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Conformal Killing Vector Fields of Riemannian Manifolds

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Abstract

The main aim of this article to study about vector fields $\mathfrak{X}(N)$ of manifold N and how these vector fields will be Killing and conformal Killing vector fields. Conformal transformation of Weyl rescaling which is conformally related to metrices from g to $\breve{g}$, Levi-Civita connection ∇, Lie derivative, torsion with tensor concept of manifold N in a multi-linear map have been treated in this paper. Finally, we have been proved Example 3.02and established the theorem 6.02 on Conformal Killing vector fields.

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I. Introduction

In this paper we have focused on some important topics such as Killing vector field X of a set of vector fields $\mathfrak{X}(N)$of manifold N, the coordinates x^{μ} of a point $p \in N$ change to

$x^{\mu} = \varepsilon x^{\mu}(p)$ under the ε countless displacement, metrices t_{ij}, vectors x^{μ}, equation of Killing, torsion tensor, Riemannian curvature tensor and some theorems, examples are derived in a different elaborate way. The Killing vector field X is produced by a one-parameter familyof transfigurations. In one consciousness the Killing vector fields illustrate way of symmetry for any manifold. A set of Killing vector fields are in general defined to be dependent if one of them is presented as a linear combination of others with constant coefficients. In this way there may be exceeding Killing vector fields than the degree of the manifold. If an infinitesimal displacement εX generates a Conformal transformation, then the vector field $X \in \mathfrak{X}(N)$ is a conformal Killing vector field.

II. $\mathbb{R}^n$ Manifolds

An $\mathbb{R}^n$ manifold is a couple (N, t) where N is a C^{∞} manifold and t is a $\mathbb{R}^n$metric on N.

Definition 2.01 Let us consider a differentiable manifold N. A poetic tensor t on N is of rank $(0,2)$ is more general vectorfield on N.

At any point $q \in N$, t satisfies the properties given below.

(i) $t_q(P,Q) = t_q(Q,P)$

(ii) $t_q(P,Q) \geq 0$, if and only if $P = 0$.

Here $P, Q \in T_q N$ and $t_q = t\big|_q$. In a nutshell $t_q$has three forms.

Definition 2.02 Let us suppose N is a manifold which is differentiable. A more generalized vector field t be on N is duplicate $\mathbb{R}^n$ metric when it holds properties (i) &(ii) and $t_q(P,Q) = 0$ for any $P \in T_q N$, then $Q = 0$.

Definition 2.03 If t be$\mathbb{R}^n$metric, all the eigenvalues are exactly definite and if t is duplicate-Riemannian some eigenvalues are opposite. k, l are eigenvalues which are opposites and the couple (k, l) is called the exponent of generalized vector fields. If $k = l$ then that is called Lorentz metric. [1]

Definition 2.04 [12] Let (N, t) is Lorentzian. The elements of TqN are classified as

(i) space like when $t(X, X) > 0 \rightarrow X$

(ii) light like when $t(X, X) = 0 \rightarrow X$

(iii) time like when $t(X, X) < 0 \rightarrow X$

Definition 2.05 [2] When a C^{∞} manifold N agree a $\mathbb{R}^n$metric t, then (N, t) is referred to as Riemannian manifold. If t is a duplicate-Riemannian metric, then (N, t) is said to be a pseudo-Riemannian manifold. If t is

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a duplicate-Riemannian metric, then (N,t) is said to be a pseudo-Riemannian manifold. If t is Lorentzian, then (N,t) is called Lorentz manifold.

Example 2.06 An m-sized Euclidean space $(\mathbb{R}^m,\delta)$ is a Riemannian manifold and an $m-$degreed Minkowski space $(\mathbb{R}^m,\eta)$ is a Lorentz manifold.

III. Covariant Derivative with Affine Connection

Let us consider two nearby tangent spaces τ_ξ and $\tau_{\xi+d\xi}$. Also let $\chi = x^i e_i(\xi) \in \tau_\xi$ and $\tilde{\chi} = \tilde{x}^i e_i(\xi + d\xi) \in \tau_{\xi+d\xi}$.

The basis vectors e_i depend on the base point. So even if the components $x^i, \tilde{x}^i$ are the same, we can not actually say that $x, \tilde{x}$ are same.

The notion of an affine connection defines some one-one correspondence between tangent spaces with nearby base points that limit to the identity when the two base points coincide.

We want to introduce a connection which is known as affine connection as a means of mapping basis vectors $\tau_{\xi+d\xi}$ to τ_ξ. The map from $e_i^{'} \in \tau_\xi$ to $\tilde{e}_i \in \tau_{\xi+d\xi}$ is close to $e_i(\xi)$, so we can represent it as

$$e_i^{'} = e_i + de_i = e_i + (de_i^j)e_j$$

But $de_i^j \to 0$ as $d\xi \to 0$.

