Dhaka Univ. J. Sci. 73(1): 39-42, 2025 (January)
ISSN 1022 - 2502, E-ISSN 2408-8528
DOI: https://doi.org/ 10.3329/dujs.v73i1.81283

Characterization of Almost Complex Manifolds and Kähler Geometry

Md. Shapan Miah, and Khondokar M. Ahmed*

Department of Mathematics, University of Dhaka, Dhaka-1000, Bangladesh.

(Received: 12 August 2024 ; Accepted: 19 December 2024)

Abstract

The primary objective of this task is to take steps an indicative immersion into the subject of complex geometry by providing several characterizations of Kähler manifolds. We have defined complex, Hermitian, almost complex and Kähler manifolds in this paper and studied some of their features. The main purpose of this article is with a view to understanding the complex, almost complex and Kähler manifolds and their relations with Lie brackets and affine connections. Finally, a theorem which is related to Kähler geometry is established.

Keywords: Geometry; Topology; Manifold; Lie Algebra; Affine Connection; Complex and Kähler Manifolds.

Subject Classification, 2010:32Q15, 53B05, 53C55, 53D05.

I. Introduction

It would not be propagation to say each person is necessarily a bit accommodate with geometry. In modern times, topology and geometry[7] are the sections that mathematics curiosity are possibly be aware of. Most probably geometry and topology are more famous because they are very easy to visualize and understand. The flexibility and generality of topological shape help to realize their concepts easily.

Once in a way, we want more solid forms than the ones narrated by topological spaces[7]. Then we move to smooth and Riemannian manifolds[6], which organize notions such as differentiability, integration and distances[6], where all features draw of real numbers. In this paper we will talk about complex manifolds and observe how their forms narrate to racial landmark of which will lead to Hermitian and Kähler manifolds[2,3,9].

Complex and Kähler manifolds[2,13] have huge applications in quantum mechanics and supersymmetry studies. In this paper we will establish different forms which have relevancy with physics[6]. In Fundamental complex analysis, the partial derivatives must have to satisfy the Cauchy-Riemann equations[6]. So we will say about both differentiability and analyticity of a function.

A complex as well as a Kähler manifold obeys a complex formation where every coordinate neighborhood $\mathbb{C}^m$ and the alteration of coordinate must have to be analytic.

In this article first we have discussed about complex manifolds, their properties, example and proved a theorem which is related to it. Then we deal with Hermitian and almost complex manifolds andtheir features[14].

Finally, we have explained the Kähler manifolds[2], some of its characteristics and proved a theoremwhich is our main interest. Then we draw our conclusion.

*Author for correspondence. e-mail: smidhdu@du.ac.bd, meznan9@du.ac.bd

II. Complex Manifolds

Definition 2.1[11] A complex-valued function $f: \mathbb{C}^m \to \mathbb{C}$ is holomorphic if $f = f_1 + if_2$ maintain the Cauchy-Riemann equations for each $z^\rho = x^\rho + iy^\rho$ as

$$\frac{\partial f_1}{\partial x^\rho} = \frac{\partial f_2}{\partial y^\rho},\ \frac{\partial f_2}{\partial y^\rho} = -\frac{\partial f_2}{\partial y^\rho}$$

A mapping $(f^1, \dots \dots, f^n): \mathbb{C}^m \to \mathbb{C}^n$ is said to be holomorphic if every function f^p , $(1 \le p \le n)$ is holomorphic.

Definition 2.2 A nonempty set M is said to be a complex manifold if (i) M being a topological space, (ii) M being comes from pairs $\{(U_i, \varphi_i)\}$, where $\{U_i\}$ is a cluster of open sets that make M and $\varphi_i: U_i \to U_i' \subseteq \mathbb{C}^m$, (iii) If U_i and U_i so that $U_i \cap U_j \neq \emptyset$, then the transition map

$\psi_{ji} = \varphi_j \varphi_i^{-1}: \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$
is holomorphic. The $dim_{\mathbb{C}}(M) = m$.

