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Classical and Higher Order Runge-Kutta Methods with Nonlinear Shooting Technique for Solving Van der Pol (VdP) Equation

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Abstract

The goal of the research work is to examine the improvement of numerical solution of VdP equation. The well-known VdP equation is governed by the second order nonlinear ODE and then solved numerically using the classical Runge-Kutta (RK) method, RK-Fehlberg method of order five, Verner method of order eight and Cash-Karp method of order six with nonlinear shooting technique. In this work, numerical simulations have been carried out using NVdP code which is written in MATHEMATICA. Also, the accuracy and efficiency of the solution of VdP equation using different RK methods with nonlinear shooting technique has been investigated. For analysis of accuracy, the approximate exact solution obtained by perturbation method is used for the comparison. It is observed that all the different RK methods give accurate result of the VdP equation. But the classical RK method shows slightly better performance than the other single step techniques.

Keywords: Runge-Kutta, Fehlberg, Verner, Cash-Karp, Shooting method.

I. Introduction

Mathematics is considered as the language of science as mathematical equation can be used as a model in almost every physical scheme and indeed any occurrence in nature. Mathematical models can be linear/nonlinear but very few of them can be resolved analytically. To solve the nonlinear problem, we use numerical techniques that give fairly accurate solutions.

Our concern in this work is to give numerical result of VdP equation of the form[1-6]

$$w'' + \mu(w^2 - 1)w' + w = 0, a \leq t \leq b \qquad (1)$$

$$w(a) = \alpha \text{ and } w(b) = \beta \qquad (2)$$

Here the "prime" denotes the derivatives with respect to t.

Here, the solution $w(t)$is an oscillator with a linear spring force and a nonlinear damping force.In this case, we consider$\mu > 0$. When$\mu = 0$, the Eq. (1) reduces to $w'' + w = 0$ and $w(t) = c_1 \cos t + c_2 \sin t$ is a solution of this second order ODE.The damping is negative for $|w| < 1$,and for this reasonit increasesthe amplitude of motion. True damping is occurred for $|w| > 1$ and the motion decreases. All of these lead to the case of oscillations, where the system begins at small w, is thrown to large w by the amplification, and is then damped back to small w.A damped oscillatory patternof the solution $w(t)$depends on the value of μ.

"Shooting method has some advantages for solving the nonlinear differential equations. It requires a minimum problem analysis and preparation. Moreover, the method is quite general and applicable to a wide variety of nonlinear differential equation"[4].For these advantages of shooting method and noticeable attention of the VdP equation, the numerical solution of VdP equation with nonlinear shooting technique is studied.

There are various methods which can be applied in solving nonlinear BVP in engineering and fluid mechanics. In this work, the VdP equation is solved by different numerical single step methods such as RK method of order four, RK-Fehlberg method, Verner method and Cash-Karp method with nonlinear shooting technique and also analysis of the numerical results is presented for different values of μ by using these different numerical methods. Finally, efficiency of different RK methods is tested with approximate exact solution which is obtained by perturbation method[6] and a conclusion has been made about the performance of the methods.

II. Shooting Method

The shooting method is a technique that is used to solve the BVP. In this method by choosing the arbitrary value for derivatives of desired function in staring time and converting BVP into initial value problem (IVP). In starting time, the method chooses an arbitrary value of the one end of the IVP. Solution starts at one end of the BVP and 'shoot' to the arbitrary value of the IVP until the boundary condition at other end hit to its correct value. More details are described in[7].

Consider the second order nonlinear boundary value problems (BVPs)[7]:

$$w'' = f(x, w, w'), a \leq t \leq b, w(a) = \alpha, \text{ and } w(b) = \beta \qquad (3)$$

Let $w(t, z_k)$ be the solution of the IVP:

$$w'' = f(x, w, w'), a \leq t \leq b, \qquad (4)$$

$$w(a) = \alpha, w'(a) = z_k$$

To approximate the solution $y(t, z)$, we define a function

$$F(z) = w(b, z) - \beta$$

If F has a root z then the outcome $w(t, z)$ of the corresponding IVP is also a solution of the BVP. In the shooting method, we will use single step methods and also Newton Raphson method.

