Dhaka Univ. J. Sci. 60(1): 61-64, 2012 (January)

Linear Equation Method of Constructing Confounding Plan in p^n - Factorial Experiment

M A Jalil

Department of Statistics, Biostatistics and Informatics, University of Dhaka, Dhaka-1000, Bangladesh

Received on 24. 03. 2011. Accepted for Publication on 28. 09. 2011

Abstract

This article describes a linear equation method of designing simultaneous confounding of one or more factorial effects in p^n - factorial experiment. It is demonstrated that it becomes easier to construct the design of simultaneous confounding of one or more factors in a p^n - factorial experiment especially when the number of factors as well as the number of levels becomes larger. It is also discussed that such approach is very much convenient from the viewpoint of computation.

I. Introduction

The practical works with factorial experiment become troublesome especially when the number of factors as well as the number of levels of each of the factors is large. This becomes more difficult if we have no required number of homogeneous plots in practice. In such situation, we are bound to use a limited number of homogeneous plots to analyze the factorial effects. As a result, some factorial effects or interactions will be mixed up with block effect, i.e. confounded. Since there is no way to avoid this, the higher order interaction effects are considered to be confounded. We usually consider one or two higher order interaction effects to be confounded to perform analysis efficiently.

Bose and Kishan (1940), Bose (1947) described the construction of p^n factorial designs using finite geometries. The treatments are represented by n-tuples $(a_1,\cdots\cdots,a_n)$ where a_i are elements of $GF(p)$. The method is available only when p is prime or prime power. A system of simultaneous confounding in 2^n factorial experiment has been described, where an intrablock subgroup is constructed with the common elements taken from the factorial effects of two incomplete blocks, each confounded with a single factorial effect (Kempthorne, 1947, 1952)[1,2]. Das (1964) described an equivalent method of Bose in which some of the treatment factors are designated as basic factors and the others as added factors. Levels of added factors are derived by combination of the levels of the basic factors over $GF(p)$. White and Hultquist (1965) extended the field method to designs with numbers of levels of treatment factors. John and Dean (1975) described the construction of a particular class of single replicate block designs, which they call generalized cyclic designs. The essential feature of the method is that the n-tuples giving the treatments of a particular block constitute an Abelian group, the intrablock subgroup. Patterson (1976) described a general computer algorithm, called DSIGN, in which levels of treatment factors are derived by linear combinations of levels of plot and block factors. The method provides finite-field, generalized cyclic and other designs. Mallick, S. A. (1973 & 1975)[3, 4] developed two systems of designing factorial effects with simultaneous confounding of two effects, one for a 3^n and the other for a 4^n - factorial experiments. In all these systems of simultaneous confounding, the design of factorial effects was based on some manipulating manner, selection by inspection the common elements (factorial effects). Jalil, M. A. et. al (1990) developed matrix method[5] of designing a single factorial effect confounded in a P^n - factorial experiment, where the level combinations are obtained by matrix operations of the levels. These methods are limited to some particular value of p in a p^n factorial experiment. The present work is a linear equation method of designing a p^n- factorial experiment of simultaneous confounding plan, where we can confound $k\,(k \ge 1)$ factorial effects.

II. The Proposed Method

In this section we highlight the development of the proposed method. Let in a p^n- factorial experiment, k - factorial effects $(k = 1, 2, 3, \cdots,)$ are to be confounded. First, we are to find the key intrablock subgroup with appropriate level combinations to confound k factorial effects. The composition of the key intrablock subgroup of level combinations confounded with k factorial effects is to be determined by the steps described below.

Step 1

Let $L = (L_1, L_2, \cdots, L_k)$ represents k linearly independent equations corresponding k - factorial effects which are to be confounded. First, we are to obtain $(n-k)$ linearly independent solutions with the level combinations satisfying the equations, $L_k = 0;\ \text{mod}\ p\ ;\ k = 1, 2, \cdots$. Let these $(n-k)$ independent solutions be denoted by $(y_1,\ y_2, \cdots,\ y_{n-k})$.

Step 2

Find $y = \sum_{i=1}^{n-k} \lambda_i y_i$, where $\lambda_i = 0, 1, \cdots, (p-1)$; are the level combinations forming the key intra-block subgroup.

