Dhaka Univ. J. Sci. 61(1): 87-92, 2013 (January)

Serret-Frenet Equations in Minkowski Space

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Received on 28.03.2012. Accepted for Published on 04.11.2012

Abstract

In this paper we established the Serret-Frenet equations in Minkowski space. These equations originally formulated in Euclidean space in $\mathbb{R}^3$, constitute a beautiful set of vector differential equations which contains all intrinsic properties of parameterized curve. From the local theory of curves in $\mathbb{R}^3$ states that a curve lies in a plane if and only if its torsion vanishes, which gives us clear geometrical insight in the notion of torsion. This theorem has two counterparts in Minkowski space that has been focused.

I. Introduction

The main objective of this paper is to look at some aspects of the differential geometry of curves in Minkowski space [2]. The article is organized as follows. In section 2 we start by setting up the Serret-Frenet equations in Minkowski space. The Serret-Frenet equation gives the derivatives with respect to the arc length parameter of the tangent, normal and binormal vectors of a curve in terms of each other. Though the Serret-Frenet equations, the evolution of a curve is completely determined, up to rigid motion, by two intrinsic scalars: the curvature $k(s)$ and the torsion $\tau(s)$. This result is known as the fundamental theorem of space curves [10]. An analogous theorem holds in Minkowski space, and proof is given in section 3. The set of equations has a solution in terms of k, curvature and θ, the angle of rotation of the osculating plane, that indirectly solves the Frenet-Serret equations, with a unique value of θ for each specified value of τ, torsion. Explicit solutions can be generated for constant θ.The equations breakdown when the tangent vector aligns to one of the unit coordinate vectors, requiring a reorientation of the local coordinate system

II. The Serret-Frenet Equations

In Euclidean space $\mathbb{R}^3$ the intrinsic geometric properties of a curve Γ(parameterized by the arc length s) are described by the Serret-Frenet equations

$$\begin{aligned} \frac{dt}{ds} &= \kappa \boldsymbol{n} \\ \frac{dn}{ds} &= -\kappa \boldsymbol{t} + \tau \boldsymbol{b} \\ \frac{db}{ds} &= -\tau \boldsymbol{n} \end{aligned} \tag{1}$$

Or, in matrix representation,

$$\begin{bmatrix} \frac{dt}{ds} \\ \frac{dn}{ds} \\ \frac{db}{ds} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} \tag{2}$$

where $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$ denote, respectively, the tangent, normal and binormal vectors of the curve Γ, which is assumed to be smooth (at least of class C^3). The triad of vectors $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$ constitute an orthonormal right handed frame defined at each point of Γ and the invariant scalars $k = k(s)$ and $\tau_1 = \tau_1(s)$ are called, respectively, the curvature and torsion of Γ. The equations (1) follows directly from the definition of the normal vector $\boldsymbol{n}$, the binormal vector $\boldsymbol{b}(\boldsymbol{b} \equiv \mathbf{t} \wedge \boldsymbol{n})$ [10].

To adapt the above formalism to Minkowski space we need to replace the Euclidean metric for the Minkowski metric $\eta_{\alpha\beta} = diag(1, -1, -1, -1)$ and define a second binormal and a second torsion $\tau_2 = \tau_2(s)$. Since usual vector products make no sense in four dimensional space we define our set of orthonormal four-vectors ($a\ tetrad$) by concomitantly requiring them to satisfy a four dimensional extension of the Frenet-Serret equations, which then governs the evolution of the tetrad. It is also convenient to restrict ourselves to timelike curves $x^\alpha = x^\alpha(s)$, i.e. those for which $\eta_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 1$, where now s denotes the arc length parameter in the sense of Minkowski metric $\eta_{\alpha\beta}$. Accordingly, if we denote the tetrad vectors by $u^\alpha_{(A)}(0 \ldots 3)$, then the orthonormality conditions read $u^\alpha_{(A)} u_{(B)\alpha} \equiv \eta_{\alpha\beta} u^\alpha_{(A)} u^\beta_{(B)} = \eta_{AB}$. It can be shown $[4, 8]$