Newton Raphson (NR) Method: NR method is a method for finding the roots of a function. The NR method begins with a real valued function J which is defined at $u \in \mathbb{R}$ and choose an initial guess u_0 for finding the root. The general formula of NR method is

$$u_{k+1} = u_k - \frac{J(u_k)}{J'(u_k)}, k = 0,1, ...$$

The process is repeated until we get our desired root. The details are also given in[7].

Introduction of Single Step method: The general R-stage RK

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is defined by[8]

$$w_{m+1} = w_m + v\, g(t_m, w_m; v),$$

$v = (b-a)/m,\ \ t_i = a + pv,\ \ p = 1, 2, \ldots, m,$ where v is called a step size.

$$g(t, w; v) = \sum_{d=1}^{R} b_d k_d$$

$$k_1 = g(t, w)$$

$$k_d = g\left(t + vc_d, w + v\sum_{l=1}^{d-1} a_{dl} k_l\right), d = 2, 3, \ldots, R$$

$$c_d = \sum_{l=1}^{d-1} a_{dl}\, , d = 2, 3, \ldots, R.$$

The coefficients c_d, a_{dl} and b_d are conveniently represented in a Butcher tableau[9]

c	**A**
	$\boldsymbol{b^T}$

=

0					
c_2	a_{21}				
c_3	a_{31}	a_{32}			
$\vdots$	$\vdots$		$\ddots$		
c_d	a_{d1}	a_{d2}	…	$a_{d,d-1}$	
	$\boldsymbol{b_1}$	$\boldsymbol{b_2}$	…	$\boldsymbol{b_{d-1}}$	$\boldsymbol{b_d}$

Runge-Kutta 4th Order Method: Classical RK methodis referred as a member of the family of Runge–Kuttamethods. It represents the solutions corresponding forthe case R = 4.

The formula of the RK4 method is as follows

$$w_{i+1} = w_i + v\,(b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4),$$

$$k_1 = g(t_i, w_i)$$

$$k_2 = g(t_i + c_2v, w_i + a_{21}k_1v)$$

$$k_3 = g(t_i + c_3v, w_i + a_{32}k_2v)$$

$$k_4 = g(t_i + c_4v, w_i + a_{43}k_3v)$$

The Butcher tableau:

0				
½	½			
½	0	½		
1	0	0	1	
	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

Runge-Kutta-Fehlberg Method (RKF5): In mathematics, the RK–Fehlberg is a procedure for solving an ordinary differential equation numerically. This method was introduced by Fehlberg[10] and is considered as the large class of RK techniques. This method is an embedded form of the RK family.

The Butcher tableau for the Fehlberg method of order four is

0						
$\frac{1}{4}$	$\frac{1}{4}$					
$\frac{3}{8}$	$\frac{3}{32}$	$\frac{9}{32}$				
$\frac{12}{13}$	$\frac{1932}{2197}$	$-\frac{7200}{2197}$	$\frac{7296}{2197}$			
1	$\frac{439}{216}$	-8	$\frac{3680}{513}$	$\frac{-845}{4104}$		
$\frac{1}{2}$	$-\frac{8}{27}$	2	$-\frac{3544}{2565}$	$\frac{1859}{4104}$	$-\frac{11}{40}$	
	$\frac{25}{216}$	0	$-\frac{1408}{2565}$	$\frac{2197}{4104}$	$-\frac{1}{5}$	0

Runge-Kutta-Verner Method (RKV8): An example of a 7-stage method with a 8-stage error estimation was developed by Verner[11]. The Butcher tableau for the Verner method is shown below:

0								
$\frac{1}{18}$	$\frac{1}{18}$							
$\frac{1}{6}$	$-\frac{1}{12}$	$\frac{1}{4}$						
$\frac{2}{9}$	$-\frac{2}{81}$	$\frac{4}{27}$	$\frac{8}{81}$					
$\frac{2}{3}$	$\frac{40}{33}$	$-\frac{4}{11}$	$-\frac{56}{11}$	$\frac{54}{11}$				
1	$-\frac{369}{73}$	$\frac{72}{73}$	$\frac{5380}{219}$	$-\frac{12285}{584}$	$\frac{2695}{1752}$			
$\frac{8}{9}$	$-\frac{8716}{891}$	$\frac{656}{297}$	$\frac{39520}{891}$	$-\frac{416}{11}$	$\frac{52}{27}$	0		
1	$\frac{3015}{256}$	$-\frac{9}{4}$	$-\frac{4219}{78}$	$\frac{5985}{128}$	$-\frac{539}{384}$	0	$\frac{693}{3328}$	
	$\frac{57}{640}$	0	$-\frac{16}{65}$	$\frac{1377}{2240}$	$\frac{121}{320}$	0	$\frac{891}{8320}$	$\frac{2}{35}$

Cash Karp Method (RKCK6): The Cash–Karp method is an algorithm for solving "ODEs numerically" and it was developed by Cash[12]. It is a member of the family of RK techniques and considered as a higher order RK method for solving ODEs.

The Butcher tableau for this method is

$$
\begin{array}{c|cccccc}
0 & & & & & & \\
\frac{1}{5} & \frac{1}{5} & & & & & \\
\frac{3}{10} & \frac{3}{40} & \frac{9}{40} & & & & \\
\frac{3}{5} & \frac{3}{10} & -\frac{9}{10} & \frac{6}{5} & & & \\
1 & -\frac{11}{54} & \frac{5}{2} & -\frac{70}{27} & \frac{35}{27} & & \\
\frac{7}{8} & -\frac{1631}{55296} & \frac{175}{512} & \frac{575}{13824} & \frac{44275}{110592} & \frac{253}{4096} & \\
\hline
 & \frac{37}{378} & 0 & \frac{250}{621} & \frac{125}{594} & 0 & \frac{512}{1771}
\end{array}
$$

III. Results and Discussion

Consider a second order nonlinear boundary value problem[7]

$$w'' = \frac{1}{8}(32 + 2t^3 - ww'), 1 \le t \le 3$$

with $w(1) = 17$, and $w(3) = 43/3$.

It is solved numerically for the validation of our proposed MATHEMATICA code by using the RK4, Fehlberg, Verner and the Cash-Karp methods with nonlinear shooting technique. The numerical results are also compared with exact solution.

$$w(t) = t^2 + \frac{16}{t}$$

It is observed that all the solutions with different numerical methods give good approximation compared with the exact solution, which is shown in Table 1. However, after comparing the results with exact solution, it can be concluded that Verner method and Cash-Karp method gives more accurate result compared to the other methods.

Table 1. Numerical results of different Runge-Kutta methods with shooting technique.

t	RK4	Fehlberg	Verner	Cash-Karp	Exact
1.0	17.00000000	17.00000000	17.00000000	17.00000000	17.00000000
1.2	14.77354215	14.77333070	14.77333333	14.77333333	14.77333333
1.4	13.38996856	13.38856706	13.38857143	13.38857142	13.38857143
1.6	12.56255367	12.55999435	12.56000000	12.56000000	12.56000000
1.8	12.13224617	12.12888226	12.12888889	12.12888888	12.12888889
2.0	12.00372040	11.99999262	12.00000000	12.00000000	12.00000000
2.2	12.11637206	12.11271934	12.11272727	12.11272727	12.11272727
2.4	12.42983769	12.42665837	12.42666666	12.42666666	12.42666667
2.6	12.91620491	12.91383766	12.91384615	12.91384615	12.91384615
2.8	13.55556307	13.55427718	13.55428571	13.55428571	13.55428571
3.0	14.33333333	14.33332491	14.33333333	14.33333333	14.33333333

