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New Traveling Wave Solutions to the Simplified Modified Camassa–Holm Equation and the Landau-Ginsburg-Higgs Equation

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Abstract

Researchers are interested in the (1+1)-dimensional Camassa-Holm and Landau-Ginzburg-Higgs equations as they allow for the study of unidirectional wave propagation in shallow waters with a flat seabed, as well as nonlinear media exhibiting dispersion systems and superconductivity. This work has effectively developed exact wave solutions to the stated models, which may have significant consequences for characterising the nonlinear dynamical behaviour related to the phenomena. The extended (G'/G)-expansion technique is employed to procure a diverse array of progressive wave solutions characterized by hyperbolic, trigonometric, and rational functions. The solutions are shown as 3D profiles with a variety of shapes, including kink, singular kink, periodic, singular periodic, etc. The physical significance of the solutions is discussed by these plots, and the approach used in this study is considered efficient and capable of finding analytical solutions for the nonlinear models.

Keywords: Extended (G'/G)-expansion method; Camassa-Holm equation; Landau-Ginzburg-Higgs equation; Traveling wave solutions; Soliton.

I. Introduction

Nonlinear evolution equations (NLEEs) are frequently employed to portray intricate physical phenomena. Nonetheless, determining closed-form solutions of traveling waves for these models, which are essential for nonlinear science and engineering, presents a significant impediment to their utilization. The proliferation of scholars investigating analytical wave solutions of NLEEs has generated significant interest in this area of study. In recent times, several efficient and direct techniques have emerged, enabling researchers to achieve a deeper perception of the underlying mechanisms of these natural phenomena. Some of the presently employed methodologies include: the exp-function process[1], nonlinear transformation method[2], sine-cosine method[3], modified simple equation technique[4], Hirota's bilinear method[5], variational iteration method[6], He’s homotopy perturbation technique[7], (G'/G)-expansion technique and its several variations[8-10], Adomian decomposition method[11], test function method[12], tanh-function procedure[13], the improved tanh process[14], generalized Kudryashov process[15] and others. Different equations have been studied using a variety of techniques. For the simplified form of modified Camassa-Holm (MCH) equation, Islam et al.[16] applied the new auxiliary equation approach, Najafi et al.[17] exploited the semi-inverse scattering method, and Alam and Akbar[18] investigated the same equation applying the generalized (G'/G)-expansion technique. Also, Liu et al.[19] utilized the (G'/G)-expansion process to analyze the specified model. Similarly, for the nonlinear Landau-Ginzburg-Higgs (LGH) equation, Kundu et al.[20] applied the Sine-Gordon expansion procedure to obtain exact solutions, while Barman et al.[21,22] examined the same equation using the Kudryashov technique and the extended tanh-function process to derive exact wave results. Additionally, Ahmad et al.[23] applied the power index approach to study this model.

As far as the authors' knowledge goes, the extended (G'/G)-expansion technique[12] has not been used in the analysis of the (1+1)-dimensional simplified MCH model and LGH equation. The aim of this paper is to employ the extended (G'/G)-expansion technique to obtain analytical wave solutions of the previously mentioned couple of equations. The paper’s structure is as follows: Section II expounds on the methodology, while Section III employs the extended (G'/G)-expansion technique to probe the solutions. Section IV presents the comparison and validations of the obtained results. Section V deliberates on the findings, accompanied by graphical depictions of the solutions. At last, Section VI furnishes concluding remarks.

II. Elucidation of the Technique

Consider an evolution equation involving spatiotemporal variables, denoted as the space variable (x) and time variable (t), and expressed as:

$$M(w, w_x, w_t, w_{xx}, w_{tt}, w_{xt}, \ldots) = 0 \quad (1)$$

Here, $w = w(x,t)$ is an unidentified wave function, and M is a polynomial that includes time and space derivatives of $w(x,t)$. To solve equation (1), we will employ the extended

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(G'/G)-expansion approach. The steps involved in this approach are as follows:

Step 1: The wave coordinate ζ serves to link the space-time coordinates x and t, as shown below:

$$w(x,t) = V(\zeta),\ \zeta = x - ct \tag{2}$$

where c is the wave displacement rate. The equation (1) can be transformed into an equation for V by using the wave variable specified in (2).

$$N(V, V', V'', V''', \ldots) = 0 \tag{3}$$

wherein N indicates a polynomial of V as well as ordinary differential coefficients of its (V) with respect to ζ, exhibiting varying orders, and the superscripts on V represent the orders of the differential coefficients.

