Dhaka Univ. J. Sci. 61(1): 93-96, 2013 (January)

On Submanifolds of (κ, μ) – Contact Metric Manifolds

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Received on 28. 03. 2012. Accepted for Published on 04. 11. 2012

Abstract

The object of the present paper is to obtain conditions of integrability of some distributions of semi-invariant submanifolds of contact metric manifolds with ξ belonging to (κ, μ) -nulity distribution.

Keywords and Phrases: (κ, μ) – Contact Metric Manifolds, Semi-invariant submanifolds, Integrable distributions.

The Mathematics Subject Classification: 53C 15, 53C 40

I. Introduction

In 1981 A. Bejancu [1] introduced the study of semi-invariant submanifolds, as a generalization of invariant and anti –invariant submanifolds of contact metric manifolds. This concept was further generalized as generic [10], almost CR [7], generalized CR [8], and almost semi-invariant [2] submanifolds. In 1983 Papaghuic [12] introduced almost semi-invariant submanifolds of almost contact metric manifolds. Submanifolds of contact metric manifolds have been studied by several authors [9], [15].

In 1995 Blair, Koufogiorgos and Papantoniou [4] introduced the notion of contact metric manifolds with ξ belonging to (κ, μ) – nullity distribution which are also known as (κ, μ) – contact metric manifolds. They obtained several results and examples of such manifolds and a full classification of these manifolds has been given by E. Boeckx [5]. Before Boeckx [5], two classes of non-Sasakian (κ, μ) – contact metric manifolds were known. The first class consists of the unit tangent sphere bundles of spaces of constant curvature, equipped with their natural contact metric structure and the second class contains all the three-dimensional unimodular Lie groups, except the commutative one, admitting the structure of a left invariant (κ, μ) – contact metric manifolds [4], [5], [13]. One of the peculiarities of these manifolds is that its tangent space give rise to three mutually orthogonal distributions corresponding to the eigenspaces of M so that these manifolds are endowed with a bi-Legendrian structure [11]. This is why the distributions of (κ, μ) -contact metric manifolds are seemed to the interesting as a field of study.

The aim of the present paper is to study the integrability of some distributions of a semi-invariant submanifold of a (κ, μ) –contact metric manifolds. The present paper is organized as follows:

In Section 2, we give the required preliminary results of contact metric manifolds and discuss about (κ, μ) – contact metric manifolds. Section 3 contains the notion of semi-invariant submanifolds. Section 4 deals with the integrability conditions of the distributions $D \oplus \xi, D^{\perp} \oplus \xi$ and D of semi-invariant submanifolds of (κ, μ) – contact metric manifolds.

II. Preliminaries

In this section, we recall some general definitions and basic formulas.

An odd dimensional Riemannian manifold $(\widetilde{M}, g)$ is said to be an almost contact metric manifold if there exists on $\widetilde{M}$ a (1, 1) tensor field ϕ, a vector field ξ and a 1 – form η such that [3]

$$\phi^2(X) = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ g(X, \xi) = \eta(X) \quad (2.1)$$

$$\phi\xi = 0,\ \eta\phi = 0,\ g(X, \phi Y) = -g(\phi X, Y), \quad (2.2)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.3)$$

for any vector fields X, Y on $\widetilde{M}$.

Such a manifold is said to be a contact metric manifold if $d\eta = \Phi$, where $\Phi(X, Y) = g(X, \phi Y)$ is called the fundamental 2–form of $\widetilde{M}$. If, in addition, ξ is a killing vector field, then the manifold is said to be a K – contact manifold. It is well known that a contact metric manifold is K – contact if and only if $\nabla_X \xi = -\phi X$, for any vector field on $\widetilde{M}$. On the other hand the almost contact metric structure of $\widetilde{M}$ is said to be normal if $N^1 = 0$, where N^1 is a tensor field of type (1, 2) given by

