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Bezier Polynomials with Applications

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Abstract

In this paper, we use the Galerkin technique for solving higher order linear and nonlinear boundary value problems (BVPs). The well-known Bezier polynomials are exploited as basis functions in the technique. To use the Bezier polynomials, we need to satisfy the corresponding homogeneous form of the boundary conditions and modification is thus needed. A rigorous matrix formulation is developed by the Galerkin method for linear and nonlinear systems and solved it using Bezier polynomials. The approximate solutions are compared to the exact solutions through tabular form. All problems are computed using the software MATHEMATICA.

Keywords: Galerkin method, linear and nonlinear BVPs, Bezier polynomials.

I. Introduction

The higher order BVPs are studied because of their mathematical importance and wide applications in applied sciences. Generally, fourth order BVPs arise in the mathematical modeling of viscoelastic and inelastic flows, deformation of beams and plates deflection theory, beam element theory and many more applications of engineering and applied mathematics[1,2]. These BVPs are solved either analytically[3] or numerically[4,5]. Sixth order BVPs arise in many real life phenomena, for example the vibration behavior of ring structures is governed by a sixth order ordinary differential equation and also in the mathematical modeling of astrophysics and the narrow convicting layers[6]. Many researchers have attempted to solve sixth order BVPs numerically. For example, Siddiqi and Akram[7] developed septic spline solutions of sixth order BVPs. Fazal-i-Haq *et al*[8] developed the solution of sixth order BVPs by collocation method using Haar wavelets. Viswanadham and Reddy[9] have used Petrove-Galerkin method for the solution of sixth-order BVPs. El-Gamel *et al*[10] have used Sinc-Galerkin method for the solution of sixth-BVPs. Some researchers have developed few methods for computing approximations to the solutions of eighth order BVPs. For example, Siddiqi and Ghazala Akram[11] used nonic spline and nonpolynomial, respectively for the solution of eighth order linear special case BVPs. Kasi and Ballem[12] used Quintic B-spline Collocation Method for solving eighth order BVPs. Bellal and Islam[13] applied Residual Galerkin technique to solve eighth order BVPs.

However, in section II of this paper, a short description on Bezier polynomials is mentioned. In section III, the formulation of the Galerkin method with Bezier polynomials as basis functions are to be presented for solving linear and nonlinear higher order BVP. The proposed formulation is verified on two linear and three nonlinear BVPs in section IV. Finally, in the last section, the conclusion of the paper is presented.

II. Bezier Polynomials

The Bezier polynomials of nth degree form a complete basis over [0, 1] and they are defined by

$$B_{i,d}(x) = \sum_{i=0}^{d}\binom{d}{i} x^i(1-x)^{d-i}P_i \text{ , } 0 \le x \le 1$$

where the binomial coefficients are given by

$$\binom{d}{i} = \frac{d!}{(d-i)!\,i!}$$

The points P_i are called control points for the Bezier curve.

For example, the first 17 Bezier polynomials of degree 16 over the interval [0, 1] are given bellow:

$$B_0(x) = (1-x)^{17}$$
$$B_1(x) = 17(1-x)^{16}x$$
$$B_2(x) = 136(1-x)^{15}x^2$$
$$B_3(x) = 680(1-x)^{14}x^3$$
$$B_4(x) = 2380(1-x)^{13}x^4$$
$$B_5(x) = 6188(1-x)^{12}x^5$$
$$B_6(x) = 12376(1-x)^{11}x^6$$
$$B_7(x) = 19448(1-x)^{10}x^7$$
$$B_8(x) = 24310(1-x)^9x^8$$
$$B_9(x) = 24310(1-x)^8x^9$$
$$B_{10}(x) = 19448(1-x)^7x^{10}$$
$$B_{11}(x) = 12376(1-x)^6x^{11}$$
$$B_{12}(x) = 6188(1-x)^5x^{12}$$
$$B_{13}(x) = 2380(1-x)^4x^{13}$$
$$B_{14}(x) = 680(1-x)^3x^{14}$$
$$B_{15}(x) = 136(1-x)^2x^{15}$$
$$B_{16}(x) = 17(1-x)x^{16}$$
$$B_{17}(x) = x^{17}$$

Since Bezier polynomials have special properties at $x = 0$ and $x = 1$: $B_{i,d}(0) = 0$ and $B_{i,d}(1) = 0$, $i = 1,2,\dots,d-1$, respectively, so that they can be used as set of basis function to satisfy the corresponding homogeneous forms of the essential boundary conditions in the Galerkin method to solve a BVP over the interval $[0, 1]$.

