Dhaka Univ. J. Sci. 61(1): 53-58, 2013 (January)

Solution of Fuzzy Non-linear Equation using Bisection Algorithm

Goutam Kumar Saha and Shapla Shirin*

Department of Mathematics, University of Dhaka, Dhaka-1000, Bangladesh

Received on 15.11.2011. Accepted for Published on 12.05.2012

Abstract

In Classical Mathematics a non-linear equation can be solved by using different types of numerical methods. In this paper a new approach has been introduced to get approximate solutions of a fuzzy non-linear equation with the help of Bisection Algorithm. A non-linear equation over linear fuzzy real numbers is called a fuzzy non-linear equation. Finally, graphical representations of the solutions has also been drawn so that anyone can achieve the idea of converging to the root of a fuzzy non-linear equation.

Keywords: Bisection method, Fuzzy non-linear equation (FNLE in short), Set of all Linear Fuzzy Real Number (denoted by LF^R), Linear Fuzzy Real Number (LFRN in short), Square root and n-th power of a fuzzy number.

I. Introduction

In algebra, it is easy to solve literal equations $f(x) = 0$ of all degrees up to and including fourth. But we are not always able to get exact solution of equations. Besides this, no general method exists for finding the roots of this equation in terms of their co-efficient. One can use Numerical Methods to compute the roots of a non-linear equation to any desired degree of accuracy. A fuzzy linear equation can also be solved directly and the method has been discussed in [4]. But this method is not capable of computing a solution of FNLE. Here it is noted that a non-linear equation over linear fuzzy real numbers is called a fuzzy non-linear equation. In this paper a new idea has been introduced to solve a fuzzy non-linear equation with the help of Bisection Algorithm. Then, an example is also discussed and therefore, the approximate solutions, which are LFRNs, computed from each iteration are shown in a tabular form. The graphical representations of these approximate solutions of the fuzzy non-linear equation are depicted to achieve the idea of converging to the root of the FNLE. Finally, the precise solution of the fuzzy non-linear equation is obtained with the help of the proposed algorithm and its graphical representation has also been shown as well.

II. Preliminaries

In this section some definitions have been discussed which are important to us for representing our main objective in the later sections.

Definition 2.1 [8] (Linear fuzzy real number, LFRN) Let $\mathcal{R}$ be the set of all real numbers and $\mu : \mathcal{R} \to [0,1]$ be a function such that

$$\mu(x) = \begin{cases} 0 \ \text{if} \ x \le a \ \text{or} \ x \ge c \\ \dfrac{x-a}{b-a} \ \text{if} \ a < x < b \\ 1 \quad \text{if} \ x = b \\ \dfrac{c-x}{c-b} \ \text{if} \ b < x < c \end{cases}$$

Then $\mu(a, b, c)$ is called a linear fuzzy real number with associated triple of real numbers (a, b, c) where $a \le b \le c$ shown in Fig.1.1.

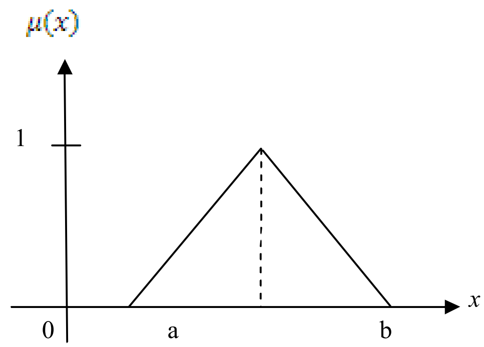

Fig. 1.1 Linear fuzzy real number $\mu(a, b, c)$.

Let $LF^{\mathcal{R}}$ be the set of all linear fuzzy real numbers. We note that any real number b can be written as a linear fuzzy real number, $r(b)$, where $r(b) = \mu(b, b, b)$ and therefore $\mathcal{R} \subseteq LF^{\mathcal{R}}$. Now it is clear that $r(b)$ represents the real number b itself. Operations on LFRN, fuzzy functions and fuzzy linear equations in $LF^{\mathcal{R}}$ are also defined in $LF^{\mathcal{R}}$ as follows.

