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# Study of Graded Algebras and General Linear Group with Lie Superalgebras and R-Algebra

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**Abstract**

Some elements of theory of $\mathbb{Z}_2$-graded rings, modules and algebras. $\mathbb{Z}_2$-graded tensor algebra, Lie superalgrbras and matrices with entries in a $\mathbb{Z}_2$-graded commutative ring are treated in our present paper. At last a **Theorem 4.4.**on the set of square matrices in the graded $R$-algebra $M_R[m|n]$ is established.

**Keywords:** $\mathbb{Z}_2$-graded rings, modules, commutative ring and graded algebras, tensor calculus, general graded linear group $GL[m|n]$, the set of graded matrices $M_R[(p+q)\times(m+n)]$ and graded $R$-algebra.

## I. Introduction

Nowadays a large body of literature is available concerning graded algebras, mainly over the real or complex numbers (usually called superalgebras), their representations, etc. Classical references are [3], [6], [7], [8], [10]. The most common notations and basic results are treated in this article.

## II. Graded Algebraic Structures

In general, given an arbitrary group $G$, we can introduce $G$-graded algebraic objects [5], [10]. Since in order to develop a 'supergeometry' only $\mathbb{Z}_2$-graded structures are needed, we shall only consider here that particular case. We shall assume as a rule that

$$graded \equiv \mathbb{Z}_2 - graded$$

*Definition* 2.1. *A ring $(R, +, \cdot)$ is said to be graded if $(R,+)$ has two subgroups $R_0$and $R_1$ such that $R = R_0 \oplus R_1$ and $R_\alpha, R_\beta \subset R_{\alpha+\beta}$ for all $\alpha,\beta \in \mathbb{Z}_2$.*

An element $a \in R$ is said to be homogeneous if either $a \in R_0$or $a \in R_1$. On the set $h(R)$ of homogeneous elements an application $|\,|$ is defined by

$$|\,|: h(R) \to \mathbb{Z}_2$$

$$a \mapsto \alpha \Leftrightarrow a \in R_\alpha.$$

The elements of degree 0 and 1 are called even and odd respectively.

Obviously, any ring $R$ can be trivially graded: $R_0 = R$, $R_1 = \{0\}$.

*Example* 2.2. Let $R$ be a $\mathbb{Z}$-graded ring, namely, $R = \bigoplus_{p\in\mathbb{Z}} \hat{R}_p$ and $\hat{R}_p.\hat{R}_q \subset \hat{R}_{p+q}$ then $R$ can be graded by takig $R_0$ as the sum of the even components and $R_1$ as the sum of the odd ones.

For any graded ring $R$, a graded commutator$<,>: R \times R \to R$ is defined by letting

$$< a,b >= ab - (-1)^{|a||b|}ba \;\forall\, a,b \in h(R) \qquad (2.1)$$

The centre of $R$ is defined as the set

$$C(R) \equiv \{a \in R| < a,b >= 0 \;\forall\, b \in R\},$$

i.e. $C(R)$ is the set of the elements of $R$ which graded – commute with any other elements.

A graded ring $R$ is said to be graded-commutative if $< a,b > = 0 \;\forall\, a,b \in R$, that is, if $C(R) = R$.

Let $R$ be a graded ring and $M$ be a left(right) $R$-module.

*Definition* 2.3. *$M$is a left (right) graded $R$-module if it has two subgroups $M_0$ and $M_1$ such that $M = M_0 \oplus M_1$ and for all $\alpha,\beta \in \mathbb{Z}_2$, one has $R_\alpha M_\beta \subset M_{\alpha+\beta}(M_\alpha R_\beta \subset M_{\alpha+\beta})$.*

If $R$ is graded-commutative, which we shall henceforth assume, we shall use the term 'graded $R$-module' without ambiguity.