that if we chose $u^{\alpha}_{(0)} = \frac{dx^{\alpha}}{ds}$, i.e. $u^{\alpha}_{(0)}$ being the components of the unit tangent vector, then we can easily construct an orthonormal basis vectors $\{u^{\alpha}_{(A)}\}$, defined along the curve, which obey the following four dimensional Serret-Frenet equations, given in matrix representation by

$$\begin{bmatrix} \frac{du_{(0)}}{ds} \\ \frac{du_{(1)}}{ds} \\ \frac{du_{(2)}}{ds} \\ \frac{du_{(3)}}{ds} \end{bmatrix} = \begin{bmatrix} 0 & k & 0 & 0 \\ k & 0 & \tau_1 & 0 \\ 0 & -\tau_1 & 0 & \tau_2 \\ 0 & 0 & -\tau_2 & 0 \end{bmatrix} \begin{bmatrix} u_{(0)} \\ u_{(1)} \\ u_{(2)} \\ u_{(3)} \end{bmatrix} \tag{3}$$

Of course the above procedure may be easily generalized to n-dimensional Riemannian (or pseudo-Riemannian) spaces by changing from ordinary differentiation to absolute differentiation [4, 8]. Here, two points are worth mentioning. First, due to the Lorentzian signature, the 4×4 matrix that governs the evolution of the tetrad vectors $u^{\alpha}_{(A)}$ is not anti-symmetric, as in the case of Euclidean signature. Secondly, in order to construct of the tetrad $\{u^{\alpha}_{(A)}\}$ it is not necessary, as a matter of fact, to assume that the curvature and the torsions have non-zero values. If $k = 0$, then the curve is a timelike geodesic and a triad of constant spacelike orthonormal vectors $\{u_{(1)}, u_{(2)}, u_{(3)}\}$ orthogonal to $\{u_{(0)}\}$ may be chosen. In this case, $\tau_1(s)$ and $\tau_2(s)$ are zero, if $k \neq 0$, but $\tau_1(s) = 0$, then we can choose an orthonormal basis $\{u_{(0)}, u_{(1)}, u_{(2)}, u_{(3)}\}$ in such a way that $u_{(2)}$ and $u_{(3)}$ are constant spacelike vectors.

III. The Fundamental Theorem in Minkowski Space

A most important result in the local theory of curves in Euclidean space $\mathbb{R}^3$, known as the fundamental theorem of curves, states the following:

Given differentiable functions $k(s) > 0$ and $\tau(s)$, there exists a regular parameterized curve Γ such that $k(s)$ is the curvature, $\tau(s)$ is the torsion of Γ. Any other curve $\bar{\Gamma}$ satisfying the same conditions differs from Γ by a rigid motion [10]. It would be natural to expect this theorem to hold when appropriately transposed to Minkowski space. In this case, a rigid motion would correspond to a Poincare transformation and the curve would be expected to be determined by the three differentiable functions $k(s) > 0, \tau_1(s)$ and $\tau_2(s)$. Here we give a simple proof of the fundamental theorem of curves in Minkowski space. We omit the proof of the existence part since it is almost the same as in the case of $\mathbb{R}^3$, requiring only minor modifications. The proof of the uniqueness part, however, differs from its counterpart in $\mathbb{R}^3$, since the latter makes use of the positiveness of the Euclidean metric. Let us first state the theorem.

Theorem 1. Given differentiable functions $k(s) > 0, \tau_1(s)$ and $\tau_2(s)$, there exists a regular parameterized timelike curve Γ such that $k(s)$ is the curvature, $\tau_1(s)$ and $\tau_2(s)$are, respectively, the first and second torsion of Γ. Any other curve $\bar{\Gamma}$ satisfying the same conditions differs from Γ by a Poincare transformation. i.e., by a transformation of the type $x'^{\mu} = \Lambda^{\mu}_{\cdot\nu} x^{\nu} + \alpha^{\mu}$, where $\Lambda^{\mu}_{\cdot\nu}$ represents a proper Lorentz matrix and α^{μ} is a constant four vector.

