choice of an integer M such that $k \geq M$ implies

$$d(f_k{}^N(x), f_0{}^N(x)) < \varepsilon/3 \text{ for all } x \in A.$$

Therefore by inequality (3) becomes

$$d(p_k(x), p_0(x)) < \varepsilon \text{ for all } x \in A.$$

Since the points $p_k(x)$ range over p_k as x ranges over A, we have shown that

$$D(p_k, p_0) < \varepsilon \quad \text{for } k \geq M.$$

This proves the convergence of p_k to p_0 in the D-metric. Hence the proof is complete. □

Theorem 2. [1] Let (X, A) be a locally compact Hausdorff quasicomplete j-bounded uniform space. Let $T_k : X \to X$ be a ϕ-contracted mapping with fixed point

y_k for any $k = 0,1,2,\cdots$, i.e.,

$$d_\alpha(T_k x, T_k y) \leq \phi_\alpha(d_{j(\alpha)}(x, y)).$$

If $\{T_k\}_{k=1}^{\infty}$ converges pointwise to y_0, then the sequence $\{y_k\}_{k=1}^{\infty}$ converges to y_0.

IV. Applications

Here we shall apply the results obtained to some initial value problems considered in [2].

Let us consider the initial value problems

$$\varphi'(t) = F_k(t, \varphi(\Delta_1(t)),\dots,\varphi(\Delta_m(t)), \varphi'(\tau_1(t)),\dots, \varphi'(\tau_n(t))) \qquad (4_k)$$

$$t > 0, \varphi(t) = \psi(t), \varphi'(t) = \psi'(t), t \leq 0,$$

where $\varphi(t)$ is the unknown function. The deviations $\Delta_i(t) = \tau_l(t), \quad i = 1,2,\dots,m; l = 1,2,\dots,n$

are of mixed type and in general case unbounded. After usual transformations, assuming

$\psi(t) = 0$ equation (4_k) can be reduced to the following one $(x(t) = \varphi'(t))$ for $t > 0$ and $\theta(t) = \psi'(t)$ for $t \leq 0$:

$$x(t) = F_k(t, \int_0^{\Delta_1(t)} x(s)ds,\dots, \int_0^{\Delta_m(t)} x(s)ds, x(\tau_1(t)),\dots, x(\tau_n(t))), \quad t > 0, x(t) = \theta(t), t \leq 0 \qquad (5_k)$$

Let $C(R^1)$ be the linear topological space consisting of all continuous function $f(t): R^1 \to R^1$ with a topology generated by a saturated family of seminorms $A = \{\|.\|K\}, \|f\|K = \sup\{|f(t)| : t \in K\}$ where $k \subset R^1$ runs over all compact subsets of R^1. In view of Theorem 2 we shall look for a solution of (5_k) in a locally compact set of functions.

Namely, let us consider the set

$c_L = \{f \in C(R^1) : |f(t) - f(\bar{t})| \leq L|t - \bar{t}|$ for every $t, \bar{t} \in R^1\}$, where the Lipschitz constant L does not depend on K. It is easy to verify that C_L is closed convex and every point has a neighbourhood with a compact closure. We shall find a solution of (5_k) in the set

$C_L{}^0 = \{f \in C_L : |f(t)| \leq r_0(t)\}$ where $r_0(t) : R^1 \to R^1_+$, $r_0(t)$ is continuous positive function on R^1

We shall make the following assumption [2]

$\Delta_i(t), \tau(t) : R^1_+ \to R^1 (R^1_+ = [0, \infty))$ are continuous

(C1) $\Delta_i(0) \leq 0, \tau_1(0) \leq 0$

and $|\nabla_i(t) - \nabla_i(\bar{t})| \leq P_i|t - \bar{t}|$,

$|\tau_1(t) - \tau_1(\bar{t})| \leq Q_1|t - \bar{t}|$.

