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On Some Characterizations of Vector Fields on Manifolds

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Abstract

The basic geometry of vector fields and definition of the notions of tangent bundles are developed in an essential different way than in usual differential geometry. ϕ-related vector fields are studied and some related properties are developed in our paper. Finally, a theorem 5.04 on our natural injection j of submanifolds which is j-related to vector field X is treated.

I. Introduction

The word bundle occurs in topology with various words before it there are vector bundles, tangent bundles and fibre bundles [2] to mention the more commonly occurring usages. The most general vector bundles and tangent bundles [2] are just certain sort of fibre bundles. However, we introduce the theory of tangent bundles, vector fields and ϕ-related vector fields. We study orientable manifolds, independent vector fields, independent global vector fields, transformation group and Lie algebra. In this paper some necessary propositions related to geometry of vector fields are treated and the theorem 5.04 has been derived.

Definition 1.01 A section X of the function $\pi : TM \to M$ associates with each point m of its domain a tangent vector Xm at m. When the domain meets the domain of a differentiable function $f: M \to \mathbb{R}$, we define a function $Xf: M \to \mathbb{R}$ on the intersection by

$$m \longmapsto (Xm)f$$

The section X is called a *vector field* [7] in M if all such functions Xf are differentiable. Its domain is necessarily an open subsets of M.

Example 1.02 Suppose that x, y are the charts of the stereographic atlas on S^2. On the intersection $U \cap V$ of their domains

$$\frac{x^1}{y^1} = \frac{x^2}{y^2} = \frac{1}{[(y^1)^2+(y^2)^2]}$$

so that

$$\frac{\partial x^1}{\partial y^1} = \frac{(y^2)^2-(y^1)^2}{[(y^1)^2+(y^2)^2]^2} = -\frac{\partial x^2}{\partial y^2};$$

$$\frac{\partial x^1}{\partial y^2} = \frac{-2y^1y^2}{[(y^1)^2+(y^2)^2]^2} = \frac{\partial x^2}{\partial y^1}$$

It can be verified that the vector fields

$$(x^1 - x^2)\frac{\partial}{\partial x^1} + (x^1 + x^2)\frac{\partial}{\partial x^2}$$

$$(-y^1 - y^2)\frac{\partial}{\partial y^1} + (y^1 - y^2)\frac{\partial}{\partial y^2}$$

agree on the intersection $U \cap V$ of their domains and so together they define a vector fields on S^2 [10]. We have defined a vector at a point m to be derivation on $\mathcal{X}(m)$. It is possible to characterize a vector field in similar way by its action on real valued functions.

Definition 1.03 Let C_U^∞ denote the set of all *differentiable functions* [1] $f: M \to \mathbb{R}$ whose domains meet a given open subset U of a manifold M. A differentiable operator χ (of first order) on U is a global R-linear function

$$\chi: C_U^\infty \to C_U^\infty$$

such that $f, g \in C_U^\infty$ then

(i) the domain of χf in the intersection U with domain of f

(ii) $\chi(fg) = f(\chi g) + (\chi f)g$

where we supposed that the domains of the function involved are not empty. If X is a vector field on U, the function $f \longmapsto Xf$ is clearly a differential operator on U.

Proposition 1.04 Any differential operator χ on an open subset U of M arises from a unique vector field on U.

Proof. Choose any point $m \in U$. Then if $h \in \mathcal{X}(m)$ it follows from (i) that the real number $(\chi^h)m$ is defined. Condition (ii) implies that $h \longmapsto (\chi^h)m$ is a derivation on $\mathcal{X}(m)$ and we denote it by X_m. The function X is defined on U by $m \longmapsto X_m$ and X is therefore a section of π. If $f \in C_U^\infty$, Xf is equal to χf and so it is differentiable. Consequently X is a vector field on U. □

Proposition 1.05 If X, Y are vector fields in M and U is an open subset of M, then $[X|U, Y|U] = [X, Y]|U$.