Definition 2.2 [8] (Operations on $LF^{\mathcal{R}}$) For given two linear fuzzy real numbers $\mu_1 = \mu(a_1, b_1, c_1)$ and $\mu_2 = \mu(a_2, b_2, c_2)$, we define addition, subtraction, multiplication and division by

1. $\mu_1 + \mu_2 = \mu(a_1 + a_2, b_1 + b_2, c_1 + c_2)$;
2. $\mu_1 - \mu_2 = \mu(a_1 - c_2, b_1 - b_2, c_1 - a_2)$;
3. $\mu_1 . \mu_2 = \mu(\min\{a_1a_2, a_1c_2, a_2c_1, c_1c_2\}, b_1b_2, \max\{a_1a_2, a_1c_2, a_2c_1, c_1c_2\})$;
4. $\frac{\mu_1}{\mu_2} = \mu_1 . \frac{1}{\mu_2}$.

*Author for correspondence

where $\frac{1}{\mu_2} = \mu\left(\min\left\{\frac{1}{a_2}, \frac{1}{b_2}, \frac{1}{c_2}\right\},\right.$

$\left.median\left\{\frac{1}{a_2}, \frac{1}{b_2}, \frac{1}{c_2}\right\}, \max\left\{\frac{1}{a_2}, \frac{1}{b_2}, \frac{1}{c_2}\right\}\right).$

Definition 2.3 [8] (Function in $LF^{\mathcal{R}}$) If $f : \mathcal{R} \to \mathcal{R}$ is a real-valued function and $\mu(a, b, c)$ is a LFRN, then the linear fuzzy real-valued function $\bar{f} : LF^{\mathcal{R}} \to LF^{\mathcal{R}}$ is defined as

$$\bar{f}(\mu(a, b, c)) = \mu(\bar{a}, \bar{b}, \bar{c}),$$

where $\bar{a} = \min\{f(a), f(b), f(c)\}$,
$\bar{b} = \text{median}\{f(a), f(b), f(c)\}$,
$\bar{c} = \max\{f(a), f(b), f(c)\}$.

We note that if $a = b = c$ then $\bar{a} = \bar{b} = \bar{c}$, i.e., $\bar{f}(r(b)) = r(f(b))$. Hence $\bar{f}$ is an extension of the function f.

Definition 2.4 [8] (Linear equation in $LF^{\mathcal{R}}$**)** A linear equation over $LF^{\mathcal{R}}$ is an equation of the form

$$\mu_1.\mu_x + \mu_2 = \mu_3\ ,$$

where the μ_i are *LFRN*s for $i = 1, 2, 3$ with a triple of unknown real numbers $(\alpha_i, \beta_i, \gamma_i)$.

Definition 2.5 [1] (Square root of a fuzzy number) Square root of a fuzzy number $\mu(a, b, c)$ is defined by

$$\sqrt{\mu(a, b, c)} = \mu(\sqrt{a}, \sqrt{b}, \sqrt{c}),$$

where $a, b, c \geq 0$.

Definition 2.6. (n-th power of a fuzzy number) A fuzzy number $\mu(a, b, c)$ of power n is defined by

$$(\mu(a, b, c))^n = \mu(a^n, b^n, c^n),$$

where $a, b, c \geq 0$.

Definition 2.7 [8] (Fuzzy non-linear equations in $LF^{\mathcal{R}}$) Fuzzy non-linear equations can be found in many applications, all the way from light diffraction to planetary orbits for example. In this section, we discuss how to solve fuzzy non-linear equations in $LF^{\mathcal{R}}$. Thereby a solution μ_x in $LF^{\mathcal{R}}$ of a fuzzy a non-linear equation $\bar{f}(\mu_x) = 0$ will be obtained where $\bar{f}: LF^{\mathcal{R}} \to LF^{\mathcal{R}}$ is a non-linear function.

For example $\mu_x^3 + \mu_x^2 - 1 = 0$ is fuzzy non-linear equation.

