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A Procedure for Solving Quadratic Programming Problems

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Abstract

In this paper, we study on the well-known procedure of quadratic programming (QP) and its corresponding linear programming (LP) problem. We then introduce a LP problem corresponding to the QP problem. Unfortunately, an unboundedness question arises into the new converting LP problem. We then modify the converted LP problem that overcomes the unboundedness. We introduce a general computer technique that can be solved the QP problem. An example is given to clarify the procedure and the computer technique.

Keywords: Linear Programming, Quadratic Programming, Karush-Kuhn-Tucker Conditions, Computer Algebra

I. Introduction

In the literature of operation research LP, a specific class of mathematical problems, in which a linear function is maximized (or minimized) subject to given linear constraints has widely recognized in this field. This LP problem is also broad enough to encompass many interesting and important applications, yet specific enough to be tractable even if the number of variables is very large. In this sense it is always interesting to convert any optimization problem into a LP problem. However, there are a number of methods for solving the linear fractional programming (LFP) problems by converting it into LP problems.

Among them the transformation technique developed by Charnes and Cooper, the simplex type algorithm introduced by Swarup and Bitranand Novae's method are widely accepted. Tantawy developed a technique with the dual solution. However, there are a very few methods for solving the QP problems to convert it into LP problems.

But when considering the real-world applications of operation research, like LP, QP is a mathematical technique for determining the optimal solutions to many business problems. So, we study here how to convert the QP problems into the LP problems. The suggested procedure in this paper depends mainly on solving QP problems, where the constraints functions are in the form of linear inequalities. We illustrate the procedure and the computer technique by an example.

Rest of the paper is organized as follows. The Section II, briefly discuss on the mathematical background for solving QP problems. The section III, describes a well-known procedure for solving QP by converting it into a LP. In Section IV, we introduce a procedure for solving QP by converting it into a new LP. Section V is based on the computational experiments. Here we also introduce a computer technique for this method by using programming language *MATHEMATICA*. Finally, we draw a conclusion in Section VI.

II. Mathematical Background

We begin this section by examining the KKT conditions for the QP and then they turn out to be a set of linear equalities and complementary constraints.

Quadratic Programming Problem

Let a QP problem be represented by the following way:

$$\begin{aligned} &\text{Minimize } f(X) = CX + \frac{1}{2}X^T QX \\ &\quad s/t:\ AX \le B \\ &\qquad\quad X \ge 0 \end{aligned} \tag{1a}$$

$C = (c_1, c_2, ..., c_n)^T$, $X = (x_1, x_2, ..., x_n)$,

$B = (b_1, b_2, ..., b_m)^T \ne 0$.

where C is an n dimensional row vector being described the coefficients of the linear terms in the objective function, and Q is an $n \times n$ symmetric and positive definite matrix describing the coefficients of the quadratic terms. The decision variables are denoted by the n dimensional column vector X , and the constraints are defined by an $m \times n$ matrix, A and an m -dimensional column vector, B of right-hand-side coefficients. Since constraints are linear then the solution space is convex.

Karush-Kuhn-Tucker Method [6,7,10]

Let z be a real valued function of n variables defined by $z = f(x_1, x_2, ..., x_n)$ and $\{b_1, b_2, ..., b_m\}$ a set of right hand side constants of $(1b)$. If either $f(x_1, x_2, ..., x_n)$ or some $g^i(x_1, x_2, \ .... \ x_n)$, $i = 1,2,...., m$ or both are non-linear, then the problem of determining the $n-$type $(x_1, x_2, ..., x_n)$ which makes z a maximum or minimum and satisfies the following conditions, is called a general non-linear programming (NLP) problem such that

$$\begin{aligned} &g^1(x_1, x_2, \ .... \ x_n)\ \{\le, \ge or =\}\, b_1 \\ &g^2(x_1, x_2, \ .... \ x_n)\ \{\le, \ge or =\}\, b_2 \\ &g^m(x_1, x_2, \ .... \ x_n)\ \{\le, \ge or =\}\, b_m \end{aligned} \tag{1b}$$

where $g^i(x_1, x_2, \ .... \ x_n)$, $i = 1,2,...., m$ are real valued functions of n variables $x_1, x_2, \ .... \ x_n$ and $x_j \ge 0$, $j = 1,2,...., m$. This method can be used to solve NLP's in when all the constraints are not equal.

In the following, a theorem is given to visualize the standard form of KKT that we have used in our algorithm. Here, we

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assume that feasible solution exists and that the constraint region is bounded.