$$N^1(X, Y) = 2d\eta(X, Y)\xi + [\phi, \phi](X, Y), \quad (2.4)$$

where $[\phi, \phi]$ denotes the Nijenhuis torsion of ϕ, given by

$$[\phi, \phi](X, Y) = ((\widetilde{\nabla}_{\phi X}\phi)Y - (\widetilde{\nabla}_{\phi Y}\phi)X) - \phi((\widetilde{\nabla}_X\phi)Y - (\widetilde{\nabla}_Y\phi)X) \quad (2.5)$$

for any vector fields X, Y tangent to $\tilde{M}$ [3]. A normal contact metric manifold $\tilde{M}$ is called a Sasakian manifold. It can be proved that an almost contact metric manifold Sasakian if and only if

$$(\tilde{\nabla}_X \phi) Y = g(X,Y)\,\xi - \eta(Y)\,X\,,$$

for any vector fields X, Y on $\tilde{M}$.

Given a contact metric manifold $\tilde{M}\ (\phi, \xi, \eta, g)$. We define a (1, 1) tensor field h by $h = \frac{1}{2} L_\xi \phi$, where L denotes Lie differentiation. Then h satisfies $h\phi = -\phi\, h$. Thus if λ is an eigenvalue of h with eigenvector X, $-\lambda$ is also an eigenvalue with eigenvector ϕX. Also we have $Trh = Tr\phi h = 0$ and $h\xi = 0$. Moreover, if $\tilde{\nabla}$ denotes the Riemannian connection of g, then the following relation holds:

$$\tilde{\nabla}_X \xi = -\phi X - \phi h X\,. \tag{2.6}$$

It is seen that the vector field ξ is a Killing vector field with respect to g if and only if $h = 0$. In this case the manifold becomes a K− contact manifold. A K− contact structure on $\tilde{M}$ gives rise to an almost complex structure on the product $\tilde{M} \times \Re$. If this almost complex structure is integrable, the contact metric manifold is Sasakian. Equivalently, a contact metric manifold becomes Sasakian if and only if

$$\tilde{R}(X,Y)\,\xi = \eta(Y)X - \eta(X)Y\,,$$

holds for all X, $Y \in T\tilde{M}$, where $\tilde{R}$ denotes the curvature tensor of the manifold $\tilde{M}$.

The (κ, μ) − nullity distribution of a contact metric manifold $\tilde{M}\ (\phi, \xi, \eta, g)$ is a distribution [4]

$$N(\kappa,\mu):\ p \rightarrow N_p(\kappa,\mu) =$$

$$\{ Z \in T_p(M) : R(X,Y)Z = \kappa(g(Y,Z)X - g(X,Z)Y) + \mu(g(Y,Z)hX - g(X,Z)hY) \}\,, \tag{2.7}$$

for any $X, Y \in T_p\tilde{M}$. Hence if the characteristic vector field ξ belongs to the (κ, μ) − nullity distribution, then we have

$$R(X,Y)\,\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY] \tag{2.8}$$

In particular, if $\mu = 0$, then the notion of (κ, μ) − nullity distribution reduces to κ − nullity distribution, introduced by S. Tanno [14].

In a (κ, μ) − contact metric manifold we have the following [4]:

$$h^2 = (\kappa - 1)\,\phi^2\,, \quad \kappa \le 1 \tag{2.9}$$

$$\tilde{\nabla}_X \phi\, Y - \phi \tilde{\nabla}_X Y = (\tilde{\nabla}_X \phi)\,(Y) = g(X + hX, Y)\,\xi - \eta(Y)\,(X + hX) \tag{2.10}$$

III. Semi-invariant Submanifolds

Let M be a submanifold of a Riemannian manifold $\tilde{M}$ with a Riemannian metric g. Then Gauss and Weingarten formulae are given respectively by [6]

$$\tilde{\nabla}_X Y - \nabla_X Y + H(X,Y),\quad (X, Y \in TM), \tag{3.1}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}{}_X N,\quad (N \in T^{\perp}M), \tag{3.2}$$

where $\tilde{\nabla}$, ∇ and $\nabla^{\perp}$ are respectively the Riemannian ,induced Riemannian and induced normal connections in $\tilde{M}, M$ and the normal bundle $T^{\perp}M$ of M respectively, and H is the second fundamental form related to A by $g(H(X,Y), N) = g(A_N X, Y)$. (3.3)