Proof. Let us assume that two time-like curves Γ and $\bar{\Gamma}$ satisfy the conditions $k(s) = \bar{k}(s), \tau_1(s) = \bar{\tau}_1(s)$ and $\tau_2(s) = \bar{\tau}_2(s), s \in I$, where I is an open interval of $\mathbb{R}$. Let $\{u^{\alpha}_{(A)}(s_0)\}$ and $\{\bar{u}^{\alpha}_{(A)}(s_0)\}$ be the Serret-Frenet tetrads at $s_0 \in I$ of Γ and $\bar{\Gamma}$, respectively. It is clear that it is always possible, by a Poincare transformation, to bring $x^{\mu}(s_0)$ of Γ into $\bar{x}^{\mu}(s_0)$ of $\bar{\Gamma}$ in such a way that $u^{\mu}_{(A)}(s_0) = \bar{u}^{\mu}_{(A)}(s_0)$. Now, the two Serret-Frenet tetrads $\{u^{\alpha}_{(A)}(s)\}, \{\bar{u}^{\alpha}_{(A)}(s)\}$ satisfy the equations

$$\begin{bmatrix} \frac{du_{(0)}}{ds} \\ \frac{du_{(1)}}{ds} \\ \frac{du_{(2)}}{ds} \\ \frac{du_{(3)}}{ds} \end{bmatrix} = \begin{bmatrix} 0 & k & 0 & 0 \\ k & 0 & \tau_1 & 0 \\ 0 & -\tau_1 & 0 & \tau_2 \\ 0 & 0 & -\tau_2 & 0 \end{bmatrix} \begin{bmatrix} u_{(0)} \\ u_{(1)} \\ u_{(2)} \\ u_{(3)} \end{bmatrix} \tag{4}$$

and

$$\begin{bmatrix} \frac{d\bar{u}_{(0)}}{ds} \\ \frac{d\bar{u}_{(1)}}{ds} \\ \frac{d\bar{u}_{(2)}}{ds} \\ \frac{d\bar{u}_{(3)}}{ds} \end{bmatrix} = \begin{bmatrix} 0 & k & 0 & 0 \\ k & 0 & \tau_1 & 0 \\ 0 & -\tau_1 & 0 & \tau_2 \\ 0 & 0 & -\tau_2 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{(0)} \\ \bar{u}_{(1)} \\ \bar{u}_{(2)} \\ \bar{u}_{(3)} \end{bmatrix} \tag{5}$$

which can be written in a more compact form as

$$\frac{du_{(A)}}{ds} = \sum_A^{\cdot B} u_{(B)} \tag{6}$$

$$\frac{d\bar{u}_{(A)}}{ds} = \sum_A^{\cdot B} \bar{u}_{(B)}$$

with $\Sigma_A^{\cdot B}$ denoting the elements of the Serret-Frenet matrix. Clearly, the two tetrads $\{u_{(A)}^{\alpha}(s)\}$, $\{\bar{u}_{(A)}^{\alpha}(s)\}$ are related by an equation of the type

$$\bar{u}_{(A)}(s) = \Lambda_A^{\cdot B}(s) u_B(s) \tag{7}$$

with the elements of the matrix $\Lambda(s)$ satisfying the condition $\Lambda_A^{\cdot B}(s_0) = \delta_B^A$, since we are assuming that $u_{(A)}^{\alpha}(s_0) = \bar{u}_{(A)}^{\alpha}(s_0)$. From (6) and (7) we obtain a system of first order differential equations for the elements of Λ given by

$$\frac{d\Lambda_A^{\cdot B}}{ds} + \Lambda_A^{\cdot C}\Sigma_C^{\cdot B} - \Sigma_A^{\cdot C}\Lambda_C^{\cdot B} = 0 \tag{8}$$

By assumption, $\Lambda_A^{\cdot B}$ are differentiable functions of the proper parameter s. From the theory of ordinary differential equations [11], we know that if we are given a set of initial conditions $\Lambda_A^{\cdot B}(s_0)$, then the above system admits a unique solution $\{\Lambda_A^{\cdot B} = \Lambda_A^{\cdot B}(s)\}$ defined in a open interval $J \subset I$ containing s_0. On the other hand, it is easily seen that $\Lambda_A^{\cdot B}(s) = \delta_B^A$ is a solution of(8). Therefore, we conclude that $\bar{u}_{(A)}(s) = u_A(s)$.