Example 2.3 The sphere $S^2$is a complex manifold which can be defined by the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

Differential Forms on Complex Manifolds

Suppose M is a differential complex manifold having dimension m. Also let ω, τ be q-forms on $\Omega_p^q(M)$ at a point p demarcating a complex q-form $\xi = \omega + i\tau$. The vector space of q-form at p is declared as $\Omega_p^q(M)^{\mathbb{C}}$. It is clear that $\Omega_p^q(M) \subset \Omega_p^q(M)^{\mathbb{C}}$ and the adjacent of ξ is $\bar{\xi} = \omega - i\tau$. The q-form will be real if $\xi = \bar{\xi}$.

Let $\omega \in \Omega_p^q(M)^{\mathbb{C}}$ $(q \le 2m)$ and *r, s* are positive integers and *r + s = q*. Also suppose that $V_i \in T_pM^{\mathbb{C}}$ $(1 \le i \le q)$ be belongs to either T_pM^+ or T_pM^-. If $\omega(V_i) = 0$ or else r of the V_i in T_pM^+ and s in T_pM^-, ω is called bidegree *(r, s)*. The pairs *(r, s)*-forms at p is expressed as $\Omega_p^{r,s}(M)$.

Definition 2.4 The outer derivative of an *(r, s)*-form

$\omega = \frac{1}{r!\,s!}\omega_{\mu_1,\dots,\mu_r,\nu_1,\dots,\nu_s} dz^{\mu_1} \wedge \dots \dots dz^{\mu_r} \wedge dz^{\nu_1} \dots \dots dz^{\nu_s}$ for $\varphi(p) = z^\mu$ can be written as

$$d\omega = \frac{1}{r!\,s!}\left(\frac{\partial}{\partial z^\lambda}\omega_{\mu_1,\dots,\mu_r,\nu_1,\dots,\nu_s} dz^\lambda + \frac{\partial}{\partial \bar{z}^\lambda}\omega_{\mu_1,\dots,\mu_r,\nu_1,\dots,\nu_s} d\bar{z}^\lambda\right) \times$$
$$dz^{\mu_1} \wedge \dots \dots dz^{\mu_r} \wedge dz^{\nu_1} \dots \dots dz^{\nu_s},$$

where $d\omega$ is the composition of *(r + 1, s)* and *(r,s + 1)* -forms. By resolving the act of in order to its target we get $d = \partial + \bar{\partial}$, where

$\partial: \Omega_p^{r,s}(M) \to \Omega_p^{r+1,s}(M), \bar{\partial}: \Omega_p^{r,s}(M) \to \Omega_p^{r,s+1}(M)$. The operators ∂ and $\bar{\partial}$ are declared as Dolbeault operators.

Theorem 2.5 If M is a complex manifold, $\omega \in \Omega^q(M)^{\mathbb{C}}$ and $\in \Omega^p(M)^{\mathbb{C}}$, then

(i) $\partial\partial\omega = \left(\partial\bar{\partial} + \bar{\partial}\partial\right)\omega = \bar{\partial}\bar{\partial}\omega = 0$

(ii) $\partial\overline{\omega} = \overline{\bar{\partial}\omega}, \qquad \bar{\partial}\overline{\omega} = \overline{\partial\omega}$

(iii) $\partial(\omega\wedge\xi) = \partial\omega\wedge\xi + (-1)^q\omega\wedge\partial\xi$

and . $\bar{\partial}(\omega\wedge\xi) = \bar{\partial}\omega\wedge\xi + (-1)^q\omega\wedge\bar{\partial}\xi.$

Proof. It is enough to prove that ω is of bidegree *(r, s)*.