Having fixed two graded $R$-modules $M$and $N$, we say that a morphism $f: M \to N$ is $R$-linear on the right if $f(xa) = f(x)a$ for all $x \in M$ and $a \in R$. Unless otherwise stated, by 'linear' we mean 'linear on the right'. Moreover, we say that $f$ has degree $|f| = \beta \in \mathbb{Z}_2$, if $f(M_\alpha) \subset N_{\alpha+\beta}$ for all $\alpha \in \mathbb{Z}_2$. The set $Hom(M,N)$ of $R$-linear morphisms $M \to N$(that will be denoted simply by $Hom(M,N)$) has a natural grading, with $f \in Hom(M,N)_\alpha$ whenever $|f| = \alpha$. If $R$ is graded-commutative, $Hom(M,N)$ is a graded $R$-module, with the multiplication rule $(af)(x) = af(x)$.

One of the most basic results in commutative ring theory, namely the Nakayama lemma, can be generalized to the graded setting. Let us define the radical of a graded-commutative ring $R$ as the graded ideal $\mathcal{R}$ obtained by intersecting all maximal graded ideals of $R$.

*Proposition* 2.4.(Graded Nakayama Lemma) *Let $R$ be a graded-commutative ring $R$, $I$ be a graded ideal contained in the radical $\mathcal{R}$ of $R$ and $M$ be a graded finitely generated $R$-module.*

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(a) *If* $IM = M$, *then* $M = 0$.
(b) *If* $N$*is a graded submodule of* $M$*and* $M = IM + N$, *then* $M = N$.
(c) *If* $x^1, \dots, x^m$ *are even elements and* $y^1, \dots, y^n$ *are odd elements in* $M$ *such that the images* $(\bar{x}^1, \dots, \bar{x}^m, \bar{y}^1, \dots, \bar{y}^n)$ *are generators of* $M/IM$*over* $R/I$, *then* $(x^1, \dots, x^m, y^1, \dots, y^n)$ *are generators of* $M$*over* $R$.

*Definition* 2.5. *A graded* $R$*module* $F$ *is said to be free if it has a basis formed by homogeneous elements.*

A basis of $F$ of finite cardinality is of type $(m, n)$, if it is formed by $m$ even elements $\{f_i^0 \in F_0 \big| i = 1, \dots, m\}$ and $n$ odd elements $\{f_\alpha^1 \in F_1 | \alpha = 1, \dots, n\}$.

We have a canonical isomorphism

$$F \simeq \left( \bigoplus_{i=1}^{m} R f_i^0 \right) \oplus \left( \bigoplus_{\alpha=1}^{n} R f_\alpha^1 \right).$$

For each pair of natural numbers $m, n$ such that $m + n = p$, the $R$-module $R^p$ can be regarded as a free graded $R$-module endowed with a basis of type $(m, n)$, by letting,

$$(R^{m+n})_0 \equiv R^{m,n} = R_0^m \oplus R_1^n;$$

$$(R^{m+n})_1 \equiv R^{\bar{m},\bar{n}} = R_0^n \oplus R_1^m \tag{2.2}$$

$R^{m+n}$equipped with this gradation will be denoted by $R^{m|n}$.

*Example* 2.6. (cf. [5]) Let $R$ be a commutative ring, and $M$ be an $R$-module. The exterior algebra of $M$over $R$, denoted by $\wedge_R M$, is a $\mathbb{Z}$-graded algebra, namely $\bigoplus_{p \in \mathbb{Z}} \wedge_R^p M$, and is alternating, i.e. $x^2 = 0$ for all $x \in \wedge_R^{2p+1} M$. If $M$is free and finitely generated, with a basis $\{e_i | i = 1, \dots, N\}$, then $\wedge_R M$ is a free finitely generated $R$-module, with a canonical basis(relative to the basis $\{e_i\}$) which can be described as follows. Let $\Xi_N$ denote the set

$$\left\{ \begin{matrix} \mu: \{1, \dots, r\} \to \\ \{1, \dots, N\} \text{strictly increasing} \end{matrix} \middle| 1 \le r \le N \right\} \cup \{\mu_0\},$$

where $\mu_0$ is the empty sequence, and let

$$\beta_\mu = e_{\mu(1)} \wedge \dots \wedge e_{\mu(r)} \text{for } \mu \neq \mu_0, \;\; \beta_{\mu_0} = 1.$$

Then $\{\beta_\mu | \mu \in \Xi_N\}$ is the canonical basis of $\wedge_R M$.