III. The Bisection Method to Solve Fuzzy Non-linear Equation

Solving nonlinear equations over the set of linear fuzzy real ($LF^{\mathcal{R}}$) numbers of the form $\bar{f}(\mu_x) = 0$ is possible with a modification of Bisection method over real numbers. To solve fuzzy nonlinear equation using this modified Bisection method over $LF^{\mathcal{R}}$, we have begun with two initial approximation for which $\bar{f}(\mu_x^{(0)})$ have opposite sign. Let us assume that two initial approximations are

$$\mu_1^{(0)} = \mu(a_1^{(0)}, b_1^{(0)}, c_1^{(0)})$$

and $$\mu_2^{(0)} = \mu(a_2^{(0)}, b_2^{(0)}, c_2^{(0)})$$

for which $\bar{f}(b_1^{(0)})$ and $\bar{f}(b_2^{(0)})$ have opposite sign (say, $\bar{f}\left(b_1^{(0)}\right) < 0$ and $\bar{f}\left(b_2^{(0)}\right) > 0$). Hence the method generates the sequence $\{\mu_x^{(n)}\}_{n=0}^{\infty}$ by

$$\mu_x^{(n)} = \frac{\mu_1^{(n-1)} + \mu_2^{(n-1)}}{2}, for\ n \geq 1 \qquad (1.1)$$

In first iteration if we get $\mu_x^{(1)} = \mu(a_1^{(1)}, b_1^{(1)}, c_1^{(1)})$, then we find $f(\mu_x^{(1)})$. If $f(\mu_x^{(1)}) = 0$ we are done. If $f(\mu_x^{(1)}) > 0$ i.e., $\bar{f}(b_1^{(1)}) > 0$ then we replace $\mu_2^{(0)}$ by $\mu_x^{(1)}$. If $\bar{f}(\mu_x^{(1)}) < 0$ i.e., $\bar{f}\left(b_1^{(1)}\right) < 0$ then we replace $\mu_1^{(0)}$ by $\mu_x^{(1)}$. Preceding in this way we find the approximations to the solution of fuzzy nonlinear equation $\bar{f}(\mu_x) = 0$.

The stopping criterion of this method is $\frac{\left|r(b_2^{(0)} - b_1^{(0)})\right|}{2^n} < \epsilon$, which gives on simplification

$$n \geq \frac{\log_e(\left|r(b_2^{(0)} - b_1^{(0)})\right| / \epsilon)}{\log_e 2} \qquad (1.2)$$

Inequality (1.2) gives the number of iterations required to achieve an accuracy ϵ. For example, if $\left|r(b_2^{(0)} - b_1^{(0)})\right| = 1$ and $\epsilon = 0.001$, then it can be seen that $n \geq 10$.

IV. Modification of Bisection Algorithm

Now the question arises, why the modification of existing algorithm is required. This is because of the formation of FNLEs which consist of fuzzy functions and LFRNs, whereas a non-linear equation consists of functions and real numbers.

To find a solution to $\bar{f}(\mu_x) = 0$ given the continuous function $\bar{f}$ on the interval $[b_1^{(0)}, b_2^{(0)}]$ where $\bar{f}(b_1^{(0)})$ and $\bar{f}(b_2^{(0)})$ have opposite signs.

INPUT: Initial approximations are

$$\mu_1^{(0)} = \mu(a_1^{(0)}, b_1^{(0)}, c_1^{(0)})$$

and $\mu_2^{(0)} = \mu(a_2^{(0)}, b_2^{(0)}, c_2^{(0)})$;

tolerance TOL; maximum number of iterations n.

OUTPUT: approximation solution

$\mu_x^{(n)} = \mu(a^{(n)}, b^{(n)}, c^{(n)})$ or message of failure.

Step-1: Set $i = 1$.

$$FA = \bar{f}(r(b_1^{(0)}))$$

Step-2: While $i \leq n$ do steps $3 - 6$.

Step-3: Set $\mu_x^{(n)} = \frac{\mu_1^{(n-1)} + \mu_2^{(n-1)}}{2}$.

(Compute $\mu_x^{(n)}$)

$$FP = \bar{f}(r(b_x^{(n)}))$$

Step-4: If $FP = 0$

or $\frac{|\mu(b^n) - \mu(b^{n-1})|}{2} < $ TOL then

OUTPUT $(\mu_x^{(n)})$.