(i) Since $d = \partial + \bar{\partial}$, so we have $0 = d^2\omega =$

$$\left(\partial + \bar{\partial}\right)\left(\partial + \bar{\partial}\right)\omega = \partial\partial\omega + \left(\partial\bar{\partial} + \bar{\partial}\partial\right)\omega + \bar{\partial}\bar{\partial}\omega$$

The three parts of right hand side are of bidegree

$(r + 2, s), (r + 1, s + 1)$ and $(r, s + 2)$ respectively and they vanish separately. This proves (i).

(ii) As we know $d\overline{\omega} = \overline{dw}$, so we have

$$\partial\overline{\omega} + \bar{\partial}\overline{\omega} = d\overline{\omega} = \overline{(\partial + \bar{\partial})\omega} + \overline{\partial\omega} + \overline{\bar{\partial}\omega}.$$

Since $\overline{\omega}$, $\overline{\bar{\partial}\omega}$ belong to bidegree $(s + 1, r)$ and $\bar{\partial}\overline{\omega}$, $\overline{\partial\omega}$ are of $(s, r + 1)$, so we can write that $\partial\overline{\omega} = \overline{\bar{\partial}\omega}$ and $\bar{\partial}\overline{\omega} = \overline{\partial\omega}$.

(iii) We can imagine ω is of bidegree (r, s) and ξ of (r', s'). So $d(\omega\wedge\xi) = d\omega\wedge\xi + (-1)^q\omega\wedge d\xi$

$$= (\partial + \bar{\partial})\omega\wedge\xi + (-1)^q\omega\wedge(\partial + \bar{\partial})\xi$$
$$= \partial\omega\wedge\xi + (-1)^q\omega\wedge\partial\xi + \bar{\partial}\omega\wedge\xi + (-1)^q\omega\wedge\bar{\partial}\xi$$
$$= \partial(\omega\wedge\xi) + \bar{\partial}(\omega\wedge\xi).$$

Where $\partial(\omega\wedge\xi)$ and $\bar{\partial}(\omega\wedge\xi)$ are bidegree of$(r + r' + 1, s + s')$ and $(r + r', s + s' + 1)$ respectively.

III. Almost Complex Manifolds

Definition 3.1[11,15] Let us suppose that M is a differentiable manifold. Also let J is a tensor field of order (1,1) for which at every point p of M we can write $J_p^2 = -1_p$. Then the pair (M, J) is defined as almost complex manifold and J is called the almost complex structure.

If (M, J) is a complex manifold, then the Nijenhuis tensor field $N: \mathcal{X}(M) \otimes \mathcal{X}(M) \to \mathcal{X}(M)$ can be defined as

$$N(X, Y) \equiv [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY].$$

The structure J is integrable if the Lie bracket of any holomorphic vector fields $X, Y \in \mathcal{X}(M)$ is also a holomorphic vector field, $[X, Y] \in \mathcal{X}^+(M)$. If M is an almost complex structure, it must be even in dimension. For this let the dimension of M is n, and also let $J: TM \to TM$ be an almost structure with complex manifold. If J^2*=-1 then* $(\det J)^2 = (-1)^n$. But when M is a real manifold. then $\det J$ is real number, thus n obviously be even if M has an almost complex structure. One can show that it must be orientable as well.

Theorem 3.2 For any $A, B \in \mathcal{X}(M)$, $N(A, B) = 0$ if and only if the structure J on a manifold M is integrable.

Proof. Let us suppose that $Z = X + iY$ and $W = U + iV$ are two elements of $\mathcal{X}^{\mathbb{C}}(M)$. We elaborate the Nijenhuis tensor field for which its operations on vector fields in $\mathcal{X}^{\mathbb{C}}(M)$ can be written as

$$N(Z, W) = [Z, W] + J[JZ, W] + J[Z, JW] - [JZ, JW]$$
$$= \{N(X, U) - N(Y, V)\} + i\{N(X, V) + N(Y, U)\} \quad \text{..........}(\mathbf{1.0})$$

Now consider that $N(A, B) = 0$ for any $A, B \in \mathcal{X}(M)$. From (**1.0**) it becomes $N(Z, W) = 0$ for all $Z, W \in \mathcal{X}^{\mathbb{C}}(M)$.