Also this is linear in $d\xi$ that is $de_i^j = \Gamma_{ki}^j(\xi)d\xi^k$.

So $e_i^{'} = e_i + \Gamma_{ki}^j(\xi)d\xi^k e_j$.

This is linear correspondence between τ_ξ and $\tau_{\xi+d\xi}$ is established by specifying on affine connection $\Gamma = \Gamma_{ki}^j$.

Definition 3.01 [3] Along tangent vector of manifold the way of specifying a derivative is called a covariant derivative and defined by

$$\nabla_X(fY) = (\nabla_X f)Y + f\nabla_X Y.$$

Example 3.02 Let $N = \mathbb{R}^2 = \{x_1, x_2\}$ a plane. ∇ is a connection on N with coordinate symbols $\Gamma_{11}^2 = x_1^2$, $\Gamma_{21}^1 = x_1 + x_2$, all others are zero. Let us compute $\nabla_X Y$, where

$$X = e^{x_2}\frac{\partial}{\partial x^1} + 3\frac{\partial}{\partial x^2}$$

$$Y = x_1\frac{\partial}{\partial x^1} + x_2\frac{\partial}{\partial x^2}$$

Proof. Given

$$X = e^{x_2}e_1 + 3e_2$$

$$Y = x_1 e_1 + x_2 e_2$$

where $e_1 = \dfrac{\partial}{\partial x^1}$, $e_2 = \dfrac{\partial}{\partial x^2}$.

Now

$$\nabla_X Y = \nabla_X(x_1e_1 + x_2e_2)$$

$$= \nabla_X(x_1e_1) + \nabla_X(x_2e_2)$$

$$= \nabla_{e^{x_2}e_1+3e_2}(x_1e_1) + \nabla_{e^{x_2}e_1+3e_2}(x_2e_2)$$

$$= \nabla_{e^{x_2}e_1}(x_1e_1) + \nabla_{3e_2}(x_1e_1) + \nabla_{e^{x_2}e_1}(x_2e_2) + \nabla_{3e_2}(x_2e_2)$$

$$= e^{x_2}\nabla_{e_1}(x_1e_1) + 3\nabla_{e_2}(x_1e_1) + e^{x_2}\nabla_{e_1}(x_2e_2) + 3\nabla_{e_2}(x_2e_2)$$

$$= e^{x_2}[e_1(x_1)e_1) + x_1\nabla_{e_1}e_1] + 3[e_2(x_1)e_1 + x_1\nabla_{e_2}e_2] +$$
$$e^{x_2}[e_1(x_1)e_2) + x_2\nabla_{e_1}e_2] + 3[e_2(x_2)e_2 + x_2\nabla_{e_2}e_2]$$

$$= e^{x_2}[1.e_1 + x_1\nabla_{e_1}e_1] + 3[0.e_1 + x_1\nabla_{e_2}e_2] +$$
$$e^{x_2}[0.e_2 + x_2\nabla_{e_1}e_2] + 3[1.e_2 + x_2\nabla_{e_2}e_2]$$

$$= e^{x_2}[e_1 + x_1\nabla_{e_1}e_1 + x_2\nabla_{e_1}e_2] + 3[e_2 + x_1\nabla_{e_2}e_2 + x_2\nabla_{e_2}e_2]$$

Now, $\nabla_{e_i}e_j = \sum_{k=1}^n \Gamma_{ij}^k e_k$

$$\therefore \nabla_{e_1}e_1 = \sum_{k=1}^2 \Gamma_{11}^k e_k \quad \text{and} \quad \nabla_{e_2}e_2 = \sum_{k=1}^2 \Gamma_{22}^k e_k$$

$$= \Gamma_{11}^1 e_1 + \Gamma_{11}^2 e_2 \quad \text{and} \quad = \Gamma_{22}^1 e_1 + \Gamma_{22}^2 e_2$$

$$= 0.e_1 + x_1^2 x_2.e_2 \quad \text{and} \quad = 0.e_1 + 0.e_2$$

$$= x_1^2 x_2.e_2 \quad \text{and} \quad = 0$$

and

$$\therefore \nabla_{e_1} e_2 = \sum_{k=1}^{2} \Gamma_{12}^{k} e_k$$

$$= \Gamma_{12}^{1} e_1 + \Gamma_{12}^{2} e_2$$

$$= (x_1 + x_2).e_1 + 0.e_2$$

$$= (x_1 + x_2).e_1$$

$$= \nabla_{e_2} e_1 .$$

Therefore,

$$\nabla_X Y = e^{x_2}[e_1 + x_1.x_1^2 x_2.e_2 + x_2.(x_1 + x_2).e_1] + 3[e_2 + x_1.(x_1 + x_2).e_1 + x_2.0]$$