The cases$R = \mathbb{R}$ and $R = \mathbb{C}$ have a particular interest and deserve ad hoc notations:

$$\wedge_{\mathbb{R}} \mathbb{R}^L \equiv B_L; \;\; \wedge_{\mathbb{C}} \mathbb{C}^L \equiv C_L \tag{2.3}$$

$B_L$is a vector space, with a canonical basis obtained from the canonical basis of $\mathbb{R}^L$ according to the above described procedure. If $\mathbb{m}_L$ is the ideal of nilpotentsof $B_L$, the vector space direct sum decomposition $B_L = \mathbb{R} \oplus \mathbb{m}_L$ defines two projections

$$\sigma: B_L \to \mathbb{R}; \quad s: B_L \to \mathbb{m}_L \tag{2.4}$$

which are sometimes called body and soul maps.

*Tensor Products:* Let us recall that we are considering a graded-commutative ring $R$. The graded tensor product of two graded $R$-modules $M, N$ is by definition the usual tensor product $M \oplus_R N$, obtained by regarding $M$ as a right module, and $N$as a left module, equipped with the gradation

$$(M \oplus_R N)_\gamma = \oplus_{\alpha+\beta=\gamma} \left\{ \sum m_i \otimes n_j | m_i \in M_\alpha, n_j \in N_\beta \right\}$$

Evidently, $M \otimes_R N$ has a natural structure of graded $R$-module:

$$a(x \otimes y) = ax \otimes y = (-1)^{|a||x|} xa \otimes y$$
$$= (-1)^{|a||x|} x \otimes ay$$
$$= (-1)^{|a|(|x|+|y|)} (x \otimes y) a. \tag{2.5}$$

The graded tensor product can be characterized as a 'universal object'. To this end, given graded $R$-modules $M, N$ and $Q$, we introduce the set $\mathcal{L}(M, N; Q)_\alpha$ (with $\alpha \in \mathbb{Z}_2$) of the graded $R$-bilinear morphisms $f: M \times N \to Q$, homogeneous of degree $\alpha$: if $f \in \mathcal{L}(M, N; Q)_\alpha$, then $f$ is a morphism of degree $\alpha$ such that $f(xa, y) = f(x, ay) = (-1)^{|a||y|} f(x, y) a$ for all $a \in R$. The set

$$\mathcal{L}(M, N; Q) \equiv \mathcal{L}(M, N; Q)_0 \oplus \mathcal{L}(M, N; Q)_1$$

is endowed with a structure of graded $R$-module by enforcing the multiplication rule $(fa)(x, y) = f(ax, y)$. In the same way, if $M_1, \dots, M_n, Q$ are graded $R$-modules, we define the graded $R$-module $\mathcal{L}(M_1, \dots, M_n; Q)$ formed by the graded $R$-multilinear morphisms $M_1 \times \cdots \times M_n \to Q$.

*Proposition* 2.7. *There are natural isomorphisms in the category* $R - G$ *Module*

$$\mathcal{L}(M, N; Q) \simeq Hom_R(M \otimes_R N, Q)$$
$$\simeq Hom_R(M, Hom_R(N, Q)).$$

*Proposition* 2.8. *Let* $M, M', M''$ *be graded* $R$*-modules; the following natural isomorphisms of graded* $R$*-modules hold:*

(a) $M \otimes_R M' \simeq M' \otimes_R M$, *achieved by the morphism*
$$x \otimes x' \mapsto (-1)^{|x||x'|} x' \otimes x;$$
(b) $(M \otimes_R M') \otimes_R M''$
$\simeq M' \otimes_R (M' \otimes_R M'')$, *achieved by the morphism*
$$(x \otimes x') \otimes x'' \mapsto x \otimes (x' \otimes x'');$$
(c) $R \otimes_R M \simeq M \simeq M \otimes_R R$.