Suppose that $Z, W \in \mathcal{X}^+(M) \subset \mathcal{X}^{\mathbb{C}}(M)$. Because of $JZ = iZ$ and $JW = iW$, we may write

After the assumption, $N(Z, W) = 0$, we get that

$[Z, W] = -iJ[Z, W]$ or $J[Z, W] = i[Z, W]$, which means that $[Z, W] \in \mathcal{X}^+(M)$. So we can conclude that the structure J is integrable.

Conversely, let us suppose is integrable. Since

$\mathcal{X}^{\mathbb{C}}(M) = \mathcal{X}^+(M) \oplus \mathcal{X}^-(M)$, we can write separately $Z, W \in \mathcal{X}^{\mathbb{C}}(M)$ as$Z = Z^+ + Z^-$ and $W = W^+ + W^-$. Then we can write,

$$N(Z, W) = N(N^+, W^+) + N(Z^+, W^-) + N(Z^-, W^+) + N(Z^-, W^-).$$

$$N(Z,W)=N(N^+,W^+)+N(Z^+,W^-)+N(Z^-,W^+)+N(Z^-,W^-).$$

Because of $JZ^{\pm}=\pm iZ^{\pm}$ and $JW^{\pm}=\pm iW^{\pm}$ it can be written that $N(Z^-,W^+)=N(Z^+,W^-)=0$. We may also write,

$$N(Z^+,W^+)=[Z^+,W^+]+J[iZ^+,W^+]+J[Z^+,iW^+]-[iZ^+,iW^+]$$

$=2[Z^+,W^+]-2[Z^+,W^+]=0.$

Since $J[Z^+,W^+]=i[Z^+,W^+]$.

Likely $N(Z^-,W^-)=0$ which proves that $N(Z,W)=0$ for any $Z,W\in\mathcal{X}^{\mathbb{C}}(M)$. Specifically, $N(Z,W)=0$ for any $Z,W\in\mathcal{X}(M)$.

IV. Hermitian Manifolds

Let us suppose that M be a complex manifold of dimension m and J is a almost complex structure. Then the triple (M,J,g) with the Riemannian metric g is called a Hermitian manifold.

V. Kähler Manifolds

Consider that (M,J,g) is a Hermitian manifold. Also demark a tensor field Ω which has operation T_pM on as

$\Omega_p(X,Y)=g_p(J_pX,Y)$ for any $X,Y\in T_pM$, where Ω is antisymmetrical that is $\Omega(X,Y)=g(JX,Y)=g(J^2X,JY)$

$=-g(JY,X)=-\Omega(X,Y).$

So we can define Ω as a two form which is also called the Kähler form.

Definition 5.1 A Hermitian manifold (M,J,g) is called a Kähler manifold if its Kähler form Ω is closed, that is $d\Omega$ is zero. The metric g is called the Kähler metric on M.

Example 5.2 Suppose $M=\mathbb{C}^m=\{(z^1,\dots\dots,z^m)\}$, where $\mathbb{C}^m$ is marked with $\mathbb{R}^{2m}$ by the identification $z^\mu\to x^\mu+iy^\mu$. Let δ be the Euclidean metric of $\mathbb{R}^{2m}$, so we can write that

$$\delta\left(\frac{\partial}{\partial x^\mu},\frac{\partial}{\partial x^\vartheta}\right)=\delta\left(\frac{\partial}{\partial y^\mu},\frac{\partial}{\partial y^\vartheta}\right)=\delta_{\mu\vartheta}\text{ and }\delta\left(\frac{\partial}{\partial x^\mu},\frac{\partial}{\partial y^\vartheta}\right)=0.$$