Step 3

After having the 1st key intrablock subgroup (B_1), we can obtain the other key intrablock subgroups of the underlying confounded factorial effect(s) using simple operations of matrix algebra described in the following section.

To obtain the second key intrablock subgroup (B_2), we will add the vector $(0 \cdots 0\ 1\ 0)$ to each of the vector elements (row vector) of the 1st key intrablock subgroup (B_1). Similarly, for the 3rd key intrablock subgroup (B_3) we are to add the vector $(0 \cdots 0\ 2\ 0)$ with the vector elements of B_1; and so on. These key intrablock subgroups are arranged in column form; the intrablock subgroups in the 1st column are the key intrablock subgroups.

Step 4

To obtain the intrablock subgroups in the second column, we will add the vector $(0 \cdots 0\ 1)$ to each of the vector elements of the key intrablock subgroups in the 1st column. Similarly, the intrablock subgroups in the next (third) column are obtained by adding the vector $(0 \cdots 0\ 2)$ to each of the vector elements of the key intrablock subgroups; and so on.

The method is illustrated with examples explained below.

Example 1. Design a plan in a 3^3- factorial experiment where the effect ABC is confounded.

[For a single confounding there will be $\frac{p^n}{p^{n-1}} = p = 3$ intrablock subgroups (incomplete blocks) each with $\frac{p^n}{p} = \frac{3^3}{3} = 3^2 = 9$ level combinations.]

Solution. Here, $p = 3\ (0,1,2)$, and the number of factors, $n = 3$; Let these factors be denoted by A, B *and* C. The number of factors to be confounded is one *i.e.* $k = 1$.

Therefore, we have $n - k = 3 - 1 = 2$ **independent non-null solution vectors** with the level combinations satisfying the symbolic equation, $L_1 = x_1 + x_2 + x_3 = 0;\ \text{mod } 3$ for ABC to be confounded.

By a simple trial we can get two such solution vectors with level combinations as given by, $y_1 = (1\ 1\ 1)$ *and* $y_2 = (1\ 0\ 2)$. Therefore, the elements of the key intrablock subgroups that consists of the treatment combinations (level combinations), are given by the linear equation,

$y = \lambda_1 y_1 + \lambda_2 y_2;\ \lambda_1, \lambda_2 = 0, 1, 2$.

The level combinations of the key intrablock subgroup are obtained by solving the equation

$y = \lambda_1 (1\ 1\ 1) + \lambda_2 (1\ 0\ 2);\ \lambda_1, \lambda_2 = 0, 1, 2$.

We can obtain the solution as follows.

$$\begin{array}{lccc}
\lambda_1 = 0, \lambda_2 = 0; & 0 & 0 & 0 \\
\lambda_1 = 0, \lambda_2 = 1; & 1 & 0 & 2 \\
\lambda_1 = 0, \lambda_2 = 2; & 2 & 0 & 1 \\
\\
\lambda_1 = 1, \lambda_2 = 0; & 1 & 1 & 1 \\
\lambda_1 = 1, \lambda_2 = 1; & 2 & 1 & 0 \\
\lambda_1 = 1, \lambda_2 = 2; & 0 & 1 & 2 \\
\\
\lambda_1 = 2, \lambda_2 = 0; & 2 & 2 & 2 \\
\lambda_1 = 2, \lambda_2 = 1; & 0 & 2 & 1 \\
\lambda_1 = 2, \lambda_2 = 2; & 1 & 2 & 0
\end{array}$$

Therefore, the key intrablock is found as,

$$B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

To obtain the second (B_2) and third (B_3) intrablock subgroups, we will add the vectors $(0\ 0\ 1)$ and $(0\ 0\ 2)$ with each of the elements (row vector) of the key intrablock subgroups B_1 respectively. Thus, the confounding plan of a 3^3 factorial experiment confounded with the factorial effect ABC is given by,

$$B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix} \text{ and } B_3 = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \\ 2 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 2 \end{bmatrix}.$$

The final plan is shown as follows.

