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Exterior Algebra with Differential Forms on Manifolds

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Abstract

The concept of an exterior algebra was originally introduced by H. Grassman for the purpose of studying linear spaces. Subsequently Elie Cartan developed the theory of exterior differentiation and successfully applied it to the study of differential geometry [8], [9] or differential equations. More recently, exterior algebra has become powerful and irreplaceable tools in the study of differential manifolds with differential forms and we develop theorems on exterior algebra with examples.

I. Introduction

Due to Elie Cartan's systematic development of the method of exterior differentiation, the alternating tensors have played an important role in the study of manifolds [2]. An alternating contravariant tensor of order r is also called an exterior vector of degree r or an exterior r-vector. The space $\Lambda^r(V)$ is called the exterior space of V of degree r. For convenience, we have the following conventions: $\Lambda^1(V) = V$, $\Lambda^0(V) = \mathbb{F}$. More importantly, there exists an operation, the exterior (wedge) product, for exterior vectors such that the product of two exterior vectors is another exterior vector. Differential forms are an important component of the apparatus of differential geometry [10], they are also systematically employed in topology, in the theory of differential equations, in mechanics, in the theory of complex manifolds and in the theory of functions of several complex variables. Currents are a generalization of differential forms, similar to generalized functions. The algebraic analogue of the theory of differential forms makes it possible to define differential forms on algebraic varieties and analytic spaces. Differential forms arise in some important physical contexts. For example, in Maxwell's theory of electromagnetism, the Faraday 2 form, or electromagnetic field strength, is $F = \frac{1}{2} f_{ab} dx^a \wedge dx^b$, where f_{ab} are formed from the electromagnetic fields. In this paper we have studied theorems on exterior algebra with differential forms.

II. Exterior Algebra

Definition 1. Suppose ξ is an exterior k-vector and η an exterior l-vector. Let

$$\xi \Lambda \eta = A_{k+1}(\xi \otimes \eta)$$

where A_{k+l} is the alternating mapping that defined in [4]. Then $\xi \Lambda \eta$ is an exterior $(k+l)$-vector, called the exterior (wedge) product of ξ and η.

Theorem 2.1. [4] The exterior product satisfies the following rules. Suppose $\xi, \xi_1, \xi_2 \in \Lambda^k(V)$, $\eta, \eta_1, \eta_2 \in \Lambda^l(V)$, $\zeta \in \Lambda^h(V)$. Then we have

(i) Distributive Law

$$(\xi_1 + \xi_2) \Lambda \eta = \xi_1 \Lambda \eta + \xi_2 \Lambda \eta$$
$$\xi \Lambda (\eta_1 + \eta_2) = \xi \Lambda \eta_1 + \xi \Lambda \eta_2$$

(ii) Anti-commutative Law

$$\xi \Lambda \eta = (1)^{kl} \eta \Lambda \xi$$

(iii) Associative Law

$$(\xi \Lambda \eta) \Lambda \zeta = \xi \Lambda (\eta \Lambda \zeta).$$

Remark 1. Suppose $\xi, \eta \in V = \Lambda^1(V)$. Then the anti-commutative law implies

$$\xi \Lambda \eta = -\eta \Lambda \xi, \quad \xi \Lambda \xi = \eta \Lambda \eta = 0.$$

Generally, if there are repeated exterior 1-vectors in a polynomial wedge product, then the product is zero.

Definition 2. Denote the formal sum $\sum_{r=0}^{n} \Lambda^r(V)$ by $\Lambda(V)$. Then $\Lambda(V)$ is a 2^n- dimensional vector space. Let

$$\xi = \sum_{r=0}^{n} \xi^r, \quad \eta = \sum_{s=0}^{n} \eta^s$$

where $\xi^r \in \Lambda^r(V)$, $\eta^s \in \Lambda^s(V)$. Define the exterior (wedge) product of ξ and η by

$$\xi \Lambda \eta = \sum_{r,s=0}^{n} \xi^r \Lambda \eta^s.$$

Then $\Lambda(V)$ becomes an algebra with respect to the exterior product and is called the exterior algebra or Grassman algebra of V.