The map $j : A \to A$ is defined as in [2], where the index set A consists all of compact subsets of R^1:

(C2) For every k = 0, 1, 2, 3,… the functions

$F_k(t, u_1,\dots u_m, v_1,\dots v_n) : R^1_+ \times R^{mn} \to R^1$

are continuous and satisfy conditions:

(C3) $|F_k(t, u_1,\dots u_m, v_1,\dots v_n)| \leq w(t)\left[1 + \sum_{i=1}^{m}|u_i| + \sum_{l=1}^{n}|v_l|\right]$

(C4) $| F_k(t, u_1,\dots u_m, v_1,\dots v_n) - F_k(t, \bar{u}_1,\dots \bar{u}_m, \bar{v}_1,\dots \bar{v}_n) | \leq \Omega\left[|u_1 - \bar{u}_1| + \dots + |u_m - \bar{u}_m| + |v_1 - \bar{v}_1| + \dots + |v_m - \bar{v}_m|\right]$

where Ω is a positive constant:

$| F_k(t, u_1,\dots u_m, v_1,\dots v_n) - F_k(\bar{t}, u_1,\dots u_m, v_1,\dots v_n) | \leq L_0|t - \bar{t}|$

where L_0 is a positive constant and

$$L_0 + \Omega\left[r_0(t)\sum_{i=1}^{m}|P_i| + \sum_{l=1}^{n}|Q_l|\right] \leq L;$$

$$w(t)\left[1+\sum_{i=1}^{m}\left|\Delta_i(t)\right|r_0(t)+nr_0(t)\right]\le .r_0(t);$$

$$\Omega(m\overline{\Delta}_k+n)<1$$

for every compact $K\subset R^1$ where $\overline{\Delta}_k$ =sup $\{\left|\Delta(t)\right| : t\in K\}$. Conditions (C3) and (C4) are the same as in [2], assuming that the initial functions have Lipschitz constants

Theorem 3. Let the assumptions (C1) – (C4) be fulfilled. If the sequence of functions $\{F_K\}_{K=1}^{\infty}$ tends pointwise to F_0, then the sequence of solutions of (5_K) tends to the solution of (5_0).

Proof. We form by the right hand side of (5_K) the sequence of operators $\{T_K\}_{K=1}^{\infty}$.

It is easy to see that T_K maps the set $C_L^0=\{f\in C_L(R^1):\left|f(t)\right|\le r_0(t), t\ge 0\}$ into itself. We shall verify only that $\{T_K f\}(t)$ has a Lipschitz constant equals to *L,* because another details of the proof are as in [2]. For $t,\bar{t}>0$ we have

$$\left|(T_k f)(t-(T_k f))(\bar{t})\right|\le L_0\left|t-\bar{t}\right|+$$

$$\Omega\left[r_0(t)\sum_{i=1}^{m}\left|\Delta_i(t)-\Delta_i(\bar{t})\right|+\sum_{l=1}^{n}L\left|\tau_l(t)-\tau_l(\bar{t})\right|\right]\le$$

$$L_0\left|t-\bar{t}\right|+\Omega\left[r_0(t)\sum_{i=1}^{m}P_i+L\sum_{l=1}^{n}Q_l\right]\left|t-\bar{t}\right|\le L\left|t-\bar{t}\right|$$

Now we can apply Theorem 2 in order to conclude that the solution of (5_K) tends to the solution of (5_0). This is possible because T_K is an equicontinuous family of operators and then pointwise convergence on compact sets implies a uniform convergence.

V. Conclusion

The stability of the fixed point sets of a uniformly convergent sequence of set-valued contractions is proved under the assumption that the maps are defined on a closed bounded subset *B* of Hilbert space and take values in the family of non-empty closed convex subsets of *B*.

The convergence of sequence of fixed points of a convergent sequence of set-valued contractions can be easily investigated in a metric space setting. By restricting the underlying space to be a Hilbert space we prove the convergence of the sequence of fixed point sets of a convergent sequence of set-valued contractions. This also extends a similar result for point valued maps [4] to the set valued case.

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