If $f: M \to P$, $g: N \to Q$ are morphisms of graded modules over a graded ring $R$, the tensor product $f \otimes g: M \otimes_R N \to P \otimes_R Q$ is the morphism defined by the condition

$$(f \otimes g)(m \otimes n) = (-1)^{|g||m|} f(m) \otimes g(n). \tag{2.6}$$

**III. Graded Algebras and Graded Tenso Calculus**

Let $R$ be a graded-commutative ring.

*Definition* 3.1. *A graded* $R$*-algebra* $P$*is a graded* $R$*-module endowed with a graded* $R$*-bilinear multiplication*

$$P \otimes P \to P$$

$$x \otimes y \mapsto x \cdot y.$$

A graded $R$-algebra $P$ is said to be graded-commutative if all graded commutators

$$< x, y > = x \cdot y - (-1)^{|x||y|} y \cdot x,$$

defined on the analogy of equation (2.1), vanish.

*Example* 3.2. The graded module $B_L(C_L)$in traduced in Example 2.6. , equipped with the exterior product, is a graded-commutative $\mathbb{R}$-algebra($\mathbb{C}$-algebra).

The graded tensor product $P \otimes_R Q$ of two graded $R$-algebras $P$and $Q$ is defined as the tensor product of the underlying $R$-modules equipped with the multiplication naturally induced by those of $P$ and

$$Q: (x_1 \otimes y_1) \cdot (x_2 \otimes y_2) = (-1)^{|y_1||x_2|}(x_1 \cdot x_2) \otimes (y_1 \cdot y_2).$$

*Definition* 3.3. *A graded Lie $R$-algebra ( or Lie $R$-superalgebra ) $\mathfrak{B}$ is a graded $R$-algebra, whose multiplication, called graded Lie bracket and denoted by* [ , ], *satisfies the following identities:*

$$[x, y] = -(-1)^{|x||y|}[y, x]; \tag{3.1}$$

$$(-1)^{|x||z|}\big[x, [y, z]\big] + (-1)^{|y||x|}\big[y, [z, x]\big] + (-1)^{|z||y|}\big[z, [x, y]\big] = 0. \tag{3.2}$$

*Remark* 3.4. *Given a graded Lie algebra $\mathfrak{B}$, its even part $\mathfrak{B}_0$is a Lie algebra over the ring $R_0$.*

*An important class of graded Lie algebras can be constructed in terms of the notion of graded derivation.*

Let Pbe a graded-commutative R-algebra.

*Definition* 3.5. *A homogeneous morphism $D \in End_R P$ is a graded derivation of $P$ over $R$ if it fulfills the following condition (called the graded Leibnitz rule)*

$$D(x \cdot y) = D(x) \cdot y + (-1)^{|x||D|} x \cdot D(y). \tag{3.3}$$

The graded $R$-submodule of $End_R P$ generated by the graded by the derivations of $P$ will be denoted by $Der_R P$, or simply $Der P$.

*Proposition* 3.6. *$DerP$, equipped with the graded Lie bracket*

$$[D_1, D_2] \equiv D_1 \circ D_2 - (-1)^{|D_1||D_2|} D_2 \cdot D_1, \tag{3.4}$$

*is a graded Lie $R$-algebra.*

By identifying $R$ with the submodule $R.\ 1 \subset P$, condition (3.4) implies that, for all $D \in DerP, D(R) = 0$. We notice that $DerP$ is a (left) graded $P$-module in a natural way, by letting $(xD)(y) = x \cdot D(y)$.

*Definition* 3.7. *A graded derivation of $P$ over $R$ with values in $M$ is a homogeneous element $D \in Hom_R(P, M)$ which fulfills a graded Leibnitz rule formally identical with equation (3.3).*

The graded $P$-submodule of $Hom_R(P, M)$ generated by the graded derivations of $P$ with values in $M$ will be denoted by $Der_R(P, M)$.