The set $\{1, e_i (1 \le i \le n),\ e_{i_1} \Lambda e_{i_2} (1 \le i_1 \le i_2 \le n), \dots, e_1 \Lambda \dots \Lambda e_n\}$ is a basis of the vector space $\Lambda(V)$. Similarly, we have an exterior algebra for the dual space V^*,

$$\Lambda(V^*) = \sum_{0 \le r \le n} \Lambda^r(V^*).$$

An element of $\Lambda^r(V^*)$ is called an exterior form of degree r or exterior r-form on V; it is an alternating $\mathbb{F}$-valued r-linear function on V.

The vector spaces $\Lambda^r(V)$ and $\Lambda^r(V^*)$ are dual to each other by a certain pairing. Suppose $v_1\Lambda\ldots\Lambda v_r \in \Lambda^r(V)$ and $v^{*1}\Lambda\ldots\Lambda v^{*r} \in \Lambda^r(V^*)$. Then

$$< v_1\Lambda\ldots\Lambda v_r, v^{*1}\Lambda\ldots\Lambda v^{*r} >= \det(< v_\alpha, v^\beta >)$$

Thus $\{e_{i_1}\Lambda\ldots\Lambda e_{i_r}, 1 \le i_1 < \cdots < i_r \le n\}$ and $\{e^{*j_1}\Lambda\ldots\Lambda e^{*j_r}, 1 \le i_1 < \cdots < i_r \le n\}$, the basis of $\Lambda^r(V)$ and $\Lambda^r(V^*)$ respectively satisfy the following relationship:

$$< e_{i_1}\Lambda\ldots\Lambda e_{i_r}, e^{*j_1}\Lambda\ldots\Lambda e^{*j_r} >= \det(< e_{i_\alpha}, e^{*j_\beta} >)$$

$$= \delta^{j_1\ldots j_r}_{i_1\ldots i_r} = \begin{cases} 1, & if\ \{j_1\ldots j_r\} = \{i_1\ldots i_r\} \\ 0, & if\ \{j_1\ldots j_r\} \ne \{i_1\ldots i_r\} \end{cases}$$

Thus these two bases are dual to each other.

Theorem 2.2. Suppose $f: V \to W$ is a linear map. Then f^* commutes with the exterior product, that is, for any $\varphi \in \Lambda^r(W^*)$ and $\psi \in \Lambda^s(W^*)$,

$$f^*(\varphi\,\Lambda\,\psi) = f^*\varphi\,\Lambda\,f^*\psi.$$

Proof. Choose any $v_1, \ldots, v_{r+s} \in V$. Then

$$f^*(\varphi\,\Lambda\,\psi)(v_1, \ldots, v_{r+s}) = \varphi\,\Lambda\,\psi\,(f(v_1), \ldots, f(v_{r+s}))$$

$$= \frac{1}{(r+s)!} \sum_{\sigma \in S(r+s)} sgn\sigma\,.\,\varphi\left(f\left(v_{\sigma(1)}\right), \ldots, f\left(v_{\sigma(r)}\right)\right).$$

$$\psi(f(v_{\sigma(r+1)}), \ldots, f(v_{\sigma(r+s)}))$$

$$= \frac{1}{(r+s)!} \sum_{\sigma \in S(r+s)} sgn\sigma\,.\,f^*\varphi(v_{\sigma(1)}, \ldots, v_{\sigma(r)}).$$

$$f^*\psi(v_{\sigma(r+1)}, \ldots, v_{\sigma(r+s)}) = f^*\varphi\,\Lambda\,f^*\psi(v_1, \ldots, v_{r+s}).$$

Therefore $f^*(\varphi\,\Lambda\,\psi) = f^*\varphi\,\Lambda\,f^*\psi$. This completes the proof of the theorem. □

Theorem 2.3. A necessary and sufficient condition for the vectors $v_1, \ldots, v_r \in V$ to be linearly dependent is $v_1\Lambda\ldots\Lambda\,v_r = 0$.

Proof. If $v_1, \ldots, v_r$ are linearly dependent, then we may assume without loss of generality that v_r can be expressed as a linear combination of $v_1, \ldots, v_{r-1}$:

$$v_r = a_1v_1 + \cdots + a_{r-1}v_{r-1}.$$

Then $v_1\Lambda\ldots\Lambda\,v_{r-1}\,\Lambda\,v_r = v_1\Lambda\ldots\Lambda\,v_{r-1}\,\Lambda\,(a_1v_1 + \cdots + a_{r-1}v_{r-1}) = 0$.