Proof. Suppose that X, Y have domains V, W respectively. Since $X|U - X$ is the zero vector field on $U \cap V$, $[X|U - X, Y]$ is the zero vector field 0 on $U \cap V \cap W$ and consequently

$$[X|U, Y] - [X, Y] = 0.$$

It follows that $[X|U, Y] = [X, Y]|U$. A similar argument shows that $[X, Y|U] = [X, Y]|U$ and together these give the result required. □

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II. The Tangent Bundle

If x is a chart of M with domain U, any vector $v \in \pi^{-1}U$ can be expressed uniquely as $\sum a^i(\partial/\partial x^i)_m$ where $a = (a^1, \dots\dots\dots, a^n) \in \mathbb{R}^n$. We therefore have an injection

$$(\tilde{x}, y): TM \to \mathbb{R}^{2n}$$

defined by $v \longmapsto (xm, a)$, whose domain is $\pi^{-1}U$ and whose range is the open set $xU \times \mathbb{R}^n$. This will be the standard chart of TM associated with the chart x of M. The coordinate functions are defined by

$$\tilde{x}^i v = x^i(\pi v), \quad y^i v = vx^i \quad (i = 1, \dots\dots\dots, n).$$

Let x' be a chart of M with domain U' and suppose that the domain of the corresponding standard chart $(\tilde{x}', y')$ of TM meets the domain of $(\tilde{x}, y)$. Then the domains of x, x' intersect and the change of coordinates $f = x' \circ x^{-1}$ is a diffeomorphisn. It follows that the corresponding change of coordinates on TM is defined on $x(U \cap U') \times \mathbb{R}^n$ by

$$(z, a) \longmapsto (fz, [j_f z]a)$$

and so it is differentiable. The standard charts therefore form an atlas which defines a C^∞ structure of dimension $2n$ on the tangent bundle TM.

Using any chart x of M and the corresponding standard chart of TM, π is represented by a function $(z, a) \longmapsto z$ whose domain is $xU \times \mathbb{R}^n$. It follows that π is a submersion of TM onto M [5].

Proposition 2.01 If $\phi: M \to M'$ is a differentiable function, its differential $\phi_*: TM \to TM'$ is also differentiable [7].

Proof. Choose a point v in the domain of ϕ_* and charts x, x' of M, M' whose domains include $\pi v, \phi(\pi v)$ respectively. Let $\Phi = x' \circ \phi \circ x^{-1}$. Using the corresponding standard charts of TM, TM' it follows from the representative for ϕ_* is defined on the neighbourhood of $(\tilde{x}v, yv)$ by

$$(z, a) \longmapsto (\Phi z, [J_\Phi z]a)$$

and is therefore differentiable at the point. □

Proposition 2.02 If p, p' are the projections of the product manifold $M \times M'$ onto M, M' respectively, the function $\lambda = (p_*, p'_*)$ is a diffeomorphism of $T(M \times M')$ onto $(TM) \times (TM')$.

Proof. λ is clearly differentiable and it is also bijection. We have only to prove that λ^{-1} is differentiable. Suppose that x, x' are charts of M, M' with domain U, U'. On the domain of the standard chart associated with the chart $w = x \times x'$ of $M \times M'$, λ is given by

$$\sum \left(a^i \frac{\partial}{\partial w^i} + b^i \frac{\partial}{\partial w^{i+n}}\right)_{(m,m')} \longmapsto \left(\sum a^i \left(\frac{\partial}{\partial x^i}\right)_m, \sum b^i \left(\frac{\partial}{\partial x'^i}\right)_{m'}\right)$$

where $i = 1, \dots\dots\dots, n$. Using the above standard chart and the product of the standard charts associated with x and x', λ^{-1} has coordinate representative

$$(z, a, z', b) \longmapsto (z, z', a, b)$$

where $z \in xU, z' \in x'U'$ and $a, b \in \mathbb{R}^n$. This function is differentiable. □

Proposition 2.03 A section $X: M \to TM$ of π is a vector field iff it is differentiable [1].