Other extensions of known theorems of the differential geometry of curves in Euclidean space $\mathbb{R}^3$ can easily be carried over into Minkowski space. As an example, let us consider the following results on curves lying in $\mathbb{R}^3$:

A curve, with non vanishing curvature, is plane if and only if its torsion vanish identically [5]. Natural extensions of this result to Minkowski space are given by the following theorem:

Theorem 2 A time-like curve Γ, with non vanishing curvature, lies in a hyperplane if and only if the second torsion vanishes identically.

Proof. Again we restrict ourselves to time-like curves. Let us start with the necessary condition. Suppose the curve Γ lies in a hyperplane. Then, by a Lorentz rotation we can align one of the coordinate axes with the normal direction to the hyperplane. For the sake of the argument, let us assume that we can bring Γ to lie, say, in the (x^0, x^1, x^2)-hyperplane. Then the parametric equations of Γ are of the form $x^{\alpha} = x^{\alpha}(s) = (x^0(s), x^1(s), x^2(s), 0)$. Let $\{e_{(\alpha)}\}$ denote the vectors of the canonical coordinate basis. Thus, in these coordinates, $u_{(0)} = \frac{dx^0}{ds}e_{(0)} + \frac{dx^1}{ds}e_{(1)} + \frac{dx^2}{ds}e_{(2)}$ and $\frac{du_{(0)}}{ds} = \frac{d^2x^0}{ds^2}e_{(0)} + \frac{d^2x^1}{ds^2}e_{(1)} + \frac{d^2x^2}{ds^2}e_{(2)}$. From (3) we have $\frac{du_{(0)}}{ds} = ku_{(1)}$. Given that $k \neq 0$ we conclude that $u_{(1)}$ has no components in the x^3-direction, i.e., $u_{(1)} = (u_{(1)}^0, u_{(1)}^1, u_{(1)}^2, 0)$. Thus $\frac{du_{(1)}}{ds} = \frac{du_{(1)}^0}{ds}e_{(0)} + \frac{du_{(1)}^1}{ds}e_{(1)} + \frac{du_{(1)}^2}{ds}e_{(2)}$, hence from the equation $\frac{du_{(1)}}{ds} = ku_{(0)} + \tau_1 u_{(2)}$ we conclude that $\tau_1 u_{(2)}^3 = 0$. If $\tau_1 = 0$, then τ_2 also must vanish, for in this case $u_{(2)}$ is chosen to be constant. If $\tau_1 \neq 0$, then $u_{(2)}^3 = 0$, hence $u_{(2)} = \{u_{(2)}^0, u_{(2)}^1, u_{(2)}^2, 0\}$. From the equation $\frac{du_{(2)}}{ds} = \frac{du_{(2)}^0}{ds}e_{(0)} + \frac{du_{(2)}^1}{ds}e_{(1)} + \frac{du_{(2)}^2}{ds}e_{(2)}$ and the third Serret-Frenet equation $\frac{du_{(2)}}{ds} = -\tau_1 u_{(1)} + \tau_2 u_{(3)}$ we are led to conclude that $\tau_2 u_{(3)}^3 = 0$. However, $u_{(3)}^3$ cannot be zero, otherwise the set of vectors $\{u_{(0)}, u_{(1)}, u_{(2)}, u_{(3)}\}$ would not be linearly independent. Therefore, τ_2 must vanish.

Let us turn to sufficient condition. Suppose that $\tau_2 = 0$. Then, the fourth Serret-Frenet equation implies that $u_{(3)}$ is a constant vector. Let us conveniently choose our coordinate system in such a way that $e_{(3)} = u_{(3)}$. Now, since $u_{(0)}$ is orthogonal to $u_{(3)}$ we must have $u_{(0)}^3 = 0$, which means that Γ lies in the hyperplane $x^3 = const$. This completes the proof. ■

Another extension of theorem 2 leads directly to the following proposition:

Proposition. A time-like curve Γ, with non-vanishing curvature, is plane if and only if first and second torsions vanish identically.