Conversely, if $v_1, \ldots, v_r$ are linearly independent, then they can be extended to a basis $\{v_1 \ldots v_r, v_{r+1}, \ldots, v_n\}$ of V. Then

$$v_1\Lambda\ldots\Lambda\,v_r\,\Lambda\,v_{r+1}\,\Lambda\ldots\Lambda\,v_n \ne 0.$$

Therefore $v_1\Lambda\ldots\Lambda\,v_r \ne 0$. This completes the proof of the theorem. □

Theorem 2.4 (Cartan's Lemma) [4]. Suppose $\{v_1, \ldots, v_r\}$ and $\{w_1, \ldots, w_r\}$ are two sets of vectors in V such that

$$\sum_{\alpha=1}^{r} v_\alpha\,\Lambda\,w_\alpha = 0. \qquad (1)$$

If $v_1, \ldots, v_r$ are linearly independent, then w_α can be expressed as linear combination of v_β:

$$w_\alpha = \sum_{\beta=1}^{r} a_{\alpha\beta}\,v_\beta \quad ; 1 \le \alpha \le r \text{ with } \quad a_{\alpha\beta} = a_{\beta\alpha}.$$

Theorem 2.5. Suppose $v_1, \ldots, v_r$ are r linearly independent vectors in V and $w \in \Lambda^p(V)$. A necessary and sufficient condition for w to be expressible in the form

$$w = v_1\,\Lambda\,\psi_1 + \cdots + v_r\,\Lambda\,\psi_r, \qquad (2)$$

where $\psi_1, \ldots, \psi_r \in \Lambda^{p-1}(V)$, is that

$$v_1\,\Lambda\ldots\Lambda\,v_r\,\Lambda\,w = 0. \qquad (3)$$

Proof. When $p + r > n$ (2) and (3) are trivially true. In the following we assume that $p + r \le n$. Necessity is obvious, so we need only show sufficiency. Extend $v_1, \ldots, v_r$ to a basis $\{v_1, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ of V. Then w can be expressed as

$$w = v_1\,\Lambda\,\psi_1 + \cdots + v_r\,\Lambda\,\psi_r + \sum_{r+1 \le \alpha_1 < \cdots < \alpha_p \le n} \xi^{\alpha_1\ldots\alpha_p}\,v_{\alpha_1}\Lambda\ldots\Lambda\,v_{\alpha_p},$$

where $\psi_1, \ldots, \psi_r \in \Lambda^{p-1}(V)$. Plugging into (3) we get

$$\sum_{r+1 \le \alpha_1 < \cdots < \alpha_p \le n} \xi^{\alpha_1\ldots\alpha_p}\,v_1\,\Lambda\ldots\Lambda\,v_r\,\Lambda\,v_{\alpha_1}\Lambda\ldots\Lambda\,v_{\alpha_p} = 0 \qquad (4)$$

Inside the summation, the terms $v_1\,\Lambda\ldots\Lambda\,v_r\,\Lambda\,v_{\alpha_1}\Lambda\ldots\Lambda\,v_{\alpha_p}$ $(r+1 \le \alpha_1 < \cdots < \alpha_p \le n)$ are all basis vectors of $\Lambda^{p+r}(V)$. Therefore (4) gives

$$\xi^{\alpha_1\ldots\alpha_p} = 0 \quad ; r+1 \le \alpha_1 < \cdots < \alpha_p \le n,$$

i.e., $w = v_1\,\Lambda\,\psi_1 + \cdots + v_r\,\Lambda\,\psi_r$. This completes the proof of the theorem. □

Theorem 2.6. Suppose v_α, w_α ; v'_α, w'_α ; $1 \le \alpha \le k$ are two sets of vectors in V. If $\{v_\alpha, w_\alpha, 1 \le \alpha \le k\}$ is linearly independent and

$$\sum_{\alpha=1}^{k} v_\alpha \Lambda\ w_\alpha = \sum_{\alpha=1}^{k} v'_\alpha \Lambda\ w'_\alpha , \qquad (5)$$

then v'_α, w'_α are linear combinations of $v_1, \dots, v_k$ and $w_1, \dots, w_k$ are also linearly independent.