(The procedure is successful.)

STOP.

Step-5: Set $i = i + 1$.

Step-6: IF $FA.FP > 0$ then setting

$$\mu_1^{(0)} = \mu_x^{(n-1)}$$

(Update $\mu_x^{(0)}$)

$$FA = FP$$

Else $\mu_2^{(0)} = \mu_x^{(n-1)}$.

Step-7: OUTPUT. ("The method failed after n iteration, n" n);

(The procedure is unsuccessful.)

STOP.

Example. Solve a fuzzy non-linear equation $\mu_x^3 + \mu_x^2 - 1 = 0$.

Solution: Suppose that $\bar{f}(\mu_x) = \mu_x^3 + \mu_x^2 - 1$.

If we use the interval

$[\mu_1^{(0)}, \mu_2^{(0)}] = [\mu(0, 0.5, 1), \mu(0.5, 1, 1.5)]$, then equation (1.1) generates the sequence $\{\mu_x^n\}$ of approximate solutions as follows:

Iteration-1: $\bar{f}(\mu_1^{(0)}) = \mu^3(0, 0.5, 1) + \mu^2(0, 0.5, 1) - 1$

$= \mu(0, 0.125, 1) + \mu(0, 0.25, 1) - 1$

$= \mu(-1, -0.375, 1) < 0$

$\bar{f}(\mu_2^{(0)}) = \mu^3(0.5, 1, 1.5) + \mu^2(0.5, 1, 1.5) - 1$

$= \mu(0.125, 1, 3.375) + \mu(0.25, 1, 2.25) - 1$

$= \mu(-0.625, 1, 4.625) > 0$

Hence root lies between $[\mu_1^{(0)}, \mu_2^{(0)}]$.

Then, we get $\mu_3^{(1)} = \frac{\mu_1^{(0)} + \mu_2^{(0)}}{2} = \mu(0.25, 0.75, 1.25)$,

Where $\bar{f}(\mu_3^{(1)}) < 0$. So, the root lies between $[\mu_2^{(0)}, \mu_3^{(1)}]$. In the same way we obtain the approximate solutions of $\bar{f}(\mu_x) = \mu_x^3 + \mu_x^2 - 1 = 0$ which are shown in the following table.

Table. 1. Approximate solutions of fuzzy nonlinear equation using Bisection algorithm.

Iteration no. n	Root's interval	Root $\mu_{n+1}^{(n)}(x)$	Sign of $\bar{f}(\mu_{n+1}^{(n)})$
1.	$[\mu_1^{(0)}, \mu_2^{(0)}]$	$\mu(0.25, 0.75, 1.25)$	$(-)$ ve
2.	$[\mu_2^{(0)}, \mu_3^{(1)}]$	$\mu(0.375, 0.875, 1.375)$	$(+)$ ve
3.	$[\mu_3^{(1)}, \mu_4^{(2)}]$	$\mu(0.3125, 0.8125, 1.3125)$	$(+)$ ve
4.	$[\mu_3^{(1)}, \mu_5^{(3)}]$	$\mu(0.28125, 0.78125, 1.28125)$	$(+)$ ve
5.	$[\mu_3^{(1)}, \mu_6^{(4)}]$	$\mu(0.265625, 0.765625, 1.265625)$	$(+)$ ve
6.	$[\mu_3^{(1)}, \mu_7^{(5)}]$	$\mu(0.2578125, 0.7578125, 1.2578125$	$(+)$ ve
7.	$[\mu_3^{(1)}, \mu_8^{(6)}]$	$\mu(0.25390625, 0.75390625, 1.25390625)$	$(\ )$ ve
8.	$[\mu_8^{(6)}, \mu_9^{(7)}]$	$\mu(0.255359375, 0.755859375, 1.255859375)$	$(+)$ ve
9.	$[\mu_9^{(7)}, \mu_{10}^{(8)}]$	$\mu(0.254382812, 0.754882812, 1.254882812)$	$(+)$ ve
10.	$[\mu_9^{(7)}, \mu_{11}^{(9)}]$	$\mu(0.254394531, 0.754394531, 1.254394531)$	$(-)$ ve
11.	$[\mu_{11}^{(9)}, \mu_{12}^{(10)}]$	$\mu(0.254638671, 0.754638671, 1.254638671)$	$(-)$ ve
12.	$[\mu_{11}^{(9)}, \mu_{13}^{(11)}]$	$\mu(0.254760741, 0.754760741, 1.254760741)$	$(-)$ ve
13.	$[\mu_{11}^{(9)}, \mu_{14}^{(12)}]$	$\mu(0.254821776, 0.754821776, 1.254821776)$	$(-)$ ve
14.	$[\mu_{11}^{(9)}, \mu_{15}^{(13)}]$	$\mu(0.254352294, 0.754852294, 1.254852294)$	$(-)$ ve
15.	$[\mu_{11}^{(9)}, \mu_{16}^{(14)}]$	$\mu(0.254367553, 0.754867553, 1.254867553)$	$(-)$ ve
16.	$[\mu_{11}^{(14)}, \mu_{17}^{(15)}]$	$\mu(0.254375182, 0.754875182, 1.254875182)$	$(-)$ ve