Step 2: We may integrate equation (3) one or multiple times, depending on the circumstance. To simplify the search for soliton solutions, we set the integral constants to 0.

Step 3: As per the extended (G'/G)-expansion technique, the solution of equation (3) is taken as follows:

$$V(\zeta) = a_0 + \sum_{i=1}^{l}[a_i(G'/G)^i + b_i(G'/G)^{-i}] \tag{4}$$

where the constants $a_0,\ a_i,\ b_i (i = 1, 2, \ldots, l)$ will be determined at a later stage and $G = G(\zeta)$ satisfies the following ODE:

$$G'' + \gamma G' + \mu G = 0 \tag{5}$$

The above equation has the following solutions:

$$(G'/G) = \begin{cases} -\frac{\gamma}{2} + \frac{\sqrt{\gamma^2-4\mu}}{2} \times \frac{C\cosh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}\zeta\right) + D\sinh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}\zeta\right)}{C\sinh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}\zeta\right) + D\cosh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}\zeta\right)}, \gamma^2 - 4\mu > 0 \\ -\frac{\gamma}{2} + \frac{\sqrt{4\mu-\gamma^2}}{2} \times \frac{C\cos\left(\frac{\sqrt{4\mu-\gamma^2}}{2}\zeta\right) - D\sin\left(\frac{\sqrt{4\mu-\gamma^2}}{2}\zeta\right)}{C\sin\left(\frac{\sqrt{4\mu-\gamma^2}}{2}\zeta\right) + D\cos\left(\frac{\sqrt{4\mu-\gamma^2}}{2}\zeta\right)}, \gamma^2 - 4\mu < 0 \\ -\frac{\gamma}{2} + \frac{D}{C+D\zeta}, \gamma^2 - 4\mu = 0 \end{cases} \tag{6}$$

wherein C and D are any constants.

Step 4: To find the natural number l in equation (4), we must carefully examine the homogeneous balance that exists between the highest order derivative term and the non-linear term of highest order in equation (3).

Step 5: By incorporating equations (4) and (5) into equation (3) and utilizing the value of l found in Step 4, we can derive an algebraic expression for (G'/G). The process of equating the coefficient of every term to zero will yield a class of algebraic equations that may be unravelled for the estimations of a_0, a_i, b_i, c and any other necessary constraints.

Step 6: By plugging the solutions provided in equation (6) and the estimations of a_0, a_i, b_i, and c into solution (4), we can derive comprehensive analytical solutions for equation (1).

III. Implementations of the Technique

Here, we implement the extended (G'/G)-expansion technique to develop the further fresh exact wave solutions for the (1+1)-dimensional simplified MCH model and LGH equation.

Modified Camassa-Holm equation

We contemplate the following simplified MCH model[16-19]:

$$w_t + 2kw_x - w_{xxt} + \beta w^2 w_x = 0 \tag{7}$$

This equation, which concerns the dispersion of water waves and has been widely studied, is particularly important in its field. It was originally developed to explain how shallow water waves propagate in a single direction over flat terrain. The parameter k, which belongs to the set of real numbers, is linked to the critical speed at which shallow water waves can travel, while $w(x,t)$, represented in non-dimensional variables, refers to the water's free surface. By using a traveling wave transformation

$w(x,t) = V(\zeta)$, where $\zeta = x - \nu t$, the following nonlinear ODE is created from equation (7).