Proof: Wedge-multiply (5) by itself k times to get

$$k!\ (v_1 \Lambda w_1 \Lambda \dots \Lambda v_k \Lambda w_k)$$
$$= k!\ (v'_1 \Lambda w'_1 \Lambda \dots \Lambda v'_k \Lambda w'_k). \qquad (6)$$

Since $\{ v_\alpha, w_\alpha ; 1 \le \alpha \le k\}$ is linearly independent, the left hand side of (6) is not equal to zero, that is, $\{ v'_\alpha, w'_\alpha ; 1 \le \alpha \le k\}$ is also linearly independent (Theorem 2.3).We can also obtain from (6) that

$$v_1 \Lambda w_1 \Lambda \dots \Lambda v_k \Lambda w_k \Lambda v'_\alpha = 0,$$

which means $\{ v_1, w_1, \dots, v_k, w_k, v'_\alpha \}$ is linearly dependent. Therefore v'_α can be expressed as a linear combination of $v_1, \dots, v_k, w_1, \dots, w_k$ .The above conclusion is also true for w'_α. This completes the proof of the theorem. □

Remark 2. For a geometrical application of theorem 2.6 refer to Chern [5].

III. Exterior Differentiation

Suppose M is an m-dimensional smooth manifold. The bundle of exterior r-forms on M

$$\Lambda^r(M^*) = \bigcup_{p\in M} \Lambda^r(T_p^*)$$

is a vector bundle on M. Use $A^r(M)$ to denote the space of the smooth sections of the exterior bundle $\Lambda^r(M^*)$: $A^r(M) = \Gamma(\Lambda^r(M^*))$.

$A^r(M)$ is a $C^\infty(M)$-module. The elements of $A^r(M)$ are called exterior differential r-forms on M. Therefore, an exterior differential r-form on M is a smooth skew-symmetric covariant tensor field of order r on M.

Similarly, the exterior form bundle $\Lambda(M^*) = \bigcup_{p\in M} \Lambda(T_p^*)$ is also a vector bundle on M. The elements of the space of its sections $A(M)$ are called exterior differential forms on M. Obviously $A(M)$ can be expressed as the direct sum

$$A(M) = \sum_{r=0}^{m} A^r(M) , \qquad (7)$$

i.e., every differential form w can be written as $w = w^0 + w^1 + \dots + w^m$, where w^i is an exterior differential i-form. The wedge product of exterior forms can be extended to the space of exterior differential form $A(M)$.

Suppose $w_1, w_2 \in A(M)$. For any $p \in M$, let

$$w_1 \Lambda w_2(p) = w_1(p) \Lambda w_2(p),$$

where the right hand side is a wedge product of two exterior forms. It is obvious that $w_1 \Lambda w_2 \in A(M)$. The space $A(M)$ then becomes an algebra with respect to addition, scalar multiplication and the wedge product. Moreover, it is a graded algebra. This means that $A(M)$ is a direct sum (8) of a sequence of vector space and the wedge product Λ defines a map

$$\Lambda : A^r(M) \times A^s(M) \to A^{r+s}(M),$$

where $A^{r+s}(M)$ is zero when $r + s > m$.

Remark 3. The tensor algebras $T(V)$ and $T(V^*)$, with respect to the tensor product $\otimes$ and the exterior algebra $\Lambda(V)$, with respect to the exterior product Λ are all graded algebras.

Theorem 3.1. [7] Suppose M is an m-dimensional smooth manifold. Then there exists a unique map $d: A(M) \to A(M)$ such that $d(A^r(M)) \subset A^{r+1}(M)$ and such that d satisfies the following:

(i) For any ω_1, $\omega_2 \in A(M)$,

$$d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2.$$

(ii) Suppose ω_1 is an exterior differential r-form. Then

$$d(\omega_1 \Lambda \omega_2) = d\omega_1 \Lambda \omega_2 + (-1)^r\ \omega_1 \Lambda d\omega_2 .$$

(iii) If f is a smooth function on M, i.e., $f \in A^0(M)$ then, df is precisely the differential of f.

(iv) If $f \in A^0(M)$, then $d(df) = 0$.

The map d defined above is called the exterior derivative.

Theorem 3.2. (Poincare's Lemma). [4] $d^2 = 0$, i.e., for any exterior differential form ω,

$$d(d\omega) = 0 .$$

Theorem 3.3. [10] Let M be a C^∞ manifold. Then the set $A^k(M)$ of all k-forms on M can be naturally identified with that of all multi-linear and alternating maps, as $C^\infty(M)$ modules, from k-fold direct product of $X(M)$to $C^\infty(M)$.