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III. Formulation of BVPs in Matrix Form

In this section, we first obtain the rigorous matrix formulation for fourth order linear BVP and then we extend our idea for solving nonlinear BVP. For this, we consider a linear fourth order differential equation given by

$$a_4\frac{d^4u}{dx^4}+a_3\frac{d^3u}{dx^3}+a_2\frac{d^2u}{dx^2}+a_1\frac{du}{dx}+a_0u$$
$$=r, a<x<b \qquad (1a)$$

Subject to the following boundary conditions

$$u(\mathrm{a})=\mathrm{A}_0,\qquad u(b)=\mathrm{B}_0,\qquad u'(a)=\mathrm{A}_1,$$
$$u'(b)=\mathrm{B}_1 \qquad (1b)$$

where $\mathrm{A}_i, \mathrm{B}_i, i=0,1$ are finite real constants and $a_i, i=0,1,2,3,4$ and r are all continuous and differentiable functions of x defined on the interval $[a, b]$. We want to solve the BVP of the form (1) by Galerkin method using the polynomials, described in section II, as trial functions.

We approximate the solution of the differential equation $(1a)$ as

$$\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n-1}\mathrm{a}_i\,B_i(x)\,, n\geq 2 \qquad (2)$$

Here $\theta_0(x)$ is specified by the essential boundary conditions, $B_i(x)$ are the Bezier polynomials which must satisfy the corresponding homogeneous boundary conditions such that $B_i(\mathrm{a})=B_i(b)=0$, for each $i=1,2,3,\ldots,n-1$.

Using (2) into $(1a)$, the Galerkin weighted residual equations are:

$$\int_{\mathrm{a}}^{b}\left[a_4\frac{d^4\tilde{u}}{dx^4}+a_3\frac{d^3\tilde{u}}{dx^3}+a_2\frac{d^2\tilde{u}}{dx^2}+a_1\frac{d\tilde{u}}{dx}+a_0\tilde{u}-r\right]B_j(x)dx=0,\ \ j=1,2,..,n-1 \qquad (3)$$

Integrating by parts the terms up to second derivative on the left hand side of (3), we get

$$\int_{\mathrm{a}}^{b}\left[a_4\frac{d^4\tilde{u}}{dx^4}\right]B_j(x)dx=$$
$$\left[a_4\frac{d^3\tilde{u}}{dx^3}B_j(x)\right]_{\mathrm{a}}^{b}-\int_{\mathrm{a}}^{b}\frac{d}{dx}\left(a_4B_j(x)\right)\frac{d^3\tilde{u}}{dx^3}dx$$
$$=\left[\frac{d}{dx}[a_4B_j(x)]\frac{d^2\tilde{u}}{dx^2}\right]_{\mathrm{a}}^{b}+\int_{\mathrm{a}}^{b}\frac{d^2}{dx^2}\left(a_4B_j(x)\right)\frac{d^2\tilde{u}}{dx^2}dx$$

[Since $B_i(\mathrm{a})=B_i(b)=0$]

$$=-\left[\frac{d}{dx}[a_4B_j(x)]\frac{d^2\tilde{u}}{dx^2}\right]_{\mathrm{a}}^{b}+\left[\frac{d^2}{dx^2}[a_4B_j(x)]\frac{d\tilde{u}}{dx}\right]_{\mathrm{a}}^{b}$$
$$-\int_{\mathrm{a}}^{b}\frac{d^3}{dx^3}\left(a_4B_j(x)\right)\frac{d\tilde{u}}{dx}dx \qquad (4)$$

$$\int_{\mathrm{a}}^{b}\left[a_3\frac{d^3\tilde{u}}{dx^3}\right]B_j(x)dx=$$
$$\left[a_3\frac{d^2\tilde{u}}{dx^2}B_j(x)\right]_{\mathrm{a}}^{b}-\int_{\mathrm{a}}^{b}\frac{d}{dx}\left(a_3B_j(x)\right)\frac{d^2\tilde{u}}{dx^2}dx$$
$$=-\left[\frac{d}{dx}[a_3B_j(x)]\frac{d\tilde{u}}{dx}\right]_{\mathrm{a}}^{b}$$
$$+\int_{\mathrm{a}}^{b}\frac{d^2}{dx^2}\left(a_3B_j(x)\right)\frac{d\tilde{u}}{dx}dx \qquad (5)$$