$$(2k-\nu)V + \nu V'' + (\beta/3)V^3 = 0 \tag{8}$$

Equation (8) yields $l = 1$, if we apply the principle of homogeneous balance between the highest order nonlinear term (V^3) and the highest order derivative (V''). Hence, the form of the solution (4) is as follows:

$$V(\zeta) = a_0 + a_1(G'/G) + b_1(G'/G)^{-1} \tag{9}$$

By inserting equations (9) and (5) into (8) and gathering the coefficients for $(G'/G)^i$and $(G'/G)^{-i}$ (where $i = 0, 1, 2, 3$), and setting them to 0, a collection of algebraic equations is obtained. Unravelling this group of equations by means of the software MAPLE produces the subsequent set of solutions:

Set-1:

$$a_0 = \pm\sqrt{\frac{6k}{\beta(4\mu-\gamma^2-2)}}\gamma,\ a_1 = \pm 2\sqrt{\frac{6k}{\beta(4\mu-\gamma^2-2)}},$$

$$b_1 = 0,\ \nu = -\frac{4k}{4\mu-\gamma^2-2} \tag{10}$$

where β, γ, μ and k are free constants.

Set-2:

$$a_0 = \pm\sqrt{\frac{6k}{\beta(4\mu-\gamma^2-2)}}\gamma,\ a_1 = 0,$$

$$b_1 = \pm 2\sqrt{\frac{6k}{\beta(4\mu-\gamma^2-2)}}\mu,\ \nu = -\frac{4k}{4\mu-\gamma^2-2} \tag{11}$$

where β, γ, μ and k are uninformed constants.

Substituting the estimations of the constants from (10) and (11) into the equation (9), we get

$$V_1(\zeta) = \pm\sqrt{\frac{6k}{\beta(4\mu-\gamma^2-2)}}\left(\gamma + 2(G'/G)\right) \tag{12}$$

$$V_2(\zeta) = \pm\sqrt{\frac{6k}{\beta(4\mu-\gamma^2-2)}}\left(\gamma + 2\mu(G'/G)^{-1}\right) \tag{13}$$

where $\zeta = x + \frac{4k}{4\mu-\gamma^2-2}t$.

The traveling wave solutions to the equation (7) are obtained from the solution (12) by putting

the results of (G'/G) given in (6) as below:

When $\gamma^2 - 4\mu > 0$, we get the hyperbolic solution:

$$V_{11}(\zeta) = \pm\sqrt{\frac{6k(\gamma^2-4\mu)}{\beta(4\mu-\gamma^2-2)}} \times \frac{C\cosh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}\zeta\right) + D\sinh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}\zeta\right)}{C\sinh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}\zeta\right) + D\cosh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}\zeta\right)} \tag{14}$$

where $\zeta = x + \frac{4k}{4\mu-\gamma^2-2}t$.

When $\gamma^2 - 4\mu < 0$, we have the trigonometric solution:

$$V_{12}(\zeta) = \pm\sqrt{\frac{6k(4\mu-\gamma^2)}{\beta(4\mu-\gamma^2-2)}} \times \frac{C\cos\left(\frac{\sqrt{4\mu-\gamma^2}}{2}\zeta\right) - D\sin\left(\frac{\sqrt{4\mu-\gamma^2}}{2}\zeta\right)}{C\sin\left(\frac{\sqrt{4\mu-\gamma^2}}{2}\zeta\right) + D\cos\left(\frac{\sqrt{4\mu-\gamma^2}}{2}\zeta\right)} \tag{15}$$

where $\zeta = x + \frac{4k}{4\mu-\gamma^2-2}t$.

When $\zeta^2 - 4\mu = 0$, we attain the rational solution:

$$V_{13}(\zeta) = \pm\sqrt{-\frac{3k}{\beta}} \times \frac{2D}{C+D\zeta} \tag{16}$$

where $\zeta = x - 2kt$.