Because of $J\frac{\partial}{\partial x^\mu}=\frac{\partial}{\partial y^\mu}$ and $J\frac{\partial}{\partial y^\mu}=-\frac{\partial}{\partial x^\mu}$ we can say that δ is a Hermitian metric. In complex field, we have

$$\delta\left(\frac{\partial}{\partial z^\mu},\frac{\partial}{\partial z^\vartheta}\right)=\delta\left(\frac{\partial}{\partial \bar{z}^\mu},\frac{\partial}{\partial \bar{z}^\vartheta}\right)=0\text{ and}$$

$$\delta\left(\frac{\partial}{\partial z^\mu},\frac{\partial}{\partial \bar{z}^\vartheta}\right)=\delta\left(\frac{\partial}{\partial \bar{z}^\mu},\frac{\partial}{\partial z^\vartheta}\right)=\frac{1}{2}\delta_{\mu\vartheta}$$

The Kähler form can be written as

$\Omega=\frac{i}{2}\sum_{\mu=1}^m dz^\mu\wedge d\bar{z}^\mu=\sum_{\mu=1}^m dx^\mu\wedge dy^\mu$, which ensures us that the above form is closed. So we get that δ is an Euclidean metric of $\mathbb{R}^{2m}$ is Kähler metric on $\mathbb{C}^m$. Since all the properties of being Kähler manifold are satisfied, hence $\mathbb{C}^m$is a Kähler manifold.

Theorem 5.3 A Hermitian manifold (M,J,G) is a Kahler manifold if and only if the structure satisfies

$$\nabla_\mu J=0$$

where ∇_μ is Levi-Civita affection related to g.

Proof. If ω is any r-form, then its differential form $d\omega$ can be expressed as

$$d\omega=\nabla\omega\equiv\frac{1}{r!}\nabla_\mu\omega_{v_1,\dots\dots,v_r}dx^\mu\wedge dx^{v_1}\wedge\dots\dots\wedge dx^{v_r}$$

We have

$$\nabla\Omega=\frac{1}{2}\nabla_\mu\Omega_{\mu v}dx^\lambda\wedge dx^\mu\wedge dx^v$$

$$=\frac{1}{2}\left(\partial_\lambda\Omega_{\mu v}-\Gamma^k_{\lambda\mu}\Omega_{kv}-\Gamma^k_{\lambda v}\Omega_{\mu k}\right)dx^\lambda\wedge dx^\mu\wedge dx^v$$

Since Γ is symmetrical, so we get

$$=\frac{1}{2}\partial_\lambda\Omega_{\mu v}dx^\lambda\wedge dx^\mu\wedge dx^v=d\Omega$$

Now,

$$(\nabla_Z\Omega)(X,Y)=\nabla_Z[\Omega(X,Y)]-\Omega(\nabla_ZX,Y)-\Omega(X,\nabla_ZY)$$

$$=\nabla_Z[g(JX,Y)]-g(J\nabla_ZX,Y)-g(JX,\nabla_ZY)$$

$$=(\nabla_Zg)[(X,Y]-\Omega(\nabla_ZJX,Y)-\Omega(J\nabla_ZX,Y$$

$$=g(\nabla_ZJX-J\nabla_ZX,Y)$$

$=g((\nabla_ZJ)X,Y)$, where $\nabla_Zg=0$ is used.

This is obvious for any X,Y,Z. So it follows that $\nabla_Z\Omega=0$ if and only if $\nabla_Z\mathrm{J}=0$. Which completes the prove.

VI. Conclusion

Complex geometry is difficult but fecund thing. It amazed us both in its at the first sights limitless altitude. It is very amazing that the toughest classical mechanics can be easily described with the help of geometry of manifolds and which are very close to complex geometry through Kähler manifolds. Complex and Kähler manifolds have great applications in theories of physics[12,13]. Our lesion on complex geometry seemed that it is an extension of differential geometry.

We are further interested to continue this study especially on the Kähler manifolds and will try to extend our research to Calabi-Yau manifolds and Hyper Kähler manifolds.

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