Since it follows the same lines of reasoning presented in the proof of theorem 2 that's why we omit the proof.

IV. Mathematical Development

A local coordinate system having the property $T = i'$ supports the definition of N :

$$N = j'\cos\theta + k'\sin\theta \tag{9}$$

The curvature k and the angle of rotation θ of the plane (containing N and T) characterize the curve. When the plane containing T and the global coordinate j is normal to k' (figure 2) then

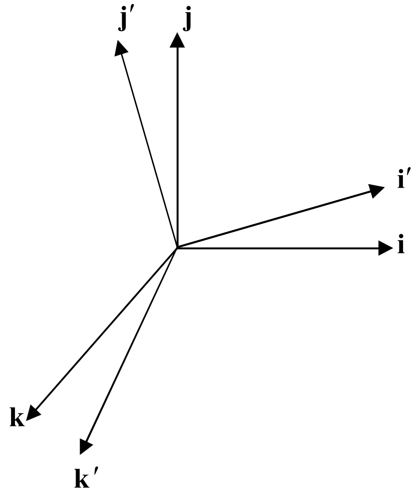

Fig. 2. Angular orientation of the local coordinate system with respect to the global coordinate system.

$$k' = \frac{T \times j}{|T \times j|} = \frac{-iT_k + kT_i}{\sqrt{1 - T_j^2}} \tag{10}$$

Equation (10) breaks down when $T = \pm j$, requiring an alternate expression. However, when $T \neq \pm j$,

$$j' = k' \times T = \frac{-iT_iT_j + j(1 - T_j^2) - kT_jT_k}{\sqrt{1 - T_j^2}} \tag{11}$$

Substituting (11) and (10) into (9):

$$N_i = \frac{1}{\kappa}\frac{dT_i}{ds} = \frac{-T_k \sin\theta - T_iT_j \cos\theta}{\sqrt{1 - T_j^2}}; \tag{12}$$

$$N_j = \frac{1}{\kappa}\frac{dT_j}{ds} = \cos\theta\sqrt{1 - T_j^2}; \tag{13}$$

$$N_k = \frac{1}{\kappa}\frac{dT_k}{ds} = \frac{T_i \sin\theta - T_jT_k \cos\theta}{\sqrt{1 - T_j^2}}; \tag{14}$$

Equation (13) can be integrated directly:

$$\int_{T_{j0}}^{T_j} \frac{dT_j}{\sqrt{1 - T_j^2}} = \int_{s0}^{s} \kappa \cos\theta d\sigma; \tag{15}$$

Leading to

$$\sin^{-1} T_j = \sin^{-1} T_{j0} + \int_{s0}^{s} \kappa \cos\theta d\sigma;$$

$$T_j = \sin\left[\sin^{-1} T_{j0} + \int_{s0}^{s} \kappa \cos\theta d\sigma\right] = \sin\delta;$$

$$T_j = T_{j0} \cos\int_{s0}^{s} k \cos\theta d\sigma + \sqrt{1 - T_j^2} \sin\int_{s0}^{s} k \cos\theta d\sigma \tag{16}$$

Equation (12) is solved by noting that $T_k = \sqrt{1 - T_j^2 - T_i^2} = \sqrt{\cos^2\theta - T_i^2}$ and introducing the variable β so that

$$T_i = \cos\delta \cos\beta; \tag{17}$$

$$T_K = \cos\delta \sin\beta; \tag{18}$$

Substituting into (12):

$$\frac{dT_i}{ds} = -\kappa \cos\theta \sin\delta \cos\beta - \cos\delta \sin\beta \frac{d\beta}{ds}$$

$$= \frac{-\kappa \sin\theta \cos\delta \sin\beta - \kappa \cos\theta \cos\delta \cos\beta \sin\delta}{\cos\delta}$$

$$= -\kappa \sin\theta \sin\beta - \kappa \cos\theta \cos\beta \sin\delta. \tag{19}$$