*Proposition* 3.8. *Let $M$ and $N$ be $R$-modules. There is a natural morphism of graded $R$-modules*

$$\phi: N \otimes M^* \to Hom(M, N)$$

*described by $\phi(n \otimes \omega)(m) = n\omega(m)$. This induces a morphism*

$$\gamma: M^* \otimes N^* \to (M \otimes N)^*$$

*whose expression is*

$$\gamma(\omega \otimes \eta)(m \otimes n) = (-1)^{|\eta||m|}\omega(m)\eta(n).$$

Both morphisms are bijective whenever $M$is free and finitely generated.

Graded Exterior Algebra : Let $M$ be a graded $R$-module and let us denote by

$$T^p M = M \underbrace{\otimes \cdots \otimes}_{P} M \tag{3.1}$$

The $p$-th tensor power of $M$, graded as usual. We can consider as in the non-graded setting the graded tensor algebra of $M$,

$$\mathcal{T}(M) = \bigotimes_{p=0}^{\infty} T^p M, \tag{3.5}$$

which is in a natural way a bigraded$R$-algebra (i.e. it has the usual $\mathbb{Z}$-gradation of the tensor algebra, together with the $\mathbb{Z}_2$-gradation it carries as a graded $R$-algebra).

The graded exterior algebra $\wedge_R M$ of $M$(denoted simply by $\wedge M$) is defined as the quotient of $\mathcal{T}(M)$ by the graded ideal $\mathfrak{I}(M)$ generated by elements of the form $m_1 \otimes m_2 + (-1)^{|m_1||m_2|} m_2 \otimes m_1$, with $m_1, m_2$ homogeneous. The product induced in $\wedge M$ by this quotient is denoted by $\wedge$ and is called the (graded) wedge product, as usual. If we let $\mathfrak{I}^p(M) = \mathfrak{I}(M) \cap T^p M$, since $\mathfrak{I}(M)$ is generated by homogeneous elements, we obtain $\mathfrak{I}(M) = \otimes_{p=0}^{\infty} \mathfrak{I}^p(M)$ and therefore,

$$\wedge M = \bigotimes_{p=0}^{\infty} \wedge^p M$$

with $\wedge^p M = T^p M / \mathfrak{I}^p(M)$.

We wish to ascertain the relationship existing between the exterior algebra $\wedge M^*$and the modules of alternating graded multilinear forms: this will be realized by a morphism analogous to the morphism

$$\gamma: M_1^* \otimes \cdots \otimes M_n^* \to (M_1 \otimes \cdots \otimes M_n)^* \simeq \mathcal{L}(M_1, \ldots, M_n; R). \tag{3.6}$$

If $F_p \in HOm(T^p M, R)$ and $F_q \in Hom(T^p M, R)$ are homogeneous graded multilinear forms, $F_p \otimes F_q$ acts on a family of homogeneous elements according to the formula:

$$(F_p \otimes F_q)(m_1, \ldots, m_{p+q}) = (-1)^{|F_q|(|m_{p+1}|+\cdots+|m_{p+q}|)} F_p(m_1, \ldots, m_n) F_q(m_{p+1}, \ldots, m_{p+q}).$$

Let $\mathcal{S}_p$ be the group of permutation of $p$ objects. For any $\sigma \in \mathcal{S}_p$ and any $F_p \in Hom(T^p M, R)$, we write, for homogeneous elements $m_1, \ldots, m_p \in M$,

$$F_p^\sigma(m_1, \ldots, m_p) = (-1)^{\Delta_1(\sigma,m)} F_p(m_{\sigma(1)}, \ldots, m_{\sigma(p)}),$$

where

$$\Delta_1(\sigma, m) = \sum_{1\le i<j\le p} \sum_{\sigma(i)>\sigma(j)} |m_{\sigma(i)}||m_{\sigma(j)}|. \quad (3.7)$$

*Definition* 3.9. *A graded multilinear form $F_p \in Hom(T^p M, R)$ is said to be alternating if $F_p^\sigma = (-1)^{|\sigma|} F_p$ for every $\sigma \in \mathcal{S}_p$, where $|\sigma|$ is the parity of the permutation $\sigma$.*