$$= e_1[e^{x_2} + x_2.(x_1 + x_2).e^{x_2} + 3x_1.(x_1 + x_2)] + e_2[x_1^3 x_2 e_2 + 3]$$

$$= [e^{x_2} + x_2.(x_1 + x_2).e^{x_2} + 3x_1.(x_1 + x_2)]\frac{\partial}{\partial x^1} + [x_1^3 x_2 e_2 + 3]\frac{\partial}{\partial x^2} .$$

IV. Torsion Tensor and Riemann Curvature Tensor

Definition 4.01 Let $\chi(N)$ be the collection of all differentiable vector fields on differentiable manifold N with connection $\mathfrak{D}$. Then the mapping

$$\tau : \chi(N) \times \chi(N) \to \chi(N) \text{ given by}$$

$$\tau(X, Y) = \mathfrak{D}_X Y - \mathfrak{D}_Y X - [X, Y]$$

where $[.,.] : \chi(N) \times \chi(N) \to \chi(N)$

$$[X, Y] = XY - YX .$$

Then $\tau(X, Y) \in \chi(N)$ is called the torsion tensor or simply torsion of the connection $\mathfrak{D}$. Also τ is a tensor of type $(1,2)$.

Definition 4.02 On affine connection ∇, torsion τ^{∇} is a map

$$\tau^{\nabla} : \chi(N) \times \chi(N) \to \chi(N) \text{ given by}$$

$$(X, Y) \mapsto \tau^{\nabla}(X, Y)$$

Theorem 4.03 For every affine connection ∇, its torsion τ^{∇} is a special vector of type $(1,2)$.

Definition 4.04 If $\tau = 0$, then the connection $\mathfrak{D}$ is said to be torsion free or symmetric. Thus if $\tau = 0$ we have $\mathfrak{D}_X Y - \mathfrak{D}_Y X = [X, Y]$. Thus the Lie bracket is also related in terms of covariant derivative. In terms of local coordinates (x^i), the component $\tau_{i,j}^{k}$ of τ is then $\tau_{i,j}^{k} \dfrac{\partial}{\partial x^k} = \tau\left(\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial x^j}\right)$. Here $\tau_{i,j}^{k}$ are tensor features.

Theorem 4.05 [2] The curvature R^{∇} for each affine connection ∇ is a tensor of rank $(1,3)$.

V. Connections with Levi-Civita

Let $\langle .,. \rangle$ denotes the Riemannian metric on a manifold N , there exists an unique affine connection ∇ satisfies

(i) $\nabla_{\eta}\xi - \nabla_{\xi}\eta = [\eta, \xi]$

(ii) $\chi\langle \eta, \xi \rangle = \langle \nabla_{\chi}\eta, \xi \rangle + \langle \eta, \nabla_{\chi}\xi \rangle$

For all $\chi, \eta, \xi \in \aleph(N)$. This affine connection ∇ is called the Connection with Levi-Civita on N .

Theorem 5.01 On a pseudo-$\mathbb{R}^n$ manifold (N, t), there exist a unique symmetric Levi-Civita connection, which is compatiable having the metric t .

Proof. Suppose $\tilde{\Gamma}$ is any arbitrary connection coefficient and $\tau_{k\,j}^{i}$ be the torsion tensor. Then we have from the definition of any connection coefficient

$$\tilde{\Gamma}_{jk}^{i} = \left\{ {i \atop jk} \right\} + K_{jk}^{i} \text{, where } i, j, k = 1,2 .$$

and $\left\{ {i \atop jk} \right\}$ are Christoffel symbols defined by

$$\left\{ {i \atop jk} \right\} = \frac{1}{2} t^{il} \left(\frac{\partial t_{kl}}{\partial x^j} + \frac{\partial t_{jl}}{\partial x^k} - \frac{\partial t_{jk}}{\partial x^l} \right).$$

Coefficient of connection are

$$\Gamma_{jk}^{i} = \tilde{\Gamma}_{jk}^{i} + \tau_{jk}^{i} .$$

Now we choose

$$\tau_{jk}^{i} = -K_{jk}^{i} \text{ so that}$$

$$\Gamma_{jk}^{i} = \left\{ {i \atop jk} \right\} = \frac{1}{2} t^{il} \left(\frac{\partial t_{kl}}{\partial x^j} + \frac{\partial t_{jl}}{\partial x^k} - \frac{\partial t_{jk}}{\partial x^l} \right).$$

By structure, the connection is obviously symmetric and clearly unique obtained a metric.