Theorem[9]

Assume that, $f(x_1, x_2, \ldots, x_n)$ and $g^i(x_1, x_2, \ldots\ x_n),\ i = 1,2,\ldots,\ m$ are differentiable functions satisfying certain regularity condition. Then $x^* = (g_1^*, g_2^*, \ldots, g_n^*)$ can be an optimal solution for the NLP only if there exit m equations such that all the KKT conditions are satisfied.

III. Converting QP into a LP[7]

The Lagrangian function for the quadratic program $(1a)$ is:

$$L(x,\mu) = cX + \frac{1}{2}X^T QX + \mu(AX - B)$$

where μ is an m-dimensional row vector. Now, the Karush-Kuhn-Tucker (KKT) conditions for a local minimum are given as follows,

$$\frac{\partial L}{\partial x_j} \geq 0,\ j = 1,2,\ \ldots\ ,n; C + X^T Q + \mu A \geq 0 \quad (2)$$

$$\frac{\partial L}{\partial \mu_i} \leq 0,\ i = 1,2,\ \ldots\ ,m \quad ;\ AX - B \leq 0 \quad (3)$$

$$x_j \frac{\partial L}{\partial x_j} = 0,\ j = 1,2,\ \ldots\ ,n\ ;\ and$$

$$X^T(C^T + QX + A^T\mu^T) = 0 \quad (4)$$

$$\mu_i g_i(X) = 0,\ i = 1,2,\ \ldots\ ,m \quad and$$

$$\mu(AX - B) = 0 \quad (5)$$

$$x_j \geq 0,\ j = 1,2,\ \ldots\ ,n \quad ;\ X \geq 0$$

$$\mu_i \geq 0,\ i = 1,2,\ \ldots\ ,m \quad ;\ \mu \geq 0.$$

To construct a more manageable form into the above form we introduce non negative surplus variables $Y \in \Re^n$ to the inequalities in (2) and nonnegative slack variables $v \in \Re^m$ to the inequalities in (3) then we have the following equations.

$$C^T + Q^T X + A^T \mu^T - Y = 0;\ AX - B + v = 0$$

The KKT conditions can now be written into the following linearly constants form:

$$Q^T X + A^T \mu^T - Y = -C^T \quad (6)$$

$$AX + v = B \quad (7)$$

$$X \geq 0, \mu \geq 0, Y \geq 0,\ v \geq 0 \quad (8)$$

$$Y^T X = 0,\ \mu v = 0 \quad (9)$$

The equation (8) restricts all the variables to be nonnegative, and the equation (9) prescribes the complementary slackness conditions. To create the appropriate linear program, we add artificial variables to each constraint and minimize their sum.

$$Minimize:\ z = \sum_{l=1}^{n+m} a_l$$

$$Q^T X + A^T \mu^T - Y + a_l = -C^T\ ,\ l = 1,2,\ \ldots\ ,\ n$$

$$AX + v + a_l = B\ ,\ l = n+1, n+2,\ \ldots\ ,\ n+m$$

$$X \geq 0, \mu \geq 0, Y \geq 0,\ v \geq 0$$

$$Y^T X = 0,\ \mu v = 0$$

We now apply the modified simplex method in the above LP to find the optimal solution and optimal value of the original QP problem.

IV. Converting QP into a New LP

We use the usual KKT conditions to build a new LP but in this case a linear objective function is also formulated from the set of linear equations and complementary slackness conditions. Unfortunately, an unboundedness challenge arises in this formulation and this challenge is alleviated by construction of an additional constraint. In this formulation, we will apply the simplex technique to find the optimal solution.

Multiply (6) by X^T, we have:
$X^T QX + X^T A^T \mu^T - X^T Y = -X^T C^T$.

By following matrix operations $Y^T X = X^T Y$ and the complementary conditions (9) implies $X^T Y = 0$ and using this into the above equations we also have:
$X^T QX + X^T A^T \mu^T = -X^T C^T$.

By rearranging this we get

$$X^T QX + X^T A^T \mu^T + X^T C^T = 0 \quad ..\ (10)$$

Similarly, we multiply (7) by μ, we have:
$\mu AX + \mu v = \mu B$. By using the complementary conditions (7) implies $\mu AX = \mu B$ and it is trivial to show that $\mu AX = X^T A^T \mu^T$. We want to eliminate $X^T A^T \mu^T$ in equation (10) then we will have $X^T QX + \mu B + X^T C^T = 0$ or this can be written as $X^T QX + \mu B + CX = 0$.

Now, $\frac{1}{2}X^T QX + \frac{1}{2}\mu B + CX = \frac{1}{2}CX$ implies $f(X) + \frac{1}{2}\mu B = \frac{1}{2}CX$.