Now, we shall characterize the exterior differentiation without using the local expression namely, we have the following theorem:

Theorem 3.4. Let M be a C^∞ manifold and $\omega \in A^k(M)$ an arbitrary k-form on M. Then for arbitrary vector fields $X_1, \dots, X_{k+1} \in X(M)$, we have

$$d\omega(X_1,\cdots,X_{k+1}) = \frac{1}{k+1}\{\sum_{i=1}^{k+1}(-1)^{i+1}X_i(\omega(X_1,\cdots,\hat{X}_i,\cdots,X_{k+1}))+\sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_1,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_{k+1})\}.$$

Here the symbol $\hat{X}_i$ means X_i omitted. In particular, the often-used case of $k = 1$ is

$$d\omega(X,Y) = \frac{1}{2}\{X_\omega(Y) - Y_\omega(X) - \omega([X,Y])\}.$$

Proof. If we consider the right hand side of the formula to be proved as a map from the (k + 1)-fold direct product of $X(M)$ to $C^\infty(M)$, we see that it satisfies the conditions of degree (k +1) alternating form as a map between modules over $C^\infty(M)$. Since it is easy to verify this by Theorem 2.8, we see that the right hand side is a (k +1) form on M. If two differential forms coincide in some neighborhood of an arbitrary point, they coincide on the whole. Then consider a local coordinate system $(U, x_1, \cdots x_n)$ around an arbitrary point $p \in M$. Let the local expression of ω with respect to the local coordinate system be

$\omega = \sum_{i_1<\cdots<i_k} f_{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. Then we have

$$d\omega = \sum_{i_1<\cdots<i_k} df_{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \quad (8)$$

From the linearity of differential forms with respect to the functions on M, it is enough to consider only vector fields X_i such that $X_i = \frac{\partial}{\partial x_j}(i = 1,\cdots,k+1)$ in a neighborhood of P. Then $[X_i, X_j] = 0$ near P. Moreover by the alternating property of differential forms, we may assume that $j_1 < \cdots < j_{k+1}$. Then, if we apply (8) to $(X_1,\cdots,X_{k+1})$, we have

$$d\omega(X_1,\cdots,X_{k+1}) = \frac{1}{(k+1)!}\{\sum_{s=1}^{k+1}(-1)^{s-1}\frac{\partial}{\partial x_j} f_{j_1\cdots j_s\cdots j_{k+1}}\}.$$

On the other hand, when we calculate the right hand side of the formula using $[X_i, X_j] = 0$, we obtain the same value. This finishes the proof. □

We can consider theorem 3.4 as a definition of the exterior differentiation that is independent of the local coordinates.

Example 1. Suppose the Cartesian coordinates in $\mathbb{R}^3$ are given by (x, y, z).

1) If f is a smooth function on $\mathbb{R}^3$, then
$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

The vector formed by its coefficients $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$ is the gradient of f, denoted by $grad\, f$.

2) Suppose $a = Adx + Bdy + Cdz$, where A, B, C are smooth functions on $\mathbb{R}^3$. Then

$$da = dA \wedge dx + dB \wedge dy + dC \wedge dz = \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right) dy \wedge dz + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) dz \wedge dx + \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx \wedge dy.$$

Let X be the vector (A, B, C), then the vector

$$\left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}, \quad \frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}, \quad \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)$$

formed by the coefficients of da is just the $curl$ of the vector field X, denoted by $curl\, X$.

3) Suppose $a = Ady \wedge dz + Bdz \wedge dx + Cdx \wedge dy$. Then

$$da = \left(\frac{\partial A}{\partial x}, \frac{\partial B}{\partial y}, \frac{\partial C}{\partial z}\right) dx \wedge dy \wedge dz$$
$$= div\, X\, dx \wedge dy \wedge dz$$

where $div\, X$ means the divergence of the vector field $X = (A, B, C)$.