Proof. Suppose that x is a chart of M whose domain U meets the domain W of X. Then $X|U = \sum A^i\left(\frac{\partial}{\partial x^i}\right)$, where the functions $A^i: M \to \mathbb{R}$ have domain $U \cap W$. $X|U$ is a vector field iff $A = (A^1, \dots\dots, A^n): M \to \mathbb{R}^n$ is differentiable. X is a vector field iff $X|U$ is a vector field for each such chart x.

On the other hand, using the chart x and the corresponding standard chart of TM, the coordinate representative F for X is defined on $x(U \cap W)$ by $z \longmapsto (z, A(x^{-1}z))$ and this is differentiable iff A is differentiable. X is differentiable iff F is differentiable for each such chart x. □

III. Independent Vector Fields

A set $X_1, \dots\dots, X_r$ of a vector fields on a given open subset U of M is said to be independent if, at each point $m \in U$, $X_1 m, \dots\dots, X_r m$ are linearly independent vectors of $T_m M$. In particular, a single vector field is independent iff it is nowhere zero [9].

For example, the vector fields $\frac{\partial}{\partial x^1}, \dots\dots\dots, \frac{\partial}{\partial x^n}$ on the domain of a chart x are an independent set; consequently independent vector fields certainly exist on any coordinate domain.

Proposition 3.01 If $\mathrm{v}: \mathbb{R}^{k+1} \times \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ is a R-linear function such that

(i) $\mathrm{v}(v, z) = 0$ implies that $v = 0$ or $z = 0$

(ii) there exists $e \in \mathbb{R}^{k+1}$ such that $\mathrm{v}(e, z) = z$ for all $z \in \mathbb{R}^{n+1}$ then S^n admits k independent vector fields.

Proof. Given an element $v \in \mathbb{R}^{k+1}$, a vector field V' is defined on $\mathbb{R}^{n+1}$ by

$$z \longmapsto \oint_z(\mathrm{v}(v, z))$$

where $\oint_z$ is the standard isomorphism of $\mathbb{R}^{n+1}$ onto $T_z\mathbb{R}^{n+1}$. If $\sigma: \mathbb{R}^{n+1} \to S^n$ is the differentiable function defined for all $z \neq 0$ by $z \longmapsto z/|z|$ and if $j: S^n \to \mathbb{R}^{n+1}$ is the natural injection, the composition

$$V = \sigma_* \circ V' \circ j$$

is easily seen to be a vector field on S^n. Choose elements $e_1, \dots\dots\dots\dots, e_k$ of $\mathbb{R}^{k+1}$ so that, together with e, they form a basis for $\mathbb{R}^{k+1}$. We shall show that the corresponding vector field $E_1, \dots\dots, E_k$ on S^n are independent.

Suppose that the real numbers $a^\alpha(\alpha = 1, \dots\dots\dots, k)$ exist so that for some $z \in S^n$.

$$\sum a^\alpha (E_\alpha z) = 0$$

As we saw that the kernel of σ_{*z} is the subspace of $T_z\mathbb{R}^{n+1}$ generated by $\oint_z z$. Consequently

$$\oint_z^{-1}(\sum a^\alpha (E'_\alpha z)) = \alpha z$$

for some $a \in \mathbb{R}$. Replacing az by $\mathrm{v}(ae, z)$ and using the definition of the vector fields E'_α, we obtain the equation

$$\mathrm{v}(\sum a^\alpha e_\alpha - ae, z) = 0.$$

Since $z \neq 0$, this implies that

$$\sum a^\alpha e_\alpha - ae = 0.$$

But $e, e_1, \dots\dots, e_k$ are linearly independent and consequently $a^\alpha = 0$ $(\alpha = 1, \dots\dots, k)$. □

Proposition 3.02 If manifolds M, M' admit k, k' independent vector fields respectively, then $M \times M'$ admit $k + k'$ independent global vector fields [8].