So the linear objective function for the QP problem becomes a linear quantity like $f(X)=\frac{1}{2}CX-\frac{1}{2}\mu B$. This can be achieved as the following LP problem:

$$Minimize \quad f(X)=\frac{1}{2}CX-\frac{1}{2}\mu B \qquad (10')$$

$$s/t: \quad QX+A^T\mu^T-Y=-C^T$$
$$AX+v=B$$
$$X\geq 0, \mu\geq 0, Y\geq 0, \ v\geq 0$$
$$Y^TX=0, \ \mu v=0$$

Unfortunately, in the above minimization problem we will have an unbounded solution due to the negative coefficients of μ in the new objective function. This is the only source of unboundedness in the above LP problem. If we modify the objective function (11) into the following

$Minimize \ f(X)=\frac{1}{2}CX-\frac{1}{2}\mu B+l*h$ by assuming $h=\frac{1}{2}\mu B$, where l is very big number for example $5{,}000$ or any other big number. Then the new LP problem from the original QP problem becomes:

$$Minimize \quad f(X)=\frac{1}{2}CX-\frac{1}{2}\mu B+l*h \qquad (11)$$

$$s/t: \quad QX+A^T\mu^T-Y=-C^T \qquad (12)$$
$$AX+v=B \qquad (13)$$
$$h=\frac{1}{2}\mu B \qquad (14)$$
$$X\geq 0, \mu\geq 0, Y\geq 0, h\geq 0, B\neq 0, \ v\geq 0$$
$$Y^TX=0, \ \mu v=0. \qquad (15)$$

After solving the above LP we will get the perfect optimal solution but the optimal value is not exact because of the additional assumption in the objective function (12). Fortunately, this optimal solution satisfies the complementary slackness conditions (15). To get the correct optimal value we have to use the optimal solution in the original objective function. Noted that the above procedure is similar to Munapo except the subject to the constraints of this converting LP system which is the big part of this paper.

V. Computational Experiments

This section is incorporated with two parts. Firstly, the algorithm of KKT is used in this section for solving inequality type NLP problems. We will also introduce a code[6] for solving such type of problems using the programming language *MATHEMATICA*. Secondly, we illustrate the solution procedure of QP problems by converting it into LP and the computer technique by using a numerical example.

Algorithm

Step 1: Input number of constraints (n), number of variables (m) and the unknowns as $\{x_1, x_2, \ .... \ x_n\}$, objective function (f) and the constraints $g_i, i=1,\dots, n$ in terms of unknowns.

Step 2: Input mm, for maximization input 0 and for minimization input 1.

Step 3: Define "Lagrange". If $mm=0$ set $l=f-\sum_{i=1}^{n}u_i g_i$ else set $l=f+\sum_{i=1}^{n}u_i g_i$.

Step 4: Make set eqs of $u_i g_i \ (i=1,....n), \ \frac{\partial l}{\partial x_i}(i=1,....n)$

Step 5: Sol = Solve[eqs].

Step 6: Discard the solutions from sol for which $g_i>0$ or $u_i<0$.

Step 7: Print feasible solution sol.

Step 8: Calculate objective function value for each elements of sol.

Step 9: For $mm=0$ find maximum value of objective functions and their corresponding index or $mm=1$ find minimization value of objective functions and their corresponding index.

Step 10: Print solution corresponding to index and the objective functional value.

Computer technique

We introduce here a computer technique for solving QP problem in this Section.

```
Needs["Miscellaneous`RealOnly`"]
n=Input["Number of constraints"];
m=Input["Number of variables"];
xs=Input["unknowns"];
f=Evaluate[Input["objctibe"] ];
For[i=1,i≤n,i++,
gi=Evaluate[Input["const"] ];
  ];
mm=Input["for max=0,min=1"];
If[mm=0,
  l=f-Sum[ui*gi,{i,1,n}],l=f+Sum[ui*gi,{i,1,n}]];
eqs=Table[ui*gi=0,{i,1,n}];
```

```
For[i=1,i≤m,i++,eqs=Insert[eqs,D[l,xs[[i]]]=0,-1]];
sol=Solve[eqs]
lengthsolution=Length[sol];ff={};ii={};
Print["feasible solutions are "];
l=1;
For[i=1,i≤lengthsolution,i++,

flag=0;For[j=1,j≤n,j++,If[(gj/.sol[[i]])>0||((uj/.sol[[i]])<0),flag=1;Break[];Print[i,flag]]
 ];
If[flag==0,feasiblesolution=sol[[i]];Print[feasiblesolution];fff=f/.feasiblesolution;Print["obj func. val=",fff];ff=Insert[ff,fff,-1];ii=Insert[ii,i,-1]];
 ]
Print["optimal"];
If[mm==0,maxmin=Max[ff],maxmin=Min[ff]];
iq=ii[[Position[ff,maxmin][[1,1]]]];
sol[[iq]]
Print["objfunc. val= ",maxmin]
```

Numerical illustrations

Example 1

Solve the following problem.

$$\begin{aligned} &Minimize\ -8x_1-16x_2+x_1^2+4x_2^2 \\ &\quad s/t:\ x_1+x_2\le 5,\ x_1\le 3,\ \ x_1\ge 0,\ x_2\ge 0 \end{aligned} \tag{17}$$

This example is taken from Jensen and Bard.