*The set* $Alt\left(M \underbrace{\times \cdots \times}_{p} M; R\right) \equiv Alt(M^p, R)$ *of all alternating graded multilinear forms is a submoduleof $Hom(T^p M, R)$; we can introduce a projection morphism, which is no more than the graded anti-symmetrization:*

$$A_p: Hom(T^p M, R) \to Alt(M^p; R)$$

$$F_p \to A_p(F_p) = \frac{1}{p!} \sum_{\sigma\in\mathcal{S}_p} (-1)^{|\sigma|} F_p^\sigma.$$

*Proposition* 2.10. *The morphism $A_p$ has the following properties:*

(a) $A_p(F_p) = F_p$*for any alternating form $F_p$;*

(b) $A_{p+q}(F_q \otimes F_p) = (-1)^{pq+|F_p||F_q|} A_{p+q}(F_p \otimes F_q)$ *for homogeneous $F_p, F_q$;*

(c) $A_{p+q}\left(A_p(F_p) \otimes F_q\right) = A_{p+q}(F_p \otimes F_q)$.

We assume that $M$ is a free and finitely generated module, so that we may identify $T^p(M^*)$with $Hom(T^p M, R)$. In this way, the morphism $A_p$ yields the exact sequence of graded $R$-modules

$$0 \to \mathfrak{I}(M^*) \to T^p M^* \xrightarrow{A_p} Alt(M^p; R) \to 0, \quad (3.8)$$

and therefore we obtain an isomorphism $\wedge^p M^* \simeq Alt(M^p; R)$. Thus, for a free and finitely generated module $M$, the homogeneous elements in the graded exterior algebra $\wedge M^*$ can be interpreted as alternating graded multilinear forms on $M$. In particular, we may interpret the wedge product of two elements $w^p \in \wedge^p M^*$ and $w^q \in \wedge^q M^*$ as a graded multilinear form, which acts on homogeneous elements $m_1, \ldots, m_{p+q}$ according to [9];

$$(\omega^p \wedge \omega^q)(m_1, \ldots, m_{p+q}) = \frac{1}{(p+q)!} \sum_{\sigma\in\mathcal{S}_p} (-1)^{|\sigma|+\Delta_2(\sigma,m,\omega^q)} \omega^p(m_{\sigma(1)}, \ldots, m_{\sigma(p)}) \omega^q(m_{\sigma(p+1)}, \ldots, m_{\sigma(p+q)})$$

where in terms of the symbol $\Delta_1(\sigma, m)$ previously defined, we get

$$\Delta_2(\sigma, m, \omega^q) = \Delta_1(\sigma, m) + |\omega^q| \textstyle\sum_{i=1}^p |m_{\sigma(i)}|. \quad (2.9)$$

## IV. Matrices

Given a graded-commutative ring $R$, an $R$-module morphism $R^{m|n} \to R^{p|q}$ can be regarded, relative to the canonical bases of $R^{m|n}$ and $R^{p|q}$, as a $(p+q) \times (m+n)$ matrix with entries in $R$,

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \quad (4.1)$$

which acts on column vectors in $R^{nt|n}$ from the left. The set $M_R[(p+q) \times (m+n)]$ of such matrices can be graded so as to be naturally isomorphic to the graded $R$-module $Hom_R(R^{m|n}, R^{p|q})$, by decreeing that:

•$X$ is even if $X_1$ and $X_4$ have even entries, while $X_2$ and $X_3$ have odd entries;

•$X$ is odd if $X_1$ and $X_4$ have odd entries, while $X_2$ and $X_3$ have even entries;

The set of matrices of the form (4.1), equipped with this gradation, will be denoted by $M_R[p|q; m|n]$. The set of square matrices $M_R[m|n]$(which are obtained by letting $p = m,\ q = n$) is a graded $R$-algebra.