[Since $B_i(\mathrm{a})=B_i(b)=0$]

$$\int_{\mathrm{a}}^{b}\left[a_2\frac{d^2\tilde{u}}{dx^2}\right]B_j(x)dx=$$
$$\left[a_2\frac{d\tilde{u}}{dx}B_j(x)\right]_{\mathrm{a}}^{b}-\int_{\mathrm{a}}^{b}\frac{d}{dx}\left(a_2B_j(x)\right)\frac{d\tilde{u}}{dx}dx$$
$$=-\int_{\mathrm{a}}^{b}\frac{d}{dx}\left(a_2B_j(x)\right)\frac{d\tilde{u}}{dx}dx$$

[Since $B_i(\mathrm{a})=B_i(b)=0$] (6)

Substituting eqns. (4) to (6) into eqn. (3) and using approximation for $\tilde{u}(x)$ given in eqn.(2) and imposing the boundary conditions given in eqn. $(1b)$, and rearranging the terms we obtain the system of equations in matrix form as

$$\sum_{i=1}^{n-1}B_{i.j}\mathrm{a}_i=F_j\,, j=1,2,\ldots,n-1. \qquad (7a)$$

where

$$B_{i.j}=\int_{\mathrm{a}}^{b}\left[\left\{-\frac{d^3}{dx^3}\left(a_4B_j(x)\right)+\frac{d^2}{dx^2}\left(a_3B_j(x)\right)-\frac{d}{dx}\left(a_2B_j(x)\right)+a_1B_j(x)\right\}\frac{d}{dx}\left(B_i(x)\right)+a_0B_i(x)B_j(x)\right]dx-\left[\frac{d}{dx}[a_4B_j(x)]\frac{d^2}{dx^2}[B_i(x)]\right]_{x=b}$$
$$+\left[\frac{d}{dx}[a_4B_j(x)]\frac{d^2}{dx^2}[B_i(x)]\right]_{x=\mathrm{a}} \qquad (7b)$$

$$F_j=\int_{\mathrm{a}}^{b}\left\{r(x)B_j(x)+\left[\frac{d^3}{dx^3}\left(a_4B_j(x)\right)-\frac{d^2}{dx^2}\left(a_3B_j(x)\right)+\frac{d}{dx}\left(a_2B_j(x)\right)-a_1B_j(x)\right]\frac{d\theta_0}{dx}-a_0\theta_0B_j(x)\right\}dx$$
$$-\left[\frac{d^2}{dx^2}[a_4B_j(x)]\right]_{x=b}\times\mathrm{B}_1$$
$$+\left[\frac{d^2}{dx^2}[a_4B_j(x)]\right]_{x=a}\times\mathrm{A}_1$$
$$+\left[\frac{d}{dx}[a_3B_j(x)]\right]_{x=b}\times\mathrm{B}_1-$$
$$\left[\frac{d}{dx}[a_3B_j(x)]\right]_{x=a}\times\mathrm{A}_1,\ \ j=1,2,\ldots,n-1.$$

Solving the system (7a), we find the values of the parameters a_i, and then substituting into (2), we get the approximate solution of the desired BVP(1).

For nonlinear fourth order BVP, we first compute the initial values on neglecting the nonlinear terms and using the system $(7a)$.Then using the Newton's iterative method we find the numerical approximations for desired nonlinear BVP. This formulation is described through the numerical examples in the next section.

IV. Numerical Examples and Results

In this section, we consider two linear and three nonlinear BVPs to verify the proposed formulation in section III. For this, we give the results for linear problems in brief depending on prescribed boundary conditions, but the nonlinear problem is illustrated in details. All computations are performed by *Mathematica*.