From the above table it is clear that $\mu_{17}^{(15)} = \mu(0.254867553, 0.754867553, 1.254867553)$ is the desired approximate root of the given equation. The graphical representation of the sequence of approximate roots are shown in Fig.1.2 and the optimum solution of FNLE is shown in Fig.1.3.

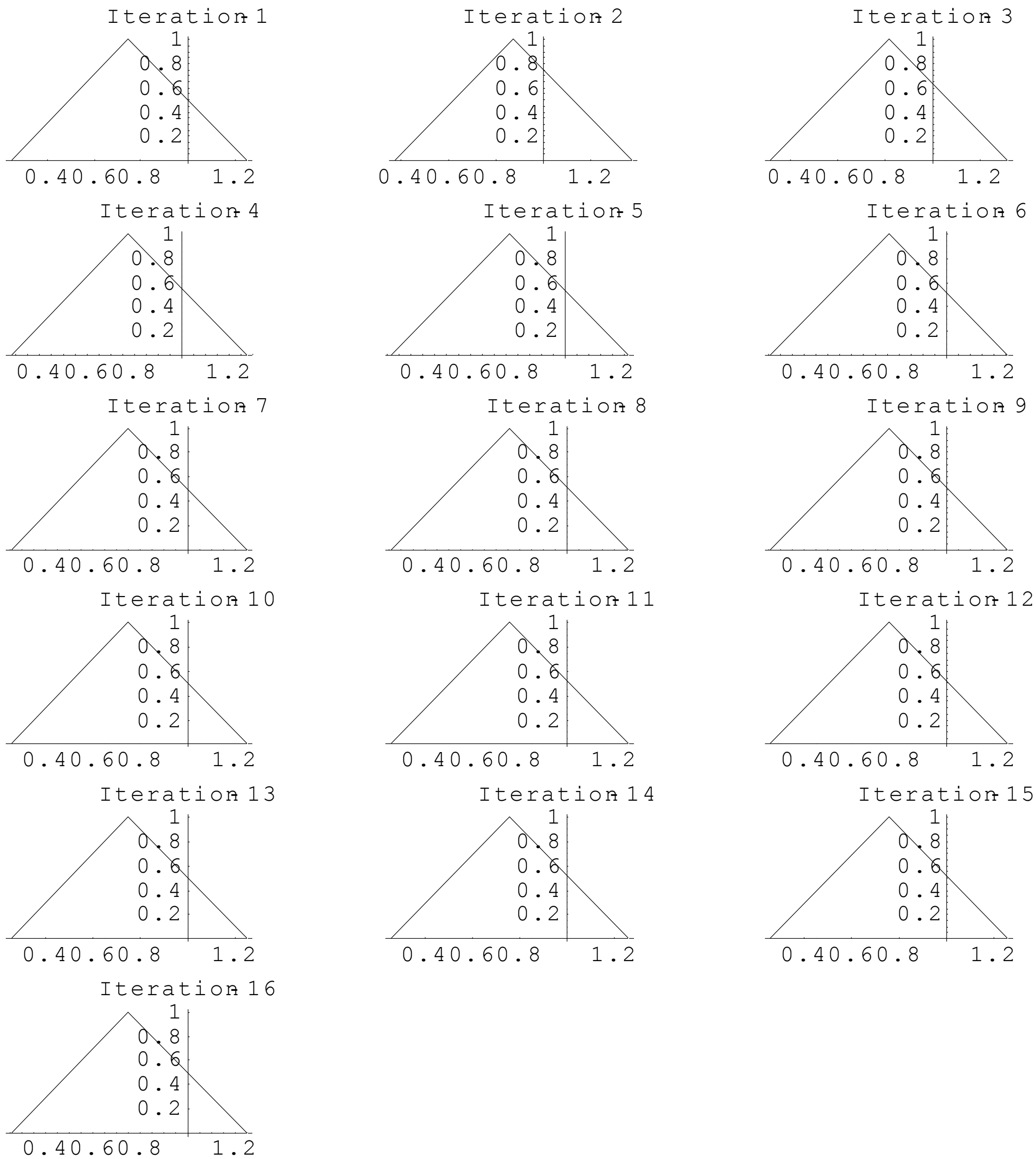

Fig. 1. 2. Graphical representations of approximate solutions of the given fuzzy non-linear equation.

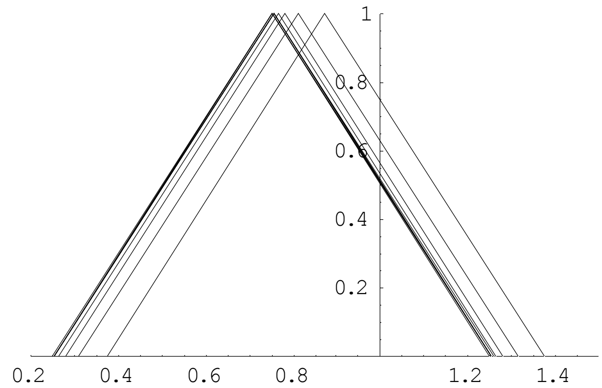

Fig. 1.3. Graphical representation of optimum solution of a fuzzy non-linear equation given in the above example.

V. Conclusion

In this paper a new approach has been introduced to solve a fuzzy non-linear equation (FNLE) with the help of Bisection Algorithm. However, by using this new approach a fuzzy non-linear equation (FNLE) is solved and also obtained the solution of the FNLE to the desired degree of accuracy which is the optimum solution of the given equation. We have also represented the approximate solutions of the FNLE graphically by which one can understand how the solutions converge to the required accuracy.

----