Here the VdP equation (1) is considered for the purpose of comparing the numerical solution based on RK4, Fehlberg, Verner and Cash-Karp method with approximate exact solution[6]using the boundary condition $w(0) = 0, w(20) = 1$ and the domain $t = 0$ to $t = 20$ is subdivided into 800 points at which the numerical approximations of VdP are estimated.

$$\text{exact solution} = a_0 \cos t + b_0 \sin t$$

$$+\mu\left(-\frac{b_0^{\,3}}{32} + \frac{b_0 a_0^{\,2}}{32} + \frac{b_0}{8}\right)\cos t - \frac{\mu\, a_0}{32}\left(5\, b_0^{\,2} - 7a_0^{\,2} + 16\right)\sin t$$

$$+\mu t\left(\frac{a_0}{2}\left(1 - \frac{a_0^{\,2} + b_0^{\,2}}{4}\right)\cos t\right) + \mu t\left(\frac{b_0}{2}\left(1 - \frac{b_0^{\,2} - a_0^{\,2}}{4}\right)\sin t\right)$$

$$+\frac{\mu\, b_0}{8}\left(\frac{b_0^{\,2} - a_0^{\,2}}{4} - 1\right)\cos 3t + \frac{\mu\, a_0}{8}\left(\frac{3\, b_0^{\,2} - a_0^{\,2}}{4}\right)\sin 3t$$

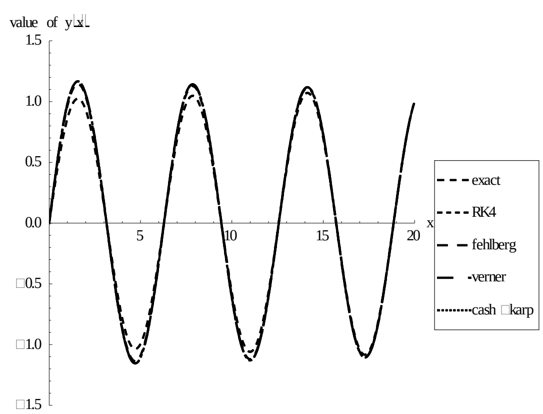

Fig. 1. Comparison of different numerical solutionswith exact solution for $\mu = 0.01$.