Similarly, the wave solutions of the equation (7) are obtained from the solution (13) by putting the results of (G'/G) given in (6) as bellow:

When $\gamma^2 - 4\mu > 0$, we get the hyperbolic solution:

$$V_{21}(\zeta) = \pm\sqrt{\frac{6k}{\beta(4\mu-\gamma^2-2)}}\left(\gamma + 2\mu\left(-\frac{\gamma}{2} + \frac{\sqrt{\gamma^2-4\mu}}{2} \times \frac{C\cosh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}\zeta\right) + D\sinh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}\zeta\right)}{C\sinh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}\zeta\right) + D\cosh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}\zeta\right)}\right)^{-1}\right) \tag{17}$$

where C, D, γ, μ, k and β are free constants and the value of ζ is given by $\zeta = x + \frac{4k}{4\mu-\gamma^2-2}t$.

When $\gamma^2 - 4\mu < 0$, we have the trigonometric solution:

$$V_{22}(\zeta) = \pm\sqrt{\frac{6k}{\beta(4\mu-\gamma^2-2)}}\left(\gamma + 2\mu\left(-\frac{\gamma}{2} + \frac{\sqrt{4\mu-\gamma^2}}{2} \times \frac{C\cos\left(\frac{\sqrt{4\mu-\gamma^2}}{2}\zeta\right) - D\sin\left(\frac{\sqrt{4\mu-\gamma^2}}{2}\zeta\right)}{C\sin\left(\frac{\sqrt{4\mu-\gamma^2}}{2}\zeta\right) + D\cos\left(\frac{\sqrt{4\mu-\gamma^2}}{2}\zeta\right)}\right)^{-1}\right) \quad (18)$$

where $\zeta = x + \frac{4k}{4\mu-\gamma^2-2}t$.

When $\gamma^2 - 4\mu = 0$, we achieve the rational solution:

$$V_{23}(\zeta) = \pm\sqrt{-\frac{3k}{\beta}}\left(\gamma + \frac{\gamma^2}{2}\left(-\frac{\gamma}{2} + \frac{D}{C+D\zeta}\right)^{-1}\right) \quad (19)$$

where $\zeta = x - 2kt$.

Landau-Ginzburg-Higgs equation

Th LGH model equation is taken as follows[20-23]:

$$w_{tt} - w_{xx} - m^2w + n^2w^3 = 0 \quad (20)$$

The electro-static potential, represented by $w(x,t)$, is influenced by both the temporal and spatial coordinates t and x respectively, with m and n serving as real constants that are not equal to zero. By using a traveling wave transformation $w(x,t) = V(\zeta)$, where $\zeta = x - ct$, equation (20) is distorted into the subsequent nonlinear ODE.

$$(c^2 - 1)\,V'' - m^2V + n^2V^3 = 0 \quad (21)$$

Equation (21) yields $l = 1$, if we adopt the principle of homogeneous balance between the highest order nonlinear term (V^3) and the highest order derivative (V''). Hence, the form of the solution (4) is as follows:

$$V(\zeta) = a_0 + a_1(G'/G) + b_1(G'/G)^{-1} \quad (22)$$

By inserting equations (22) and (5) into equation (21) and gathering the coefficients for $(G'/G)^i$and $(G'/G)^{-i}$ (where $i = 0, 1, 2, 3$), and setting them to 0, a group of algebraic equations is obtained. Unravelling this family of equations by means of the software MAPLE produces the subsequent set of solutions:

Set-1:

$$a_0 = \pm\frac{m\gamma}{n\sqrt{\gamma^2-4\mu}},\ a_1 = \pm\frac{2m}{n\sqrt{\gamma^2-4\mu}},$$

$$b_1 = 0,\ c = \pm\sqrt{\frac{\gamma^2-2m^2-4\mu}{\gamma^2-4\mu}} \quad (23)$$

where λ, μ,$m$and n are random constants.

Set-2:

$$a_0 = \pm\frac{m\gamma}{n\sqrt{\gamma^2-4\mu}},\ a_1 = 0,\ b_1 = \pm\frac{2\mu m}{n\sqrt{\gamma^2-4\mu}},$$

$$c = \pm\sqrt{\frac{\gamma^2-2m^2-4\mu}{\gamma^2-4\mu}} \quad (24)$$

where γ, μ,$m$and n are any constants.