The usual notation of trace and determinant of a matrix can be expended to the matrices in $M_R[m|n]$, thus obtaining the concepts of graded trace and Berezinian (also called supertrace and superdeterminant respectively). For any matrix $X \in M_R[p|q\,; m|n]$, regarded as a morphosm$X: R^{m|n} \to R^{m|n}$, we define the graded transpose of $X$-denoted by $X^{gt}$-as the matrix corresponding to the morphism $X^*: (R^{p|q})^* \to (R^{m|n})^*$dual to $X$. With reference to equation (4.1), one obtains the following relations, where the superscript $t$ denotes the usual matrix transportation:

$$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}^{gt} = \begin{cases} \begin{pmatrix} X_1^t & X_2^t \\ -X_3^t & X_4^t \end{pmatrix} \text{if } |X| = 0 \\ \begin{pmatrix} X_1^t & -X_3^t \\ X_2^t & X_4^t \end{pmatrix} \text{if } |X| = 1 \end{cases} \quad (4.2)$$

The graded transportation behaves naturally with respect to matrix multiplication:

$$(XY)^{gt} = (-1)^{|X||Y|} Y^{gt} X^{gt}.$$

The graded trace of $X$ is the element $StrX = \sum_i a_i^*(a^i) \in R$. Alternatively, one can give a direct characterization by letting, for all homogeneous $X \in M_R[m|n]$,

$$Str = TrX_1 - (-1)^{|X|}TrX_4 \tag{4.3}$$

where $Tre$ designates the usual trace operation. The graded trace determines an $R$-module morphism $Str: M_R[m|n] \to R$, which is natural with respect to graded transportation and matrix multiplication:

$$Str(X^{gt}) = StrX$$

$$Str(XY) = (-1)^{|X||Y|}Str(YX). \tag{4.4}$$

Let us notice that, by denoting by $I_{m|n}$ the identity matrix, one has $Str\, I_{m|n} = m - n$.

In order to extend the notion of determinant, we must consider the subgroup $GL_R[m|n]$ of the matrices in $M_R[m|n]$ corresponding to an even invertible endomorphisms. $GL_R[m|n]$is the natural extension of the notion of general linear group, so that it will be called the general graded linear group.

*Proposition* 4.1. *A matrix $x \in M_R[m|n]_0$ is in $GL_R[m|n]$if and only if $X_1 \in GL_R[m|0]$ and $X_4 \in GL_R[0|n]$, i.e. $X$ is invertible if and only if $X_1$ and $X_4$ are invertible as ordinary matrices with entries in $R_0$.*

*Definition* 4.2. [1], [3], [4] *Let $X \in GL_R[m|n]$. the Berezinian of $X$ is the element in $GL_R[1|0]$ given by*

$$BerX = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$$

$$= \det(X_1 - X_2X_4^{-1}X_3)\,(\det X^{-1}). \tag{4.5}$$

*Proposition* 4.3. *The mapping $Ber : GL_R[m|0] \to GL_R[0|n]$ is a group morphism, that coincides with the determinant whenever $n = 0$:*

$$Ber(XY) = BerX\; BerY \quad \forall\, X, Y \in GL_R[m|n] \tag{4.6}$$

**Theorem 4.4.** *A matrix in $X \in M_R[m|n]_0$ is invertible if and only if $\sigma(X) \in GL[m+n]$.*

*Proof.* The 'only if' part is trivial, since $\sigma$ is ring morphism. To show the converse, it suffices to prove that a matrix $Z \in M_{B_L}[p|0]_0$ is invertible as a matrix with entries in $(B_L)_0$ if $\sigma(Z)$ is invertible. In the case $p = 1$ this is a consequence of the fact that in $B_L$ the morphism $\sigma$ is the natural projection $(B_L)_0 \to (B_L)_0/(n_L)_0$. The result is easily extended to $p > 1$ by inclusion. ■

## V. Conclusion

We start with given an arbitrary group $G$ and introducing $G$-graded algebraic objects and for a given graded-commutative ring $R$ and $R$-module morphism can be regarded, relative to the canonical bases of relative to the canonical bases of $R^{m|n}$ and $R^{p|q}$, as a $(p+q) \times (m+n)$ matrix with entries in $R$, $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$, which acts on column vectors in $R^{nt|n}$ from the left. Finally, this article induces a **Theorem 4.4.**on a matrix of graded $R$-algebra. This paper will be helpful for other researchers.

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