Example 5.02 Let the geodesic of the surface of a sphere is $(ds)^2 = a^2(d\theta)^2 + a^2 \sin^2 \theta (d\phi)^2$. The non-zerocoefficients of the Levi-Civita connection coefficients are

$$\Gamma^1_{22} = -\sin\theta\cos\theta \text{ and } \Gamma^2_{12} = \cot\theta .$$

Example 5.03 The geodesics on a plane in polar coordinates are straight lines and the geodesics is $(ds)^2 = (du)^2 + u^2(dv)^2$ and the non-zero Christoffel symbols are

$$\Gamma^1_{22} = -u \text{ and } \Gamma^2_{12} = \frac{1}{u}.$$

VI. Killing Vector Fields with Conformality

Let $\aleph$ be a global C^∞ vector field in space time N. The $\aleph$ is called Conformal Killing Vector field when it holds

$$L_{\aleph} t_{ab} = L\psi t_{ab}$$

where ψ is scalar valued and known as Conformal factor. If the Conformal factor become constant, the Conformal Killing Vector (CKV) become Homthetic CKV reduce to Killing vector.

Let us suppose that (N,t) be a manifold of Riemannian and $X \in \aleph(N)$. A tiny dislocationεX, where ε is a very small, creates an equal dimension, then the vector field X is called a Killing vector field. If $f : x^\mu \to x^\mu + \varepsilon X^\mu$ is an isometry. It holds

$$\frac{\partial(x^k + \varepsilon X^k)}{\partial x^\mu}\frac{\partial(x^\lambda + \varepsilon X^\lambda)}{\partial x^\nu} t_{k\lambda}(x + \varepsilon X) = t_{\mu\nu}(x)$$

Example 6.01 Let x^k be the identicals of $(\mathbb{R}^m, \delta)$. Then $(\mathbb{R}^2, \delta)$ has three such type fields. The Minkowski spacetime $(\mathbb{R}^4, \eta)$, the Killing equation yields $\partial_\mu X_\vartheta + \partial_\vartheta X_\mu = 0$

In fact$\mathcal{L}_D \delta_{\mu i} = \partial_\mu x^i \delta_{ij} + \partial_j x^\lambda \delta_{\mu\lambda} = 2\delta_{i\mu}$.

Theorem 6.02 Let N be a manifold and $X \in N$. Then X is a conformal Killing vector field ifwe have $f \in C^\infty(N)$ for which$\mathcal{L}_X t = ft$, if and only if $X_{\hat{\alpha}} = X^c - f\nabla$, where $\alpha = \hat{t}(X)$.

Proof. If X is a Conformal Killing vector field $\exists$ a function $f \in C^\infty(N)$ such that $\mathcal{L}_X t = ft$ and then

$$\theta_{\tau\mathcal{L}X} t = f\theta_{\tau g} (1)$$

We know the relation

$$i(X_{\hat{\alpha}} - X^c)\omega\tau_t = d(i(X^c)\theta_{\tau_t}) + i(X^c)d\theta_{\tau_t} = \mathcal{L}_{X^c}\theta t$$

$$= \theta_{X^c}\tau_t = \theta_{\mathcal{T}\mathcal{L}_X t} (2)$$

We also know the relation

$$i(\Delta)\omega\tau_t = -\theta_{\tau_t} (3)$$

Using the relation (2), from (1) we get,

$$i(X_{\hat{\alpha}} - X^c)\omega\tau_t = f\theta_{\tau_t}.$$

Using relation (3), from (1) we get

$$i(X_{\hat{\alpha}} - X^c)\omega\tau_t = i(f\Delta)\omega\tau_t.$$

As ωT_g is nondegenerate we find

$$X_{\hat{\alpha}} - X^c = -f\Delta .$$

Conversely, if $\exists$ a function $f \in C^\infty(N)$ such that $X_{\hat{\alpha}} - X^c = -f\Delta$, then

$$i(X_{\hat{\alpha}} - X^c)\omega\tau_t = -i(f\Delta)\omega\tau_t = f\theta_{\tau_t}.$$

As a consequence of relation (3) we obtain that $\theta_{\mathcal{T}\mathcal{L}_X t} = f\theta\tau$, which implies

$\mathcal{L}_X t = ft$. Hence proved.

VII. Conclusion

In this paper, some significant and basic definitions, examples and theorems which are unavoidable. Finally, in section III, the Example 3.04 has been proved which is related with affine connection of manifold N and in section VI, the theorem 6.02 has been established. This modern approach to prove these example and theorem will be so sound for further research in Conformal Killing vector fields on the Riemannian Manifolds.

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