1. Bashar M. A. and S. Shirin, *Squares and square roots of continuous fuzzy numbers,* Dhaka Univ. J. Sci., **53(2)**, July 2005, 131 – 140.
2. Burden R. L. and J. D. Faires, *Numerical Analysis*, 7th edition, 55-63.
3. Dubois D., H. Prade*, System of linear fuzzy constraints*, Fuzzy Sets and System **13** (1982), 1 - 10.
4. Klir G. J., B. Yuan, *Fuzzy sets and fuzzy logic Theory and Applications*, Prentice-Hall of India Private Limited, New Delhi, (1997), 1-117.
5. Neggers J., H. Kim, *On Linear Fuzzy Real Numbers,* Manuscript for book under development, 2007.
6. Negoita C. V.*, Fuzziness in management*, OPSA/TIMS, Miami, 1970.
7. Prevo R, *Entropies of families of fuzzy random variables: an introduction to an in-depth exploration of several classes of important examples, Dissertation*, University of Alabama, 2002.
8. Rogers F. and Y. Jun, *Fuzzy Nonlinear Optimization for the Linear Fuzzy Real Number System,* International Mathematical Forum, 4, 2009, no. **12**, 589 - 596.
9. Rogers F., J. Neggers, Y. Jun, *Method for optimizing linear problems with fuzzy constrains*, International Mathematical Forum 3 (2008), 1141 - 1155.
10. Zimmermann H. J., *Fuzzy mathematical programming*, Comput. Ops. Res.**10** (1983), 291 - 298.