Moreover, if ϕ is a $(1,1)$ tensor field on $\tilde{M}$, for $X \in TM$ and $N \in T^{\perp}M$ we have

$$(\tilde{\nabla}_X \phi)Y = ((\nabla_X P)Y - A_{FY}X - tH(X,Y)) +$$

$$((\nabla_X F)Y + H(X,PY) \quad fH(X,Y)) \tag{3.4}$$

$$(\tilde{\nabla}_X \phi)N = ((\nabla_X t)N - A_{FN}X - PA_N X) - ((\nabla_X f)N + H(X,tN) - FA_N X) - FA_N X) \tag{3.5}$$

where

$$\phi X = PX + FX,\ (PX \in TM, FX \in T^{\perp}M \tag{3.6}$$

$$\phi N = tN + fN,\ (tN \in TM, fN \in T^{\perp}M \tag{3.7}$$

and

$$(\nabla_X P)Y \equiv \nabla_X PY - P\nabla_X Y, \qquad (\nabla_X FY) \equiv \nabla^{\perp}{}_X FY - F\nabla_X Y$$

$$(\nabla_X t)N \equiv \nabla_X tN - t\nabla^{\perp}{}_X N, \qquad (\nabla_X f)N \equiv \nabla_X{}^{\perp} FN - \nabla^{\perp}{}_X N$$

Let M be a submanifold of an almost contact metric manifold. If $\xi \in TM$, then we can write $TM = \{\xi\} + \{\xi\}^{\perp}$, where $\{\xi\}$ is the distribution spanned by ξ and $\{\xi\}^{\perp}$ is the complementary distribution of $\{\xi\}$ in M. Then we have

$$P\xi = 0 = F\xi,\ \ \eta \circ P = \eta \circ F = 0 \tag{3.8}$$

$$P^2 + tF = -I + \eta \otimes \xi,\ \ FP + fF = 0 \tag{3.9}$$

$$f^2 + Ft = -I,\ \ tf + PT = 0 \tag{3.10}$$

Definition 3.1. A submanifold M of a (κ,μ)-contact metric manifold $\tilde{M}$ with $\xi \in TM$ is called a semi-invariant submanifold [2] of M if there exists two differentiable distributions D and $D^{\perp}$ on M such that

(1) $TM = D \oplus D^{\perp} \oplus \{\xi\}$,
(2) the distribution D is invariant by ϕ, that is ,$\phi(D) = D$, and
(3) the distribution $D^{\perp}$ is anti-invariant by ϕ, that is ,$\phi(D^{\perp}) \subset T^{\perp}M$.

Here,

$$D \oplus \{\xi\} = \ker\{F\}, \quad D^{\perp} \oplus \{\xi\} = \ker\{P\} \tag{3.11}$$

Definition 3.2. If for $X, Y \in D, \nabla_X Y \in D$ then the distribution D is called autoparallel. If a distribution is autoparallel then it is integrable.

IV. Integrability of the Distributions $D \oplus \{\xi\}$ and $D^{\perp} \oplus \{\xi\}$

Interchanging X and Y in (2.10) we get

$$(\tilde{\nabla}_Y \phi)(X) = g(Y + hY, X)\xi - \eta(X)(Y + hY). \tag{4.1}$$

Subtracting the above equation from (2.10) we obtain

$$\tilde{\nabla}_X \phi Y - \tilde{\nabla}_Y \phi X - \phi[X,Y] = g(X + hX, Y)\xi - g(Y + hY, X)\xi - \eta(Y)(X + hX) + \eta(X)(Y + hY). \tag{4.2}$$

From above it follows that

$$-\phi[X,Y] = \tilde{\nabla}_Y \phi X - \tilde{\nabla}_X \phi Y + g(X + hX, Y)\xi - g(Y + hY, X)\xi - \eta(Y)(X + hX) + \eta(X)(Y + hY). \tag{4.3}$$