Placing the values of the constants from (23) and (24) into the result (22), we get

$$V_1(\zeta) = \pm\frac{m}{n\sqrt{\gamma^2-4\mu}}\left(\gamma + 2(G'/G)\right) \quad (25)$$

$$V_2(\zeta) = \pm\frac{m}{n\sqrt{\gamma^2-4\mu}}\left(\gamma + 2\mu(G'/G)^{-1}\right) \quad (26)$$

where $\zeta = x \pm\sqrt{\frac{\gamma^2-2m^2-4\mu}{\gamma^2-4\mu}}t$.

The progressive wave solutions of the LGH model are attained from the solution (25) by putting the results of (G'/G) given in (6) as follows:

When $\gamma^2 - 4\mu > 0$, we get the hyperbolic solution:

$$V_{11}(\zeta) = \pm\frac{m}{n} \times \frac{C\cosh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}\zeta\right) + D\sinh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}\zeta\right)}{C\sinh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}\zeta\right) + D\cosh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}\zeta\right)} \quad (27)$$

where $\zeta = x \pm\sqrt{\frac{\gamma^2-2m^2-4\mu}{\gamma^2-4\mu}}t$.

When $\gamma^2 - 4\mu < 0$, we have the trigonometric solution:

$$V_{12}(\zeta) = \pm\frac{im}{n} \times \frac{C\cos\left(\frac{\sqrt{4\mu-\gamma^2}}{2}\zeta\right) - D\sin\left(\frac{\sqrt{4\mu-\gamma^2}}{2}\zeta\right)}{C\sin\left(\frac{\sqrt{4\mu-\gamma^2}}{2}\zeta\right) + D\cos\left(\frac{\sqrt{4\mu-\gamma^2}}{2}\zeta\right)} \quad (28)$$

where $\zeta = x \pm\sqrt{\frac{\gamma^2-2m^2-4\mu}{\gamma^2-4\mu}}t$.

Likewise, the analytical wave solutions of the LGH model are achieved from the solution (26) by putting the results of (G'/G) given in (6) as follows:

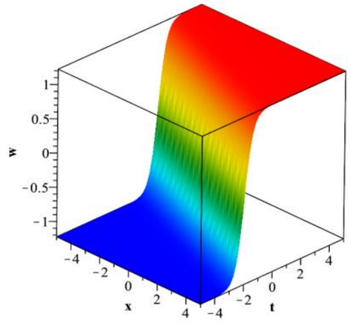

Fig. 1. Kink soliton of solution (14) for $C = 0$, $D \neq 0$, $k = -2.5$, $\beta = 5$, $\gamma = 2$ and $\mu = 0.5$.

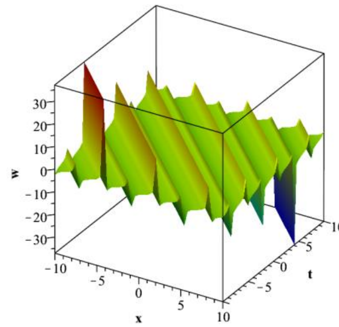

Fig. 2. Periodic soliton of solution (15) for $C = 0$, $D \neq 0$, $k = 0.5$, $\beta = 5$, $\gamma = 4$ and $\mu = 5$.

When $\gamma^2 - 4\mu > 0$, we get the hyperbolic solution:

$$V_{21}(\zeta) = \pm \frac{m}{n\sqrt{\gamma^2-4\mu}}\left(\gamma + 2\mu\left(-\frac{\gamma}{2} + \frac{\sqrt{\gamma^2-4\mu}}{2} \times \frac{C\cosh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}\zeta\right) + D\sinh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}\zeta\right)}{C\sinh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}\zeta\right) + D\cosh\left(\frac{\sqrt{\gamma^2-4\mu}}{2}\zeta\right)}\right)^{-1}\right) \tag{29}$$

where $\zeta = x \pm \sqrt{\frac{\gamma^2-2m^2-4\mu}{\gamma^2-4\mu}}\,t$.