Example 1. Consider the fourth order linear differential equation[1,3,4,5]

$$\frac{d^4u}{dx^4} + xu + (8 + 7x + x^3)e^x = 0, 0 < x < 1 \quad (8a)$$

subject to the boundary conditions

$$u(0) = u(1) = 0, u''(0) = 0, \quad u''(1) = -4e \quad (8b)$$

whose analytical solution is,

$$u(x) = x(1 - x)e^x.$$

Solution.

Applying the method illustrated in section III, we approximate $u(x)$ in a form

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n-1} a_i B_i(x), n \geq 2 \quad (9)$$

Here $\theta_0(x) = 0$ is specified by the *Dirichlet* boundary conditions of equation $(8b)$.

Now the parameters $a_i, (i = 1,2, \dots, n - 1)$ satisfy the linear system

$$\sum_{i=1}^{n-1} B_{i.j} a_i = F_j, j = 1,2, \dots, n - 1 \quad (10a)$$

where

$$B_{i.j} = \int_0^1 \left[\frac{d^2B_i(x)}{dx^2}\frac{d^2B_j(x)}{dx^2} + xB_i(x)B_j(x)\right] dx$$

$$F_j = -\int_0^1 \left[(8 + 7x + x^3)e^x B_j(x)\right] dx - 4e\frac{dB_j}{dx}(1), j = 1,2, \dots, n - 1. \quad (10b)$$

Solving the system $(10a)$ we obtain the values of the parameters and then substituting these parameters into equation (9), we get the approximate solution of the BVP (8) for different values of n.

The numerical results obtained by our method are given in Table 1.

The maximum absolute error 7.83×10^{-16} is found by the present method using 15 Bezier polynomials. On the other hand, the maximum absolute errors have been found by Islam and Hossain[4], Kasi *at el*[5] are 2.50×10^{-13} and 5.99×10^{-6}, respectively.

Example 2. Consider the fourth ordernonlinear differential equation[4,5]

$$\frac{d^4u}{dx^4} = \sin x + \sin^2 x - \left(\frac{d^2u}{dx^2}\right)^2, 0 \leq x \leq 1 \quad (11a)$$

subject to the boundary conditions

$$u(0) = 0, \quad u(1) = \sin 1, \quad u''(0) = 0,$$

$$u''(1) = -\sin 1 \quad (11b)$$

The analytical solution of this BVP is, $u(x) = \sin x$.

Solution.

Applying the method illustrated in section III, we approximate $u(x)$ in a form

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n-1} a_i B_i(x), n \geq 2 \quad (12)$$

Here $\theta_0(x) = x\sin(1)$ is specified by the essential boundary conditions of equation $(11\ b)$.

Now the parameters $a_i, (i = 1,2, \dots, n - 1)$ satisfy the system

$$(B + C)A = F \quad (13a)$$

where the elements of A, B, C, F are $a_i, b_{i,j}$,

$c_{i,j}, f_j$ respectively, given by

$$b_{i,j} = \int_0^1 \left[\frac{d^2B_i(x)}{dx^2}\frac{d^2B_j(x)}{dx^2} + 2\frac{d^2\theta_0}{dx^2}\frac{d^2B_i(x)}{dx^2}B_j(x)\right] dx \quad (13b)$$

$$c_{i,j} = \sum_{k=1}^{n-1} a_k \int_0^1 \left(\frac{d^2B_i(x)}{dx^2}\frac{d^2B_k(x)}{dx^2}B_j(x)\right) dx$$

$$f_j = \int_0^1 \left[(\sin x + \sin^2 x)B_j(x) + \frac{d^2\theta_0}{dx^2}\frac{d^2B_j(x)}{dx^2} - \left(\frac{d^2\theta_0}{dx^2}\right)^2 B_j(x)\right] dx - \sin(1)\frac{dB_j}{dx}(1) \quad (13c)$$

The initial values of these coefficients a_i are obtained by applying Galerkin method to the BVP neglecting the nonlinear term in $(13a)$. That is, to find initial coefficients we solve the system

$$BA = F \quad (14)$$

whose matrices are constructed from

$$b_{i,j} = \int_0^1 \left[\frac{d^2B_i(x)}{dx^2}\frac{d^2B_j(x)}{dx^2}\right] dx \quad (14a)$$

$$f_j = \int_0^1 \left[(\sin x + \sin^2 x)B_j(x) + \frac{d^2\theta_0}{dx^2}\frac{d^2B_j(x)}{dx^2}\right] dx - \sin 1\frac{dB_j}{dx}(1) \quad (14b)$$

Once the initial values of the coefficients a_i are obtained from equation (14), they are substituted into equation $(13a)$ to obtain new estimates for the values of a_i. This iteration process continues until the converged values of the unknown parameters are obtained. Substituting the final values of the parameters into equation (12), we obtain an approximate solution of the BVP (11).