Proof. Suppose that $X_\alpha(\alpha = 1, \dots\dots, k)$ and $X'_\beta(\beta = 1, \dots\dots, k')$ are independent vector fields on M, M' respectively and that $0, 0'$ are the zero vector fields on these manifolds. We shall show that the vector fields $\lambda^{-1} \circ (X_\alpha \times 0')$ and $\lambda^{-1} \circ (0 \times X'_\beta)$ on $M \times M'$ are independent.

If (m, m') is any given point of $M \times M'$, the function

$$T_m M \oplus T_{m'}M' \to T_{(m,m')}(M \times M')$$

induced by λ^{-1} is an isomorphism. Consequently if the real numbers a^α, b^β exit such that

$$\sum a^\alpha \lambda^{-1}(X_\alpha m, 0'm') + \sum b^\beta \lambda^{-1}(0m, X'_\beta m')$$

is the zero vector of $T_{(m,m')}(M \times M')$ then

$$(\sum a^\alpha (X_\alpha m), \sum b^\beta (X'_\beta m'))$$

must be the zero vector of $T_m M \oplus T_{m'}M'$ and so $a^\alpha = 0$ $(\alpha = 1, \dots\dots, k)$ and b^β $(\beta = 1, \dots\dots, k')$. □

IV. Orientable Manifolds

An n-frame e in a real vector space V is an ordered set $e_1, \dots\dots, e_n$ of linearly independent vectors of V. When V has dimension n, any two such frames e, e' are related by

$$e'_j = \sum A_i e_i \quad (i, j = 1, 2, \dots\dots\dots, n)$$

where $A = [A^i_j]$ is an non-singular matrix. The equivalence relation

$$e \sim e' \quad \text{iff } \det A > 0$$

partitions the n-frames into two equivalence classes is called an orientation of V. An orientation in an a differentiable manifold M is a function $\theta: m \mapsto \theta_m$, where θ_m is an orientation of $T_m M$, which satisfies the following differentiability condition. Each point in the domain of θ admits a neighbourhood U on which the values of θ are determined by the values of a parallelization on U [8]. If a manifold admits a global orientation it is said to be orientable.

Proposition 4.01 Let θ be an orientation on a manifold M and let θ' be an orientation in M with domain U. If U is connected then θ' is a restriction of either θ or $-\theta$.

Proof. Let S be the set of points $m \in U$ for which $\theta'_m = \theta_m$. Choose a point $s \in S$. The functions θ, θ' are determined on neighbourhoods V, V' of s by parallelizations X_i, X'_i $(i = 1, \dots\dots, n)$ respectively. On $V \cap V'$

$$X'_j = \sum A^i_j X_i$$

where det $A: M \to \mathbb{R}$ is a differentiable function such that (det A) $s > 0$. Thus det $A > 0$ on some neighbourhood of s and so $\theta = \theta'$ on this neighbourhood. Consequently S is an open subset of M and hence of the subspace U. The set $U - S$ is the set of points $m \in U$ for which $\theta'_m = -\theta_m$ and the same argument shows that this set is open in U. Because U is connected, S is either the empty set or U itself. Consequently θ' is a restriction either of $-\theta$ or of θ. □

Proposition 4.02 If $\phi: M \to M'$ is locally a diffeomorphism, an orientation θ' in M' with domain U' determines an orientation θ in M with domain $U = \phi^{-1}U'$.

Proof. Suppose that M and M' have dimension n. Given a point $m \in U$, choose in n-frame $e'_1, \dots\dots\dots, e'_n$ at $m' = \phi m$ which agrees with θ'. Since ϕ_{*m} is an isomorphism, the vectors $\phi^{-1}_{*m} e'_i$ form an n-frame at m and so they determine an orientation θ_m at m. θ_m is clearly independent of the choice of the n-frame at m'.