$$M = \begin{bmatrix} 0 & 0 & 0 & & 0 & 0 & 0 & & 0 & 0 & 2 \\ 1 & 0 & 3 & & 1 & 0 & 0 & & 1 & 0 & 1 \\ 2 & 0 & 1 & & 2 & 0 & 2 & & 2 & 0 & 0 \\ 1 & 1 & 1 & & 1 & 1 & 2 & & 1 & 1 & 0 \\ 2 & 1 & 0 & & 2 & 1 & 1 & & 2 & 1 & 2 \\ 0 & 1 & 2 & & 0 & 1 & 0 & & 0 & 1 & 1 \\ 2 & 2 & 2 & & 2 & 2 & 0 & & 2 & 2 & 1 \\ 0 & 2 & 1 & & 0 & 2 & 2 & & 0 & 2 & 0 \\ 1 & 2 & 0 & & 1 & 2 & 1 & & 1 & 2 & 2 \end{bmatrix}$$

We cite another example for clearing the ideas.

Example 2. Design a plan in a 3^3 - factorial experiment where the effects ABC and ABC^2 are confounded simultaneously.

Solution. [In a confounding of two factorial effects in a 3^3 factorial experiment, there will be $\frac{p^n}{p^{n-2}} = \frac{3^3}{3^{3-2}} = 9$ intrablock subgroups each with $\frac{p^n}{p^2} = \frac{3^3}{3^2} = 3$ level combinations.]

Here, the generalized interactions are, $ABC \times ABC^2 = AB$ and $ABC \times (ABC^2)^2 = C$.

The symbolic equations representing the factorial effects ABC and ABC^2, which are to be confounded are given by, $L_1 = x_1 + x_2 + x_3 = 0 \text{ (mod 3)}$ *and* $L_2 = x_1 + x_2 + 2x_3 = 0 \text{ (mod 3)}$.

Here, the number of factors is $n = 3$ and the number of factorial effects to be confounded is $k = 2$. Therefore, $n - k = 3 - 2 = 1$ independent solution vector with a combination of the levels, which satisfy both the equations are $L_1 = 0$ and $L_2 = 0$ (mod 3). Let such a solution vector is obtained by a simple algebra, given by, $y_1 = (1\ 2\ 0)$. Hence the elements of the intrablock subgroup are, $y = \lambda_1 y_1 = \lambda_1 (1\ 2\ 0),\ \lambda_1 = 0, 1, 2$; the key intra-block is given

by, $B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$.

To obtain the second and third intrablock subgroups B_2 and B_3, we add the row vectors $(0\,1\,0)$ and $(0\ 2\ 0)$ to each of the vector elements of the key intrablock subgroup B_1 respectively. Thus, we get,

$$B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 2 & 0 \end{bmatrix} \text{ and } B_3 = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

We can find the fourth block B_4 by adding the vector element $(0\,0\,1)$ with each of the vector elements of the key intrablock B_1. Sinilarly, the intrablock B_5 can be found by adding the vector element $(0\,0\,1)$ with each of the vector elements of the second block B_2. This way, we can obtain the following results.

B_4 is obtained by adding the vector $(0\,0\,1)$ to each of the vector elements of B_1;

B_5 is obtained by adding the vector $(0\,0\,1)$ to each of the vector elements of B_2;

B_6 is obtained by adding the vector $(0\,0\,1)$ to each of the vector elements of B_3; and

B_7 is obtained by adding the vector $(0\ 0\ 2)$ to each of the vector elements of B_1;

B_8 is obtained by adding the vector $(0\ 0\ 2)$ to each of the vector elements of B_2;

B_9 is obtained by adding the vector $(0\ 0\ 2)$ to each of the vector elements of B_3;

Thus, we have the complete layout of nine intrablock subgroups as shown below.