Equation (19) simplifies to

$$\frac{d\beta}{ds} = \frac{\kappa \sin\theta}{\cos\delta}; \tag{20}$$

Or,

$$\beta = \beta_0 + \int_{s0}^{s} \frac{\kappa \sin\theta}{\cos\delta} d\sigma = \cos^{-1}\left(\frac{T_i}{\cos\delta}\right); \tag{21}$$

so that

$$T_i = T_{i0} \frac{\cos\delta}{\cos\delta_0} \cos\int_{s0}^{s} \frac{\kappa \sin\theta}{\cos\delta} d\sigma - T_{k0} \frac{\cos\delta}{\cos\delta_0} \cos\int_{s0}^{s} \frac{\kappa \sin\theta}{\cos\delta} d\sigma; \tag{22}$$

where

$$\cos\delta = \sqrt{1 - T_{j0}^2} \cos\int_{s0}^{s} \kappa \cos\theta d\sigma - T_{j0} \sin\int_{s0}^{s} \kappa \cos\theta d\sigma. \tag{23}$$

The solution for T_k follows from (18) and (20):

$$T_k = T_{k0}\frac{\cos\delta}{\cos\delta_0}\cos\int_{s0}^{s}\frac{\kappa\sin\theta}{\cos\delta}d\sigma - T_{i0}\frac{\cos\delta}{\cos\delta_0}\cos\int_{s0}^{s}\frac{\kappa\sin\theta}{\cos\delta}d\sigma; \tag{24}$$

It can be easily verified that (16), (22), and (24) meet the requirement:

$$\kappa = \left|\frac{dT}{ds}\right|. \tag{25}$$

Generating an expression for the torsion τ requires first computing N by substituting (16) - (18) into (12) - (14):

$$N_i = -\cos\theta\sin\delta\cos\beta - \sin\beta\sin\theta; \tag{26}$$

$$N_j = \cos\theta\cos\delta; \tag{27}$$

$$N_k = -\cos\theta\sin\delta\sin\beta + \cos\beta\sin\theta; \tag{28}$$

Next, $B = T \times N$:

$$B_i = \sin\delta\sin\theta\cos\beta - \cos\theta\sin\beta; \tag{29}$$

$$B_j = -\cos\delta\sin\theta; \tag{30}$$

$$B_k = \sin\delta\sin\theta\sin\beta + \cos\theta\cos\beta; \tag{31}$$

Equation (32) expresses the torsion as a function of θ:

$$\tau = \left|\frac{dB}{ds}\right| = \frac{d\theta}{ds} - \kappa\tan\delta\sin\theta. \tag{32}$$

Equation (33) expresses τ in terms of components of T and B:

$$\tau = \frac{d\theta}{ds} + \frac{\kappa T_j B_j}{1 - T_j^2}. \tag{33}$$

Integrating (33) leads to the following expression for θ:

$$\theta = \theta_0 + \int_{s0}^{s}\left(\tau - \frac{\kappa T_j B_j}{1 - T_j^2}\right)d\sigma. \tag{34}$$

Equation (34) indicates a unique value of θ for each specified value of τ when $T_j \neq \pm 1$. Thus, (18), (22), and (24) indirectly solve Serret-Frenet equations.

The angle θ can also be expressed in terms of components of T, N, B:

$$\theta = -\sin^{-1}\frac{B_j}{\sqrt{1 - T_j^2}} = \cos^{-1}\frac{N_j}{\sqrt{1 - T_j^2}} = -\tan^{-1}\frac{B_j}{N_j}. \tag{35}$$

An explicit solution often results when θ is constant. Setting $T_{i0} = 1$, so that $\beta_0 = \delta_0 = 0$ (and setting $s_0 = 0$) leads to

$$\delta = \int_0^s \kappa(\sigma)\cos\theta_0 d\sigma; \tag{36}$$

$$\beta = 2\tan\theta_0 \tanh^{-1}(\tan\delta/2); \tag{37}$$

so that

$$T_i = \cos[2\tan\theta_0 \tanh^{-1}(\tan\delta/2)]\cos\int_0^s \kappa(\sigma)\cos\theta_0 d\sigma; \tag{38}$$

$$T_j = \sin\int_0^s \kappa(\sigma)\cos\theta_0 d\sigma; \tag{39}$$

$$T_k = \sin[2\tan\theta_0 \tanh^{-1}(\tan\delta/2)]\cos\int_0^s \kappa(\sigma)\cos\theta_0 d\sigma; \tag{40}$$