In view of (3.6), (4.3) takes the form

$$-P[X,Y] - F[X,Y] = \tilde{\nabla}_Y PX + \tilde{\nabla}_Y FX - \tilde{\nabla}_X PY - \tilde{\nabla}_X FY + g(X + hX, Y)\xi - -g(Y + hY, X)\xi - \eta(Y)(X + hX) + \eta(X)(Y + hY). \tag{4.4}$$

By virtue of (3.1) and (3.2), (4.4) yields

$$-P[X,Y] - F[X,Y] = H(Y, PX) + \nabla_Y PX - A_{FX} Y + \nabla^{\perp}{}_Y FX - g(Y + hY, X)\xi - \eta(Y)(X + hX) + \eta(X)(Y + hY) \tag{4.5}$$

In any contact metric manifold we have

$$hX = \tfrac{1}{2}(L_\xi F)X + \tfrac{1}{2}(L_\xi F)X \tag{4.6}$$

By virtue of (4.5) and (4.6) we have

$$H(X, PY) - \nabla_X PY + A_{FY} X - \nabla^{\perp}{}_X FY + g(X + hX, Y)\xi - g(Y + hY, X)\xi + \eta(X)\tfrac{1}{2}(L_\xi P)Y - \eta(Y)\tfrac{1}{2}(L_\xi P)X + \eta(X)\tfrac{1}{2}(L_\xi F)Y - \eta(Y)\tfrac{1}{2}(L_\xi F)X + \eta(X)Y - \eta(Y)X. \tag{4.7}$$

For $X, Y, \xi \in TM$, comparing the tangential and normal components we obtain from (4.7)

$$-P[X,Y] = \nabla_Y PX - A_{FX} Y - \nabla_X PY + A_{FY} X + g(X + hX, Y)\xi - g(Y + hY, X)\xi + \eta(X)\tfrac{1}{2}(L_\xi P)Y - \eta(Y)\tfrac{1}{2}(L_\xi P)X + \eta(X)Y - \eta(Y)X - n(Y)X \tag{4.8}$$

$$-F[X,Y] = H(Y, PX) - H(X, PY) + \nabla^{\perp}{}_Y FX - \nabla^{\perp}{}_X FY + \eta(X)\tfrac{1}{2}(L_\xi F)Y - \eta(Y)\tfrac{1}{2}(L_\xi F)X. \tag{4.9}$$

Let $X, Y \in D^{\perp} \oplus \{\xi\} = \ker\{P\}, PX = 0, PY = 0$. Then in view of (4.8) we can state the following:

Theorem 4.1. The distribution $D^{\perp} \oplus \{\xi\}$ of a semi-invariant submanifold of a (κ,μ)-contact metric manifold is integrable if and only if

$$A_{FY} X + \eta(X)Y = A_{FX} Y + \eta(Y)X.$$

Let $X, Y \in D \oplus \{\xi\}$. Then for $D + \{\xi\} = \ker\{F\}, FX = 0$ and $FY = 0$. Hence in view of (4.9) we can state the following:

Theorem 4.2. The distribution $D \oplus \{\xi\}$ of a semi-invariant submanifold of a (κ,μ)-contact metric manifold is integrable if and only if

$$H(Y, PX) - H(X, PY)$$

Changing X by ϕX in (2.10) we get

$$(\tilde{\nabla}_{\phi X} \phi)Y = g(\phi X + h\phi X, Y)\xi - \eta(Y)(\phi X + h\phi X) \tag{4.10}$$

Interchanging X and Y in (4.10) we get

$$(\tilde{\nabla}_{\phi Y} \phi)X = g(\phi Y + h\phi Y, X)\xi - \eta(X)(\phi Y + h\phi Y) \tag{4.11}$$

Subtracting (4.10) from (4.11) we get

$$((\tilde{\nabla}_{\phi X} \phi)Y - (\tilde{\nabla}_{\phi Y} \phi)X) = g(\phi X + h\phi X, Y)\xi - \eta(Y)(\phi X + h\phi X) - g(\phi Y + h\phi Y, X)\xi + \eta(X)(\phi Y + h\phi Y) \tag{4.12}$$