We shall show that the function $\theta: m \mapsto \theta_m$ is an orientation in M with domain U. Given $m \in U$, choose a neighbourhood V of m such that $\psi = \phi|V$ is a diffeomorphism and a parallelization $X'_1, \dots\dots\dots, X'_n$ in M' which agrees with θ' an a neighbourhood of ϕm. The vector fields $\psi_*^{-1} \circ X'_i \circ \phi$ form a parallelization which determines θ on a neighbourhood of m. □

V. ϕ-related Vector Fields

If $\phi: M \to M'$ is a global differential function then the vector fields X, X' on M, M' respectively are said to be ϕ-related if

$$\phi_* \circ X = X' \circ \phi .$$

We may express this condition in terms of the action of these vector fields on differentiable functions [6].

Proposition 5.01 Vector fields X, X' on M, M' are ϕ-related iff

$$X(f \circ \phi) = (X'f) \circ \phi$$

where $f: M \to \mathbb{R}$ is any differentiable function which composes with ϕ.

Proof. Suppose that X, X' are ϕ-related. The functions $X(f \circ \phi)$, $(X'f) \circ \phi$ both have domain ϕ^{-1}(domain f). If m is any point of this domain

$$X_m(f \circ \phi) = (\phi_* X_m) f = X'_{\phi m} f.$$

The condition is therefore necessary.

Suppose that it is satisfied. Choose a point $m \in M$ and $f \in C_m^\infty$ where $m' = \phi m$. Since

$$X_m(f \circ \phi) = \big(X(f \circ \phi)\big)m = (X'f)m',$$

It follows that $(\phi_* X_m) f = X'_{m'} f$ for all $f \in C_m^\infty$ and so $\phi_* \circ X = X' \circ \phi$ at m. X, X' are therefore ϕ-related. □

Proposition 5.02 Every vector field on a quotient manifold M' of a paracompact manifold M is the projection of a projectable vector field on M.

Proof. If X' is a given vector field on M' and m is any chosen point of M it is easy to construct a vector field Y on a sufficiently small neighbourhood U of m with the required property that on U, $\mu_* \circ Y = X' \circ \mu$. To do this, choose charts x, y of M, M' whose domains U, V include $m, \mu m$ respectively so that the function $y \circ \mu \circ x^{-1}$ is defined on xU by $(z, w) \longmapsto z$. That such a choice is possible is consequence on U

$$\mu_* \circ \frac{\partial}{\partial x^\alpha} = \frac{\partial}{\partial y^\alpha} \circ \mu \quad (\alpha = 1, 2, \dots\dots, l)$$

and so, if $X' = \sum \psi^\alpha \left(\frac{\partial}{\partial x^\alpha}\right)$ on V, our requirements are satisfied by the vector field

$$Y = \sum (\psi^\alpha \circ \mu) \frac{\partial}{\partial x^\alpha}.$$

The collection of such neighbourhoods U corresponding to all points of M is a covering of M. Since M is a paracompact, we can choose a partition of unity $\{\phi_\alpha\}$ $(\alpha \in A)$ subordinate to this covering. The support C_α of ϕ_α is contained in a neighbourhood U_α and we denote the corresponding vector field on U_α by Y_α. The vector field $\phi_\alpha Y_\alpha$ on U_α and the zero vector field on $M - C_\alpha$ agree on the intersection $U_\alpha - C_\alpha$ of their domains and so together they define a vector field X_α on M. Since the collection $\{C_\alpha\}$ is locally finite, any given point M has a neighbourhood on which all except a finite number of these vector fields X_α are zero. Consequently, the sum

$$X = \sum X_\alpha \qquad (\alpha \in A)$$

is defined and is differentiable on M and so it is a vector field on M.

We complete the proof by showing that this vector field X is projectable and that its projection is X'. Choose any point $m \in M$ and denote by B the finite subset of A such that $\phi_\beta m \neq 0$ iff $\beta \in B$. Then

$$Xm = \sum X_\beta m = \sum (\phi_\beta m)(Y_\beta m) \quad (\beta \in B)$$

and so, because B is finite

$$\mu_*(Xm) = \sum (\phi_\beta m)\left(\mu_*(Y_\beta m)\right) = \sum (\phi_\beta m) X'(\mu m).$$

Since $\sum \phi_\beta m = \sum \phi_\alpha m = 1$ it follows that $\mu_*(Xm) = X'(\mu m)$.