The numerical results obtained by our method are displayed in Table 2.

The maximum absolute error 7.99×10^{-15} is found by the present method using 15 Bezier polynomials. On the other hand, the maximum absolute errors have been found by Islam and Hossain[4], Kasi *at el*[5] are 1.36×10^{-5} and 9.91×10^{-10}, respectively.

Example 3. Consider the sixth order linear differential equation[7,8]

$$\frac{d^6u}{dx^6} - u = 6\cos x\,,\ 0 < x < 1$$

subject to the boundary conditions

$$u(0) = 0, \quad u(1) = 0, \quad u'(0) = -1,$$
$$u'(1) = \sin 1\,,\ u''(0) = 2,\ u''(1) = 2\cos 1$$

whose analytical solution is:

$$u(x) = (x-1)\sin x.$$

The numerical results obtained by our method are displayed in Table 3.

The maximum absolute error 5.41×10^{-15} is obtained by the present method using 16 Bezier polynomials. On the other hand, the maximum absolute errors have been found by Siddiqi and Akram[7], Fazal *at el*[8] are 6.48×10^{-9} and 6.1582×10^{-11}, respectively.

Example 4. Consider the sixth order nonlinear differential equation[9]

$$\frac{d^6u}{dx^6} + u^2e^{-x} = e^{-x} + e^{-3x},\ 0 < x < 1$$

subject to the boundary conditions

$$u(0) = 1, \quad u(1) = \frac{1}{e}, \quad u'(0) = -1,$$
$$u'(1) = -\frac{1}{e}\,, u''(0) = 1, \quad u''(1) = \frac{1}{e}$$

whose analytical solution is, $u(x) = e^{-x}$.

The numerical results obtained by our method are shown in Table 4.

The maximum absolute error 4.39×10^{-11} is found by the present method using 13 Bezier polynomials. On the other hand, the maximum absolute error has been obtained by Kasi[9] is 3.5167×10^{-6}.

Example 5. Consider the eighth order nonlinear differential equation[2,13]

$$\frac{d^8u}{dx^8} = 7!\left(e^{-8u} - \frac{2}{(1+x)^8}\right),\ 0 < x < e^{\frac{1}{2}} - 1 \qquad (15)$$

subject to the boundary conditions

$$u(0) = 0, \quad u\left(e^{\frac{1}{2}} - 1\right) = \frac{1}{2}, \quad u'(0) = 1,$$
$$u'\left(e^{\frac{1}{2}} - 1\right) = e^{-\frac{1}{2}}, \quad u''(0) = -1,$$
$$u''\left(e^{\frac{1}{2}} - 1\right) = -e^{-1}\,, \quad u'''(0) = 2,$$
$$u'''\left(e^{\frac{1}{2}} - 1\right) = 2e^{-\frac{3}{2}}$$

whose analytical solution is

$$u(x) = \ln(1+x).$$

To use Bezier polynomials over $[0,1]$, we convert the BVP (15) to an equivalent BVP on $[0,1]$ by letting $x = \left(e^{\frac{1}{2}} - 1\right)x$. Thus, the BVP (15) is equivalent to the BVP:

$$\frac{d^8u}{dx^8} = 7!\left(e^{\frac{1}{2}} - 1\right)^8\left(e^{-8u} - \frac{2}{(1+x)^8}\right),\ 0 < x < 1$$

subject to the boundary conditions

$$u(0) = 0, \quad u(1) = \frac{1}{2}, \quad u'(0) = \left(e^{\frac{1}{2}} - 1\right),$$
$$u'(1) = e^{-\frac{1}{2}}\left(e^{\frac{1}{2}} - 1\right), \quad u''(0) = -\left(e^{\frac{1}{2}} - 1\right)^2,$$
$$u''(1) = -e^{-1}\left(e^{\frac{1}{2}} - 1\right)^2, \quad u'''(0) = 2\left(e^{\frac{1}{2}} - 1\right)^3,$$
$$u'''(1) = 2e^{-\frac{3}{2}}\left(e^{\frac{1}{2}} - 1\right)^3$$

The numerical results obtained by our method are shown in Table 5.