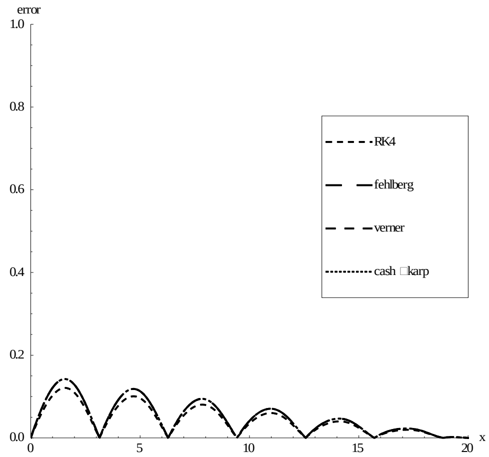

Fig. 2. Variations of absolute errors with t for $\mu = 0.01$

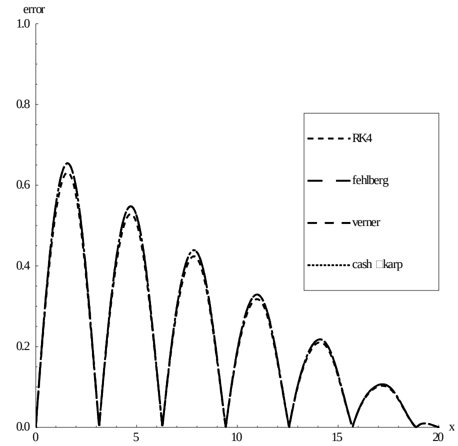

Fig. 4. Variations of absolute errors with t for $\mu = 0.05$

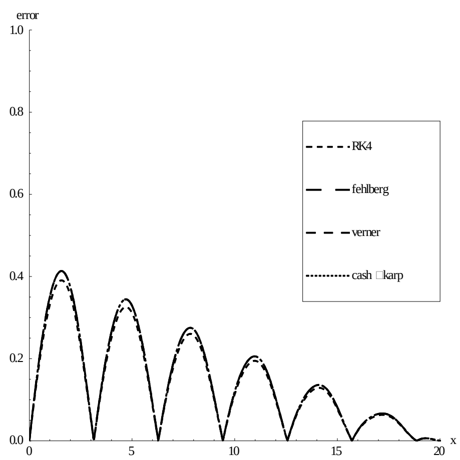

Fig. 3. Variations of absolute errors with t for $\mu = 0.03$

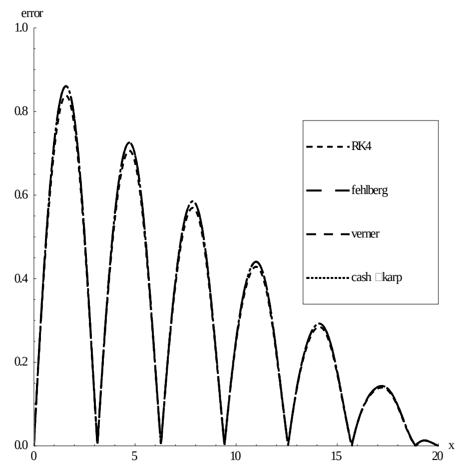

Fig. 5. Variations of absolute errors with t for $\mu = 0.07$

Table 2. Numerical results for $\mu = 0.01$

t	RK4	Fehlberg	Verner	Cash-Karp	Exact[6]
0.0	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000
1.0	0.964356893	0.982964194	0.982964200	0.982964200	0.861666664
2.0	1.041108344	1.060206887	1.060206895	1.060206895	0.931693565
3.0	0.161164437	0.163942303	0.163942305	0.163942305	0.145255152
4.0	-0.861229393	-0.875274773	-0.875274778	-0.875274778	-0.783643996
5.0	-1.090077663	-1.106782028	-1.106782036	-1.106782036	-0.993778289
6.0	-0.316933833	-0.321442587	-0.321442593	-0.321442593	-0.290629249
7.0	0.742374109	0.752253999	0.752254000	0.752254000	0.687723641
8.0	1.116515466	1.130232334	1.130232342	1.130232342	1.036991452
9.0	0.464226350	0.469421575	0.469421583	0.469421583	0.433154689
10.0	-0.610379140	-0.616665631	-0.616665631	-0.616665631	-0.575549292
11.0	-1.120231106	-1.130571207	-1.130571214	-1.130571214	-1.059976995
12.0	-0.600168294	-0.60504928	-0.60504929	-0.60504929	-0.569876961
13.0	0.468057669	0.47147048	0.471470477	0.471470477	0.449152636
14.0	1.101500202	1.108285729	1.108285735	1.108285735	1.061763201
15.0	0.722154994	0.725803846	0.725803857	0.725803857	0.697907191
16.0	-0.318392621	-0.319766917	-0.31976691	-0.31976691	-0.310907731

17.0	-1.061047125	-1.06431157	-1.064311575	-1.064311575	-1.041802022
18.0	-0.827905348	-0.829522556	-0.829522568	-0.829522568	-0.814473149
19.0	0.164478382	0.164729227	0.164729218	0.164729218	0.163476736
20.0	1.000019138	0.999999960	0.999999962	0.999999962	1.000000000

The following observation can be made from the above findings:

- The solution of VdP equation (1) can be found only for the parameter $0.01 \leq \mu \leq 0.07, a = 0, b = 20$ and $\beta = 1$. For $\mu > 0.07$, many difficulties such as machine overflow in finding the numerical solutions of Eq. (1) is observed. But if we take $a = 0, b = 60$ and $\beta = 0.1$, thenthe results can be obtainedusing different single steps methods as discussed above for large values of μ, where μ can be considered up to 40.
- Solution of VdP equation (1) shows relatively same graphical behaviorwith oscillatory pattern compared with the approximate exact solution[6] for every value of μ. This is shown in Fig. 1 for $\mu = 0.01$.
- Figs. 2-5 show that the classical RK4 method gives relativelylower absolute error compared to the other methodsfor allvalues of μ. However, theabsolute errors obtained by the Fehlberg, Verner and Cash-Karp methods are almost same for different values ofμ.Also, if we divide the interval into 200 interior points, then RK4 gives more accurate results which is shown in Fig. 5; for $\mu = 0.01$. That is if we increase the step sizes then RK4 give less absolute error and the other methods give relatively same error as earlier.
- The numerical results of different methods with approximate exact solution presented in Table2 also show that RK4 method gives slightly better result compared to other ones.

IV. Conclusions

In this study, a second order nonlinear Van der Pol equation with boundary conditions have been considered. This nonlinear equation has been solved successfully by employing classical RK, RK-Fehlberg, Verner, and the Cash-Karp methods with nonlinear shooting method to obtain the numerical solutions. The numerical results display the accuracy and efficiency of the proposed techniques by using MATHEMATICA code in solving the VdP equation. For the comparison purpose, the approximate exact solution which is obtained by perturbation method[6] has been used and also respective error bounds are presented. From the graphical representations, it is seen that the oscillatory pattern of the solution of VdP equation relatively same for different parameters of μ by using the different single step methods. Also, it is found that the solutions obtained by classical RK technique are more accurate than RKF5, RKV8 and RKCK6 methods for different values of μ. But for smaller time-step, all the discussed methods have given relatively same results. So, we may conclude that that all the numerical methods with shooting technique give good numerical solution of VdP equation.

References

1. Tsatsos M., 2006. Theoretical and Numerical study of the Van der Pol equation. Dissertation, Thessaloniki.
2. Chen, J. H. & W. Chen, 2008.Chaotic Dynamics of the Fractionally Damped Van der Pol Equation, Chaos, Solitons and Fractols, **35**,188-198.
3. Vahidi, A. R., Z. Azimzadeh & S. Mohammandifar,2012. Restarted Adomian Decomposition Method for Solving Duffing-van Der Pol equation, Applied Mathematical Sciences, **6(11)**,499-507.
4. Ha S. N., 2001. A Nonlinear Shooting Method for Two-point Boundary Value Problems, Pergamon, Computers and Mathematics with Applications, **42**, 1411-1420.
5. Cartwright, M. L., 1960.Balthazar Vander pol, J. London Math. Soc., **35**, 367–376.
6. Abbasi, N. M., 2009. Solving the Van derPol Nonlinear Differential Equation using First Order Approximation Perturbation Method.
7. Burden, R. L. & F.J.Douglas, Numerical Analysis, **9th**Edition, 678- 684.
8. S. Endre, 2014. Numerical Solution of Ordinary Differential Equations, Mathematical Institution, University of Oxford, 4-19.
9. Butcher, J. C., 1996.A History of Runge-Kutta Method, Applied numerical mathematics,**20**, 247-260.
10. Fehlberg E., 1969.Low-order Classical Runge-KuttaFormulas with Step Size Control and their Application to Some Heat Transfer Problems, NASA Technical Report 315.
11. Verner J. H., 1978. Explicit Runge-Kutta Methods with Estimates of the Local Truncation Error, SIAM J.Numer.Anal,**15(4)**,772-779.
12. Cash J. R. & A. H. Karp, 1990. “A Variable Order Runge-Kutta Method for Initial Value Problems with Rapidly Varying Right-hand Sides”, ACM Transactions on Mathematical Software, **16**, 201-222.