From theorems, two fundamental formulas in a vector calculus follow immediately. Suppose f is a smooth function on $\mathbb{R}^3$ and X is a smooth tangent vector field on $\mathbb{R}^3$. Then

$$\begin{cases} curl\,(\,grad\, f\,) = 0, \\ div\,(\,curl\, X\,) = 0. \end{cases}$$

Theorem 3.5. Suppose ω is a differential 1-form on a smooth manifold M. X and Y are smooth tangent vector fields on M. Then

$$< X \wedge Y,\, d\omega > = X < Y, \omega > - Y < X, \omega > - < [X, Y], \omega >$$

Proof: Given

$< X \wedge Y , d\omega > = X < Y ,\omega > - Y < X ,\omega > - < [X ,Y] ,\omega >$ (9)

Since both sides of (9) are linear with respect to ω , we may assume that ω is a monomial

$\omega = g\,df$; where f and g are smooth functions on $M \Rightarrow d\omega = dg \wedge df$

L.H.S: $< X \wedge Y , d\omega > = < X \wedge Y , dg \wedge df >$

$$= \begin{vmatrix} < X ,dg > & < X , df > \\ < Y , dg > & < Y , df > \end{vmatrix}$$

$$= \begin{vmatrix} Xg & Xf \\ Yg & Yf \end{vmatrix} = Xg.Yf - Xf.Yg$$

R.H.S:

$X < Y ,\omega > - Y < X ,\omega > - < [X ,Y] ,\omega > = X < Y ,g\,df > - Y < X ,g\,df > - < [X ,Y] ,g\,df = X(g\,Yf) - Y(g\,Xf) - g[X ,Y] f = Xg.Yf + g\,XYf - Yg.Xf - g\,YXf - g\,XYf + g\,YXf = Xg.Yf - Xf.Yg$

Therefore L.H.S = R.H.S. This completes the proof of the theorem. □

IV. Differential Forms

The most important tensors are differential forms. The main reason for their importance in the fact under mild compactness assumptions, it is possible to define the integration of a form of degree k on a manifold of dimension k.

Definition 3. A differential form of degree k on a manifold M is a smooth section of the bundle $\Lambda^k(M)$ and we denoted by $\Gamma(\Lambda^k M) = \Omega^k M$.

For a vector space the exterior product of $f \in \Lambda^k E^*$ and $g \in \Lambda^l E^*$ is the $(k + l)$ antisymmetric form defined by $f \wedge g = \frac{1}{k!\,l!} \mathrm{Ant}(f \otimes g)$.

Example 2. Let $w \in \Omega^1(S^2)$ be a differential 1-form on S^2 such that for any $\varphi \in SO(3)$, $\varphi^* w = w$ holds. Then $w = 0$.

Exterior forms are more interesting than tensors, for the following reasons: we shall define on

$$\sum_{k=0}^{dimM} \Omega^k M$$

a "natural" differential operator that is depending only on the differential structure of *M.* This operator gives information on the topology of the manifolds.

Remark 4. Using Theorem 3.5 the Frobenius condition for r-dimensional distributions method can be rephrased in its dual form. Suppose $L^r = \{X_1, \dots, X_r\}$ is a smooth r-dimensional distributions on M. Then for any point $p \in M$, $L^r(p)$ is an r-dimensional linear subspace of T_p. Let

$$(L^r(p))^{\perp} = \{ \omega \in T_p^* \big| < X ,\omega > = 0 \ for\ any\ X \in L^r(p) \}.$$

$(L^r(p))^{\perp}$ is certainly $(m - r)$-dimensional subspace of T_p^*, called the annihilator subspace of $L^r(p)$. In a neighborhood of an arbitrary point, there exist $m - r$ linearly independent differential 1-forms $\omega_{r+1}, \dots, \omega_m$ that span the annihilator subspace $(L^r(p))^{\perp}$ at any point p in the neighborhood. In fact, L^r is spanned by r linearly independent smooth tangent vector fields $X_1, \dots, X_r$ in a neighborhood. Therefore there exist $m - r$ smooth tangent vector fields $X_{r+1}, \dots, X_m$ such that $\{X_1, \dots, X_r, X_{r+1}, \dots, X_m\}$ is linearly independent everywhere in that neighborhood.

Suppose $\{\omega_1, \dots, \omega_r, \omega_{r+1}, \dots, \omega_m\}$ are the dual differential 1 -forms in that neighborhood. Then at every point p, $(L^r(p))^{\perp}$ is spanned by $\omega_{r+1}, \dots, \omega_m$. Locally the distribution L^r is therefore equivalent to the system of equations

$$\omega_s = 0, \quad r + 1 \le s \le m,$$

Often called a Pfaffian system of equations [1].