The maximum absolute error 9.44×10^{-13} is found by the present method using 14 Bezier polynomials. It is observed that the accuracy is found nearly 9.980×10^{-10} by Hossain and Islam[13] and 1.00135×10^{-5} by Kasi and Ballem[12].

Table 1. Numerical results of example 1.

x	Exact values	11 Legendrepolynomials[4]		15 Bezier polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	0.0000000000	0.0000000000	0.000000000	0.0000000000	0.000000000
0.1	0.0994653826	0.0994653827	4.65×10^{-14}	0.0994653826	4.24×10^{-17}
0.2	0.1954244413	0.1954244412	1.28×10^{-13}	0.1954244413	1.39×10^{-17}
0.3	0.2834703495	0.2834703352	1.78×10^{-13}	0.2834703495	1.11×10^{-16}
0.4	0.3580379274	0.3580379268	2.83×10^{-14}	0.3580379274	2.22×10^{-17}
0.5	0.4121803176	0.4121803136	2.50×10^{-13}	0.4121803176	7.83×10^{-16}
0.6	0.4373085121	0.4373085179	1.87×10^{-13}	0.4373085121	6.15×10^{-16}
0.7	0.4228880686	0.4228880686	8.54×10^{-14}	0.4228880686	2.33×10^{-16}
0.8	0.3560865486	0.3560865486	2.01×10^{-13}	0.3560865486	2.78×10^{-17}
0.9	0.2213642800	0.2213642834	1.44×10^{-13}	0.2213642800	8.17×10^{-17}
1.0	0.0000000000	0.0000000000	0.000000000	0.0000000000	0.000000000

Table 2. Numerical results of example 2 using 5 iterations.

x	Exact values	12 Legendre polynomials[4]		15 Bezier polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	0.0000000000	0.0000000000	0.000000000	0.0000000000	0.00000000
0.1	0.0998334166	0.0998334162	3.34×10^{-10}	0.0998334166	2.10×10^{-15}
0.2	0.1986693308	0.1986693307	5.05×10^{-10}	0.1986693308	7.99×10^{-15}
0.3	0.2955202067	0.2955202060	5.72×10^{-11}	0.2955202067	2.82×10^{-16}
0.4	0.3894183423	0.3894183423	2.82×10^{-11}	0.3894183423	3.90×10^{-16}
0.5	0.4794255386	0.4794255386	9.91×10^{-10}	0.4794255386	4.27×10^{-16}
0.6	0.5646424734	0.5646424733	1.43×10^{-11}	0.5646424734	4.46×10^{-16}
0.7	0.6442176872	0.6442176870	1.85×10^{-10}	0.6442176872	4.97×10^{-16}
0.8	0.7173560909	0.7173560909	8.77×10^{-11}	0.7173560909	3.44×10^{-16}
0.9	0.7833269096	0.7833269090	4.99×10^{-11}	0.7833269096	1.85×10^{-16}
1.0	0.8414709848	0.8414709848	0.000000000	0.8414709848	0.00000000

Table 3. Numerical results of example 3.