Next we can easily get $\phi((\tilde{\nabla}_Y \phi)X = \phi\tilde{\nabla}_Y \phi X - \phi^2 \tilde{\nabla}_Y X$
$= \phi(\nabla_Y \phi X) + H(Y, \phi X)) + \tilde{\nabla}_Y X - \eta(\tilde{\nabla}_Y X)\xi$ (4.13)

In view of (4.13), it follows that

$$\phi((\tilde{\nabla}_Y \phi)X - (\tilde{\nabla}_X \phi)Y) = -[X,Y] + \eta([X,Y]\xi) + P(\nabla_Y \phi X - \nabla_X \phi Y) + + F(\nabla_Y \phi X - \nabla_X \phi Y) + \phi(H(Y, \phi X) - H(X, \phi Y)) \tag{4.14}$$

In view of (4.12) and (4.14) we obtain by using (2.4) and (2.5)

$$N^{\perp}(X,Y) = 2d\eta(X,Y)\xi + g(\phi X + h\phi X, Y)\xi - \eta(Y)(\phi X + h\phi X) - g(\phi Y + h\phi Y, X)\xi + \eta(X)(\phi Y + h\phi Y) + [X,Y] - \eta([X,Y]\xi) - P(\nabla_Y \phi X - \nabla_X \phi Y) -$$

$$F(\nabla_Y \phi X - \nabla_X \phi Y) - \phi(H(Y,\phi X) - H(X,\phi Y)) \quad (4.15)$$

Let $X, Y \in D \oplus \{\xi\} = \ker\{F\}$. Hence $FX = 0$ and $FY = 0$. In order to make the distribution $D \oplus \{\xi\}$ integrable one should have $N^1 \in D \oplus \{\xi\}$ and $H(Y, PX) = H(X, PY)$. But from (3.6) we see that

$$F(\nabla_Y PX - \nabla_X PY) \in D \oplus \{\xi\} \quad (4.16)$$

In view of the expression of $N^1(X,Y)$, we can conclude the following:

Theorem 4.3. In a semi-invariant submanifold of a (κ,μ)-contact metric manifold if the distribution $D \oplus \{\xi\}$ is integrable then

$$F(\nabla_Y PX - \nabla_X PY) \neq 0 \quad (4.17)$$

Proof. Applying (2.10) in the equation (2.5) we obtain

$$[\varphi,\phi](X,Y) = g(\phi X + h\phi X, Y)\xi - \eta(Y)(\phi X + h\phi X) - g(\phi Y + h\phi Y, X)\xi + \eta(X)(\phi Y + h\phi Y) - \phi(g(X + hX)\xi) + \eta(Y)(X + hX) + \phi(g(Y + hY, X)\xi) - \phi(\eta(X)(Y + hY)) \quad (4.18)$$

If possible, let D be integrable. Then for [X, Y] $\in D$, it follows that $d\eta(X,Y) = 0$ and $[\varphi,\phi](X,Y)] \subset D$.

Now in view of (4.18) it follows that

$$\eta(N^1(X,Y)) = \eta([\varphi,\phi](X,Y) + 2d\eta(X,Y)\xi)$$
$$= g(\phi X + h\phi X, Y) - g(\phi Y + h\phi Y, X) + \eta(Y)\eta(X + hX)$$
$$= g(\phi X + h\phi X, Y) + g(Y - hY, \phi X) + \eta(Y)\eta(X + hX) \quad (4.19)$$

Replacing Y by PX we obtain from above

$$0 = \eta(N^1(X,PX)) = g(PX + hPX, PX) + g(PX - hPX, PX)$$
$$, PX) = 2g(PX,PX) \neq 0 \quad (4.20)$$

Thus we arrive at a construction for our assumption that is D is integrable. This leads us to state the following:

Theorem 4.4. The distribution D of a semi-invariant submanifold of a (κ,μ)-contact metric manifold is not integrable.

Since every autoparallel distribution is integrable it follows that the distribution D is not autoparallel because it is not integrable. Hence we can state the following:

Corollary 4.1. The distribution D of a semi-invariant submanifold of a (κ,μ)-contact metric manifold is not autoparallel.

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