When $\gamma^2 - 4\mu < 0$, we have the trigonometric solution:

$$V_{22}(\zeta) = \pm \frac{m}{n\sqrt{\gamma^2-4\mu}}\left(\gamma + 2\mu\left(-\frac{\gamma}{2} + \frac{\sqrt{4\mu-\gamma^2}}{2} \times \frac{C\cos\left(\frac{\sqrt{4\mu-\gamma^2}}{2}\zeta\right) - D\sin\left(\frac{\sqrt{4\mu-\gamma^2}}{2}\zeta\right)}{C\sin\left(\frac{\sqrt{4\mu-\gamma^2}}{2}\zeta\right) + D\cos\left(\frac{\sqrt{4\mu-\gamma^2}}{2}\zeta\right)}\right)^{-1}\right) \tag{30}$$

where $\xi = x \pm \sqrt{\frac{\gamma^2-2m^2-4\mu}{\gamma^2-4\mu}}\,t$.

IV. Comparison and Validations

Comparison with Liu et al.[19], and Barman et al.[22]

In this subsection, we will discuss the comparison between the solutions obtained for the MCH equation and those presented by Liu et al.[19]. Furthermore, we will also examine the solutions obtained for the LGH equation and compare them with the findings presented by Barman et al.[22]

With (G'/G)-expansion method[19]

To align with the MCH equation, we initially set $w = u$ and $\beta = a$ in equations (7) and similarly in (10)-(19). Our solutions (14)-(16) encompass all the solutions presented in [19]. However, our solutions (17)-(19) introduce new solutions not found in[19].

With extended tanh-function technique[22]

To obtain the same LGH equation, we initially set $w = u$, $m = a$ and $n = b$ in equation (20), as well as in equations

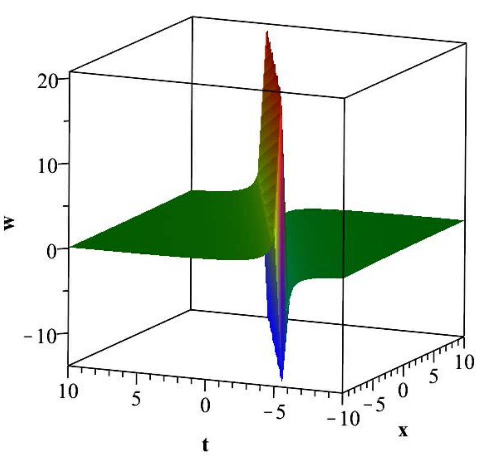

Fig. 3. Singular kink soliton of solution (19) for $C = 0$, $D \neq 0$, $k = 1$, $\beta = -1$, and $\gamma = 2$.

(23)-(30). It becomes evident that the solution presented in (47) of reference [22] corresponds to a specific form of our solution in (27), where $C = 0$, $D \neq 0$, $k = \frac{\sqrt{\gamma^2 - 4\mu}}{2}$ and $\beta = 1$. Notably, our solutions in (28)-(30) are not derived in[22].

Validations

The correctness of the acquired results has been confirmed by substituting them into the original governing models.