Exact result

The optimal solution is $(x_1,x_2)=(3,2)$ and the optimal value -31.

Illustrations using Section III

Following the Section III, the data are given below:

$C=(-8,-16)^T$, $X=(x_1,x_2)$, $B=(5,3)^T$ and the Q matrix, $\begin{pmatrix}2 & 0\\ 0 & 8\end{pmatrix}$ is symmetric and positive definite so the KKT conditions are necessary and sufficient for a global optimum with the matrix $A=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$. Also, $Y=(y_1,y_2)$, $\mu=(\mu_1,\mu_2)$, and $v=(v_1,v_2)$. The linear constraints (6) and (7) take into the following form.

$$\begin{aligned} 2x_1 + \mu_1+\mu_2-y_1 &= 8\\ 8x_2+\mu_1 - y_2 &= 16\\ x_1+x_2 + v_1 &= 5\\ x_1 + v_2 &= 3 \end{aligned} \tag{18}$$

To create the appropriate LP, we add artificial variables to each constraint and minimize their sum.

$$\begin{aligned} &Minimize\ a_1+a_2+a_3+a_4\\ &s/t:\ 2x_1+\mu_1+\mu_2-y_1 + a_1 = 8\\ &\quad 8x_2+\mu_1 - y_2 + a_2 = 16\\ &\quad x_1+x_2 + v_1 \quad a_3 = 5\\ &\quad x_1 + v_2 + a_4 = 3\\ &all\ \text{var}iables\ge 0\ \ and\ complementary\ conditions. \end{aligned} \tag{19}$$

Applying the modified simplex technique to this LP, the optimal solution is $(x_1,x_2)=(3,2)$ and the optimal value -31.

Illustrations using Section IV

Following Section IV, the data are same like Section III. Similarly, the first two constrained of the equation (13) and (14) are same like equation (18). The only additional constraint in the new LP is (15), which gives us $2.5\mu_1+1.5\mu_2=h$. Hence, the linear constraints (13) to (15) take into the following form.

$$\begin{aligned} 2x_1 + \mu_1+\mu_2-y_1 &= 8\\ 8x_2+\mu_1 - y_2 &= 16\\ x_1+x_2 + v_1 &= 5\\ x_1 + v_2 &= 3\\ 2.5\mu_1+1.5\mu_2 &= h \end{aligned} \tag{20}$$

Taking a large number $l=5{,}000$ we have the new linear objective function

$-4x_1-8x_2-2.5\mu_1-1.5\mu_2+5000h$

from the original QP problem by following the equation (12) .

Finally, to create the appropriate LP from the QP in Section IV, we have the following minimization function with linear constraints.

$$\begin{aligned} &Minimize\ \ -4x_1-8x_2-2.5\mu_1-1.5\mu_2+5000h\\ &s/t:\\ &2x_1 + \mu_1+\mu_2-y_1 = 8\\ &\quad 8x_2+\mu_1 - y_2 = 16\\ &x_1+x_2 + v_1 = 5\\ &x_1 + v_2 = 3\\ &\quad 2.5\mu_1+1.5\mu_2 = h\\ &x_1,x_2,\mu_1,\mu_2,y_1,y_2,v_1,v_2,h\ge 0. \end{aligned} \tag{21}$$

The optimal solution of the above LP (21) by the simplex method is given by:

$x_1 = 3, x_2 = 2, \mu_1 = 0, \mu_2 = 2, h = 3$ and

$y_1 = y_2 = v_1 = v_2 = 0$ but the current objective value is 14969. The solution is optimal because it satisfies the complementary slackness conditions (16) . Now, using the current optimal solution in the original QP problem we have the objective function value -31.

Coding output

feasible solutions are

{X1 → 3., X2 → 2., U_1 → 0., U_2 → 2.}

objfunc. val= -31.

VI. Conclusion

This paper studied on the well-known procedure of QP and its corresponding LP problems. We then introduced a new LP problem corresponding to the original QP problem. An unboundedness question was created into the new converting LP problem but after some modification, we overcome the unboundedness question. We also introduced a general computer technique that can be solved the QP problem. We illustrated the solution procedure of QP problems by converting it into LP and the computer technique by using a numerical example.

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