x	Exact values	10 Bezier polynomials		16 Bezier polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	0.0000000000	0.0000000000	0.000000000	0.0000000000	0.00000000
0.1	−0.0898500750	−0.0898500697	7.81×10^{-8}	−0.0898500750	5.41×10^{-15}
0.2	−0.1589354646	−0.1589354607	3.96×10^{-9}	−0.1589354646	4.22×10^{-15}
0.3	−0.2068641447	−0.2068641420	4.41×10^{-8}	−0.2068641447	2.94×10^{-15}
0.4	−0.2336510054	−0.2336510041	1.49×10^{-9}	−0.2336510054	1.75×10^{-15}
0.5	−0.2397127693	−0.2397127690	1.53×10^{-10}	−0.2397127693	5.83×10^{-16}
0.6	−0.2258569894	−0.2258569906	9.14×10^{-9}	−0.2258569894	5.27×10^{-16}
0.7	−0.1932653062	−0.1932653085	4.72×10^{-9}	−0.1932653062	1.61×10^{-15}
0.8	−0.1434712181	−0.1434712216	2.22×10^{-10}	−0.1434712181	2.58×10^{-15}
0.9	−0.0783326910	−0.0783326953	6.04×10^{-9}	−0.0783326910	3.41×10^{-15}
1.0	0.0000000000	0.0000000000	0.000000000	0.0000000000	0.00000000

Table 4. Numerical results of example 4 using 6 iterations.

x	Exact values	6 Legendre polynomials[9]		13 Bezier polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.0	1.0000000000	1.0000000000	0.000000000	1.0000000000	0.0000000
0.1	0.9048374180	0.9048371114	3.34×10^{-7}	0.9048374180	3.81×10^{-12}
0.2	0.8187307531	0.8187302595	5.05×10^{-7}	0.8187307531	8.34×10^{-12}
0.3	0.7408182207	0.7408177846	5.72×10^{-7}	0.7408182207	1.74×10^{-11}
0.4	0.6703200460	0.6703285281	2.82×10^{-6}	0.6703200460	3.23×10^{-11}
0.5	0.6065306597	0.6065325363	9.91×10^{-6}	0.6065306597	4.39×10^{-11}
0.6	0.5488116361	0.5488171372	1.43×10^{-6}	0.5488116361	4.16×10^{-11}
0.7	0.4965853038	0.4965807868	1.85×10^{-6}	0.4965853038	2.18×10^{-11}
0.8	0.4493289641	0.4493201549	8.77×10^{-6}	0.4493289641	7.48×10^{-12}
0.9	0.4065696597	0.4065691363	4.99×10^{-7}	0.4065696597	1.91×10^{-11}
1.0	0.3678794412	0.3678794412	0.000000000	0.3678794412	0.00000000

Table 5. Numerical results of example 5 using 7 iterations.

x	Exact values	11 Bernstein polynomials[13]		14 Bezier polynomials	
		Approximate	Abs. Error	Approximate	Abs. Error
0.00000	0.0000000000	0.0000000000	0.000000000	0.0000000000	0.00000000
0.06487	0.0628547398	0.0628547389	1.25×10^{-10}	0.0628547398	5.09×10^{-13}
0.12974	0.1219913568	0.1219913557	2.09×10^{-10}	0.1219913568	1.89×10^{-13}
0.19462	0.1778251645	0.1778251639	8.32×10^{-10}	0.1778251645	1.37×10^{-13}
0.25949	0.2307057366	0.2307057355	1.21×10^{-10}	0.2307057366	6.11×10^{-14}
0.32436	0.2809298081	0.2809298080	1.72×10^{-11}	0.2809298081	9.44×10^{-13}
0.38923	0.3287517292	0.3287517292	9.98×10^{-11}	0.3287517292	7.65×10^{-14}
0.45410	0.3743905511	0.3743905506	7.03×10^{-10}	0.3743905511	6.12×10^{-15}
0.51898	0.4180371649	0.4180371637	2.22×10^{-10}	0.4180371649	5.88×10^{-13}
0.58385	0.4598581862	0.4598581861	4.23×10^{-11}	0.4598581862	5.55×10^{-13}
0.64872	0.5000000000	0.5000000000	0.000000000	0.5000000000	0.000000000

V. Conclusion

In this paper, we have solved numerically higher order linear and nonlinear boundary value problems using Galerkin method. For nonlinear problems, when the order is higher the results will be better. The nonlinear BVPs take long time in testing and calculating to get more accurate results. The well-known Bezier polynomials have been exploited as basis functions in the method. These methods enable us to approximate the solutions at every points of the domain of integration. The concentration has given not only on the performance of the results but also on the formulations. We may notice that the formulations of this paper are very easy to understand and may be implemented to solve for any higher order BVP. The computed solutions are compared with the exact solutions and we have found a good agreement. The method presented in this paper performs better than other existing methods.

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