By (9), we have

$< X_\alpha \wedge X_\beta ,d\omega_s > = X_\alpha < X_\beta ,\omega_s > - X_\beta < X_\alpha ,\omega_s > -< [X_\alpha, X_\beta] ,\omega_s > = -< [X_\alpha, X_\beta] ,\omega_s >.$

Hence the distribution $L^r = \{X_1, \dots, X_r\}$ satisfies the Frobenius condition

$$[X_\alpha ,X_\beta] \in L^r, \quad 1 \le \alpha,\beta \le r$$

if and only if $< X_\alpha \wedge X_\beta ,d\omega_s > = 0$, $1 \le \alpha,\beta \le r$, $r + 1 \le s \le m$.

Theorem 4.1. [3] Suppose L^r is an r-dimensional distribution satisfying the Frobenius condition on a manifold M. Then through any point $p \in M$, there exists a maximal integral manifold $\mathcal{L}(p)$ of L^r such that any integral manifold of L^r through p is an open submanifold of $\mathcal{L}(p)$ with respect to the topology O.

The term maximal integral manifold in this theorem means that it is not proper subset of another integral manifold [6].

Suppose $f: M \to N$ is a smooth map from a smooth manifold M to a smooth manifold N. Then it induces a linear map between the spaces of exterior differential forms: $f^*: A(N) \to A(M)$.

In fact, f induces a tangent mapping $f_* : T_p(M) \to T_{f(p)}(N)$ at every point $p \in M$ and the definition of the map $f^*: A(N) \to A(M)$ for each homogeneous part of $A(N)$ and $A(M)$ as follows:

If $\beta \in A^r(N), r \geq 1$, then $f^*\beta \in A^r(M)$ such that for any r smooth tangent vector fields $X_1, \dots, X_r$ on M,

$$< X_1 \Lambda \dots \Lambda X_r, f^*\beta >_p = < f_* X_1 \Lambda \dots \Lambda f_* X_r, \beta >_{f(p)}, p \in M \quad (10)$$

where $< , >$ is the pairing. If $\beta \in A^0(N)$, we define $f^*\beta = \beta o f \in A^0(M)$ the map f^* distributes over the exterior product, that is, for any $\omega, \eta \in A(N)$,

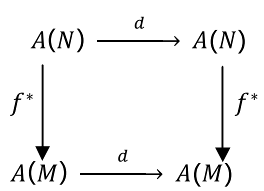
$$f^*(\omega \Lambda \eta) = f^*\omega \Lambda f^*\eta.$$

The importance of the induced map f^* also rests on the fact that it commutes with the exterior derivative d.

Theorem 4.2. Suppose $f: M \to N$ is a smooth map from a smooth manifold M to a smooth manifold N. Then the induced map $f^*: A(N) \to A(M)$ commutes with the exterior derivative d, that is,

$$f^* o\, d = d\, o\, f^*: A(N) \to A(M). \quad (11)$$

In other words, the following diagram commutes.



Proof: Since both f^* and d are linear, we need only consider the operation of both sides of (11) on a monomial β.

First suppose β is a smooth function on N i.e., $\beta \in A^0(N)$. Choose any smooth tangent vector field X on M. Then it follows from (11) that

$$< X, f^*(d\beta) > \; = \; < f_*X, d\beta > \; = \; f_*X(\beta) \; = \; X(\beta\, o\, f) \; = \; < X, d(f^*\beta) >.$$

Therefore $f^*(d\beta) = d(f^*\beta)$.

Next suppose $\beta = u\, dv$, where u, v are smooth functions on N. Then

$$f^*(d\beta) = f^*(du \Lambda dv) = f^*du \Lambda f^*dv$$
$$= d(f^*u) \Lambda d(f^*v) = d(f^*\beta).$$

Now assume that (11) holds for exterior differential forms of degree $< r$. We need to show that it also holds for exterior differential r-forms. Suppose

$$\beta = \beta_1 \Lambda \beta_2,$$

where β_1 is a differential 1-form on N and β_2 is an exterior differential $(r-1)$ form on N. Then by the induction hypothesis we have

$$d\, o\, f^*(\beta_1 \Lambda \beta_2) = d(f^*\beta_1 \Lambda f^*\beta_2) = d(f^*\beta_1) \Lambda f^*\beta_2 - f^*\beta_1 \Lambda d(f^*\beta_2)$$
$$= f^*(d\beta_1 \Lambda \beta_2) - f^*(\beta_1 \Lambda d\beta_2)$$
$$= f^* o\, d(\beta_1 \Lambda \beta_2).$$

This completes the proof of the theorem. □

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