Proposition 5.03 G is a transformation group acting on a manifold M and $G\backslash M$ is a quotient manifold. A sufficient condition for a vector field X on M to be projectable is that it is invariant under all the transformations ϕ_g where $g \in G$. If M and $G\backslash M$ have the same dimension this condition is necessary.

Proof. Since $\mu \circ \phi_g = \mu$ it follows that $\mu_* \circ \phi_{g*} = \mu_*$ and so X is invariant under ϕ_g,

$$\mu_* \circ X \circ \phi_g = \mu_* \circ \phi_{g*} \circ X = \mu_* \circ X.$$

If this is true for all $g \in G$ it implies that $\mu_* \circ X$ is an invariant of the equivalence relation on M and so X is projectable.

Conversely, suppose that X is projectable. Then $\mu_* \circ X \circ \phi_g = \mu_* \circ X$ and therefore $\mu_* \circ X \circ \phi_g = \mu_* \circ \phi_{g*} \circ X$ for all $g \in G$. Thus the two vectors $X(gm)$ and $\phi_{g*}(Xm)$ at gm have the same images under μ_*. But if M and $G\backslash M$ have the same dimension, μ_* induces an isomorphism on each tangent vector space and so

$$X(gm) = \phi_{g*}(Xm)$$

for all $m \in M$. This mean that $X \circ \phi_g = \phi_{g*} \circ X$ for all $g \in G$ and so X is invariant under all the transformation ϕ_g. □

Theorem 5.04 M' is a submanifold of M and $j: M' \to M$ is the natural injection. A vector field X in M is tangent to M' iff there exists a vector field X' in M' which is j-related to X.

Proof. We denote the domain of X by U. Suppose that each a vector field X' exists, so that $j_* \circ X' = X \circ j$. Then if $m \in M' \cap U$

$$Xm = j_*(X'm) \in j_*(T_m M')$$

and so X is tangent to M'.

Conversely, suppose that X is tangent to M'. Since j_{*m} is an isomorphism, we can define a section X' of $\pi': TM' \to M'$ by

$$m \longmapsto (j_{*m})^{-1}(Xm)$$

where $m \in M' \cap U$. To show that X' is differentiable at such a point m, then set up charts x, y of M, M' at m such that $y^\alpha = x^\alpha \circ j$ $(\alpha = 1, \dots\dots, l)$. Then

$$X'y^\alpha = X'(x^\alpha \circ j) = (Xx^\alpha) \circ j$$

and such functions are differentiable at m. It follows that X' is a vector field in M'. It is easily seen to be j-related to X. □

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1. Auslanser, L. and R. E. Mackenzir, Introduction to differentiable manifolds, McGraw-Hill, New York, 1963.
2. Ahmed, K.M. and M.S. Rahman, On the principal Fiber Bundles and Connections, Bangladesh Journal of Scientific and Industrial Research, 2006.
3. Chevaalley, C. 'Theory of Lie groups', Princeton University Press, 1946.
4. Gottschalk, W.H. and G.A. Hedlund, Topological dynamics, Amercian mathematical publications, 1955.
5. Hirsch, M. W. 'On imbedding differentiable manifolds in Euclidean space' Ann. Math. 73(1961), 566-571.
6. Jacobson, 'Lie algebras', Interscience, New York, 1963.
7. Kobayashi, S. and K. Nomizu, Foundations of differential geometry, volume I, Interscience, New York, 1963.
8. Lang, S. 'Introduction to differentiable manifolds', Interscience, New York, 1962.
9. Millnor, J. W. 'On manifolds homeomorphic to the 7-sphere' Ann. Math. 64(1956), 399-405.
10. Millnor, J. W ' Differentiable structures on spheres' Am. J. Math. 81(1959), 962-972.