$$\begin{bmatrix} & 0 & 0 & 0 & & 0 & 0 & 1 & & 0 & 0 & 2 \\ B_1 & 1 & 2 & 0 & B_4 & 1 & 2 & 1 & B_7 & 1 & 2 & 2 \\ & 2 & 1 & 0 & & 2 & 1 & 1 & & 2 & 1 & 2 \\ & & & & & & & & & & & \\ & 0 & 1 & 0 & & 0 & 1 & 1 & & 0 & 1 & 2 \\ B_2 & 1 & 0 & 0 & B_5 & 1 & 0 & 1 & B_8 & 1 & 0 & 2 \\ & 2 & 2 & 0 & & 2 & 2 & 1 & & 2 & 2 & 2 \\ & & & & & & & & & & & \\ & 0 & 2 & 0 & & 0 & 2 & 1 & & 0 & 2 & 2 \\ B_3 & 1 & 1 & 0 & B_6 & 1 & 1 & 1 & B_9 & 1 & 1 & 2 \\ & 2 & 0 & 0 & & 2 & 0 & 1 & & 2 & 0 & 2 \end{bmatrix}$$

It is easy to verify that, considering the blocks in columns we have,

$(B_1 + B_2 + B_3) Vs.\ (B_4 + B_5 + B_6)\ Vs.\ (B_7 + B_8 + B_9)$;

confounds a generalized effect, $ABC \times (ABC^2)^2 = C$;

Considering the blocks in rows we have,

$(B_1 + B_4 + B_7) Vs.\ (B_2 + B_5 + B_8)\ Vs.\ (B_3 + B_6 + B_9)$;

confounds a generalized effect, $ABC \times ABC^2 = AB$;

Considering the blocks in I-totals we have,

$(B_1 + B_5 + B_9) Vs. (B_2 + B_6 + B_7)\ Vs. (B_3 + B_4 + B_8)$;

confounds, the effect ABC^2; and
Considering the blocks in J-totals we have,

$(B_1 + B_6 + B_8) Vs. (B_2 + B_4 + B_9)\ Vs. (B_3 + B_5 + B_7)$; confounds the effect ABC.

III. Conclusion

In this article, we have introduced a general method of simultaneous confounding in p^n factorial experiment solving linear equation and operations of matrix algebra. It becomes easier and rewarding than any methods available in the construction of simultaneous confounding in p^n factorial experiments. It is quite straightforward to develop algorithm for such a method. The method is appropriate in general for any value of n, the number of factors and for any possible value of p, the levels of the factors. The method is restricted to p^n symmetrical factorial experiment.

....

1. Bose, R. C. and K. Kishan, (1940); On the problem of confounding in general symmetrical design. Sankhya 5, 21-36
2. Bose, R. C. (1947); Mathematical theory of the symmetrical factorial design. Sankhya, 8, 107-166
3. Das, M. N. (1964); A somewhat alternative approach for construction of symmetrical factorial designs and obtaining maximum numbers of factors. Calcutta Statististical Association Bulletin 13, 1-17
4. Dean, A. M. and J. A. John, (1975); Single replicate factorial experiments in generalized cyclic designs:I. Symmetrical arrangements. JRSS, B 37, 63-71
5. Jalil, M. A; S. A Mallick, M. S. Rahman, (July, 1990); Matrix method of Designing Layout in p^n - Factorial Experiments. Dhaka University Studies, B, **38 (2)**, 123-128.
6. Jalil, M A and S. A. Mallick, (2010); Matrix Method of Designing Simultaneous Confounding two factorial effects in 3^n - Factorial Experiment. The Dhaka University Journal of Science (accepted).
7. Jalil, M A and S. A. Mallick, (2010); Matrix Method of Designing Simultaneous Confounding in a 5^n - Factorial Experiment. The Dhaka University Journal of Science (submitted).
8. Kempthorne, O. (1952); The design and Analysis of Experiments. John Wiley & Sons, Inc., New York
9. Kempthorne, O. (1947); A Simple Approach to Confounding and Fractional Replication in factorial experiments. Biometrika, 34, 255-272
10. Mallick, S. A. (1973); A system of Simultaneous Confounding of Two Effects in a 3^n - Factorial Design. Dacca University Studies, Pt. B. **XXI**, 33-36
11. Mallick, S. A. (1975); A system of Simultaneous Confounding of Two Effects in a 4^n - Factorial Design. Dacca University Studies, Pt. B. **XXIII (1)**, 7-10
12. Patterson, H. D. (1976); Generation of factorial designs. JRSS B **37**, 175-179