The torsion becomes

$$\tau(s) = -\kappa(s)\sin\theta_0 \tan\int_0^s \kappa(\sigma)\cos\theta_0 d\sigma \tag{41}$$

As an example, when

$$\kappa = \kappa_0 e^{-s^2}, \tag{42}$$

$$T_i = \cos\left[\frac{\kappa_0\sqrt{\pi}}{2}erf(s)\cos\theta_0\right]\cos \left[2\tan\theta_0 \tanh^{-1}\left(\tan\left[\frac{\kappa_0\sqrt{\pi}}{4}erf(s)\cos\theta_0\right]\right)\right]; \tag{43}$$

$$T_j = \sin\left[\frac{\kappa_0\sqrt{\pi}}{2}erf(s)\cos\theta_0\right]; \tag{44}$$

$$T_k = \cos\left[\frac{\kappa_0\sqrt{\pi}}{2}erf(s)\cos\theta_0\right]\sin \left[2\tan\theta_0 \tanh^{-1}\left(\tan\left[\frac{\kappa_0\sqrt{\pi}}{4}erf(s)\cos\theta_0\right]\right)\right]; \tag{45}$$

$$\tau(s) = -\kappa_0 e^{-s^2} \sin\theta_0 \tan\left[\frac{\kappa_0\sqrt{\pi}}{2} erf(s)\cos\theta_0\right] \quad (46)$$

When $\kappa = \kappa_0$ but $\theta \neq \theta_0$, the solution will typically involve undetermined integrals. For example, when $\kappa = \kappa_0$ and $\theta = \kappa_0 s$,

$$T_i = \cos(\sin\kappa_0 s)\cos\int_0^s \frac{\kappa_0 \sin\kappa_0\sigma}{\cos(\sin\kappa_0\sigma)} d\sigma; \quad (47)$$

$$T_j = \sin(\sin\kappa_0 s); \quad (48)$$

$$T_k = \cos(\sin\kappa_0 s)\sin\int_0^s \frac{\kappa_0 \sin\kappa_0\sigma}{\cos(\sin\kappa_0\sigma)} d\sigma; \quad (49)$$

and

$$\tau(s) = \kappa_0 - \kappa_0 \tan(\sin\kappa_0 s)\sin\kappa_0 s. \quad (50)$$

When $\kappa = \kappa_0$ and $\theta = \theta_0$, (38)-(41) become:

$$T_i = \cos(\kappa_0 s\cos\theta_0)\cos\left[2\tan\theta_0 \tanh^{-1}\left(\tan\left[\frac{\kappa_0 s}{2}\cos\theta_0\right]\right)\right]; \quad (51)$$

$$T_j = \sin(\kappa_0 s\cos\theta_0); \quad (52)$$

$$T_k = \cos(\kappa_0 s\cos\theta_0)\sin\left[2\tan\theta_0 \tanh^{-1}\left(\tan\left[\frac{\kappa_0 s}{2}\cos\theta_0\right]\right)\right]; \quad (53)$$

$$\tau(s) = -\kappa_0 \sin\theta_0 \tan(\kappa_0 s\cos\theta_0) \quad (54)$$

When $\theta_0 = \pi/2$, $\tau(s) = 0$, confining T and N to a plane. When T align with $j, \tau \rightarrow \infty$ in (54), and the equations break down.

V. Conclusion

Due to the fundamental theorem of curves, if a curve represents the motion of a particle, one can look at the Serret-Frenet equations as containing complete information on the dynamics of the particle. Such correspondence between the geometry of curves and the dynamics of particle can be nicely explored in the context of special relativity to study the intrinsic geometry of world lines in Minkowski space. In fact, compared to the Newtonian formalism, special relativity is a more natural setting for a description of motion through the Serret-Frenet equations, since the worldliness of particles are usually parameterized by the arc length parameter s, turning to the equations into a much simpler form.

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