V. Results and Discussion

By employing the extended (G'/G)-expansion approach, the examination for the nonlinear (1+1)-dimensional simplified MCH model and LGH equation in the current investigation substantiates several innovative wave solutions while also rediscovering certain outcomes that have been previously reported. The solutions obtained possess free parameters, which could be valuable for revealing intricate physical phenomena or discovering novel phenomena. Graphical depictions of the solutions perform a vital task in depicting the inner workings of complex nonlinear occurrences. Figures 1-5 display visual representations of some of the acquired results for specific estimations of the subjective constants, using Maple, a commercially available software. These figures exhibit various three-dimensional profiles, including periodic, singular periodic kink, anti-kink, singular kink, etc. The solution (14) signifies the kink soliton for $C = 0$, $D \neq 0$,$k = -2.5$, $\beta = 5$, $\gamma = 2$ and $\mu = 0.5$within $-5 \leq x, t \leq 5$, visible in Fig. 1. The result (15) characterizes the periodic soliton for $C = 0$, $D \neq 0$,$k = 0.5$, $\beta = 5$, $\gamma = 4$ and $\mu = 5$ in the range $-10 \leq x, t \leq 10$, shown in Fig. 2. The result (19) yields singular kink for $C = 0$, $D \neq 0$,$k = 1$, $\beta = -1$, $\gamma = 2$ within $-10 \leq x, t \leq 10$ and presented in Fig. 3. The result (27) expresses the anti-kink soliton for $C = 0$, $D \neq 0$, $m = n = 1$, $\gamma = 4$ and $\mu = 3$ within the range $-10 \leq x, t \leq 10$ and displayed in Fig. 4. The result (30) represents singular periodic soliton for $C = 0$, $D \neq 0$, $m = 1$, $n = 1$, $\gamma = 1.6$ and $\mu = 5$ within the boundary $-10 \leq x, t \leq 10$and exposed in Fig. 5. Kink solitons, anti-kink solitons, singular kink solitons, periodic solitons, and singular periodic solitons find versatile applications in wave propagation in shallow waters with a flat seabed and nonlinear media exhibiting dispersion systems and superconductivity. Kink and anti-kink solitons describe localized disturbances and depressions, respectively, in shallow waters, modelling solitary waves and inverse solitary waves. Singular kink solitons capture extreme wave events and sharp gradients in wavefronts. Periodic solitons represent periodic wave patterns, while singular periodic solitons describe irregular periodic patterns, useful for understanding complex oceanic phenomena and flux pinning in superconducting materials. These solitons offer valuable insights into the dynamics of nonlinear systems and wave behaviours in diverse physical contexts.

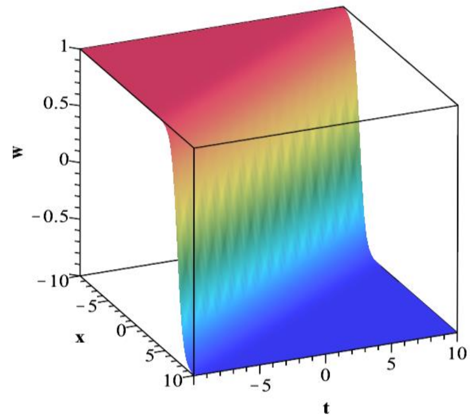

Fig. 4. Anti-kink soliton of solution (27) for $C = 0, D \neq 0,$ $m = n = 1, \gamma = 4$ and $\mu = 3$.

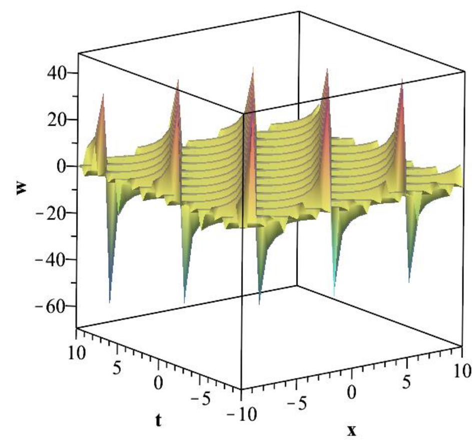

Fig. 5. Singular periodic soliton attained from solution (30) for $C = 0, D \neq 0, m = 1, n = 1, \gamma = 1.6$ and $\mu = 5$.

VI. Conclusion

This article successfully identified the travelling wave solutions of the (1+1)-dimensional Camassa-Holm and Landau-Ginzburg-Higgs equations in hyperbolic, trigonometric, and rational function forms, utilizing the extended (G'/G)-expansion scheme. The results obtained are generally advanced, and the predetermined estimates of the related constraints provide various acknowledged soliton solutions, as well as several other solitons that could be advantageous to investigate in numerous possible uses in the fields of science and engineering. The extended (G'/G)-expansion approach demonstrates its directness, conciseness, and simplicity through software for symbolic calculation, such as Maple or Mathematica, making it a more accessible and effective tool compared to other methods in obtaining analytical wave solutions for numerous nonlinear time dependent problems. To ensure the accuracy of the acquired solutions, the authors utilized Maple to cross-check the results with the original equation, further enhancing confidence in the findings.

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