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A Simulation Based Comparative Study to Find Efficient Parameter Estimation Methods for Weibull Distribution

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Abstract

This paper aims to find efficient methods for estimating the parameters (shape $=\alpha$, scale $=\beta$) of Weibull distribution in different situations. The maximum likelihood estimation method (MLE), the median rank regression method (MRR), the least square method (LSM) and the weighted least square method (WLSM) are considered for the estimation of the parameters. The root mean square error (RMSE) criterion is used to measure the relative efficiency of the estimators experimentally (Monte Carlo simulation). From the simulation study, it is observed that the MLE produces the lowest RMSE, irrespective of all sample sizes, for decreasing hazard function $(\alpha << \beta)$ (α is considerably smaller than β) and roughly linear hazard function with a positive slope ($\alpha > 1$) . When $(\alpha >> \beta)$ the WLSM produces the lowest RMSE for small sample sizes $(n \le 40)$ but for large sample sizes it is the MLE, irrespective of all types of hazard functions. When $(\alpha, \beta \to 1)$, the WLSM produces the lowest RMSE for small sample sizes $(n \le 40)$ and the MLE for large sample sizes irrespective of all types of hazard functions. This pattern becomes reversed when α and β have the larger value. Only the MLE gets stuck when the hazard function is parallel to $Y-$axis $(\alpha >> \beta)$ and the WLSM is suitable in such a situation (lowest RMSE) irrespective of all sample sizes. Finally, the utility of simulation results have been illustrated by analyzing two real-life data sets.

Keywords: Weibull distribution, the maximum likelihood method, the least square method, the weighted least square method, median rank regression method, Kolmogorov-Smirnov test.

I. Introduction

The Weibull density is one of the widely used distributions in statistics to model the life time data. It has a wide range of applicability because of its flexible hazard function (both increasing and decreasing hazard rate based on α value). The density function of Weibull distribution is defined as

$$f(x) = \frac{\alpha}{\beta^{\alpha}} (x)^{\alpha-1} \exp\left(\left(\frac{x}{\beta}\right)^{\alpha}\right);\ x, \alpha, \beta > 0.$$

An efficient application of Weibull distribution solely depends on using an efficient estimation method to estimate its parameters.The maximum likelihood estimation method (MLE), the median rank regression method (MRR), the least square method (LSM) and the weighted least square method (WLSM) are commonly used for the estimation of the parameters. There are several studies available in the literature related to finding the efficient methods to estimate the Weibull parameters. For example, Bergman [1], Lu, Chen and Wu[2] recommend the WLSM over LSM to estimate the parameters of Weibull distribution. Wu, Zhou and Li[3] conducted a simulation study to compare the LSM, the WLSM, the MLE and the MOM and they recommended the MLE in general but the WLSM for small sample sizes. All the studies mentioned so far considered only one combination of α and β values in their simulation settings, more specifically, they considered only $\alpha = 10$ and $\beta = 1$. Nielsen[4] recommended the MRR for small sample sizes but the MLE in general after comparing the MLE, the MRR and the MOM. From the comparison study conducted by Chu and Ke[5], where LSM and the MLE are compared, it was found that the LSM significantly outperforms the MLE when the sample size is small. Pobočíková and Sedliačková[6] compared the LSM, the WLSM, the MLE and the MOM based on Monte Carlo simulation and they recommend the MLE in general but the WLSM for small sample sizes.

In their simulation study, they considered only a few combinations of α and β values. More specifically, they considered $\alpha < 1, \alpha = 1$ and $\alpha > 1$ for a specific scale parameter $\beta = 1$. They kept the scale parameter fixed as it does not change the shape of the density function. The reason for choosing $\alpha < 1, \alpha = 1$ and $\alpha > 1$ is that they just considered decreasing, constant and increasing hazard functions. Their study did not focus on the scenarios that arise due to different types of increasing hazard functions. Different types of increasing hazard functions arise due to the combined effect of α and β parameter values shown in Figure 2. From the literature review, it is observed that there is no study which considers all the methods mentioned above altogether for comparative study. Furthermore, there is a scope for conducting an extensive simulation study in every study by considering different combinations of α and β values.

Motivated by these lacks, we aim to conduct a comparative study by covering all the lacks found in the literature to find efficient methods for estimating the Weibull parameter in different situations. In our extensive simulation study, we have considered a fully crossed experimental design to

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create different combinations of α and β values. Table 1 presents the different combinations of α and β values for which simulated data sets have been created for comparing different estimation methods. We have chosen α and β values in such a way that different situations of Weibull distribution are covered. For each combination of α and β values, we have also considered here different sample sizes to investigate the effect of sample size on different estimation methods.

Different scenarios of Weibull distribution considered in Table 1 can be categorized into 3 categories: (i) $\alpha < \beta$ (lower triangle elements of Table 1) (ii) $\alpha = \beta$ (diagonal elements of Table 1) and (iii) $\alpha > \beta$ (upper triangle elements of Table 1). Figure 1 shows the three different shapes of Weibull probability density function (pdf) which can be seen for different combination of α and β considered in scenarios (i)-(iii). The key to comprehend the nature of Weibull distribution is the shape parameter. The nature of the hazard functions can also be characterized by the shape parameter of Weibull density. Figure 2 shows the hazard functions of Weibull distribution for the parameter combinations of Table 1. For different combination of α and β values, four different shapes of hazard function can be seen, which is discussed in detail in section 2.

Table 1. Different combinations of α and β values (fully crossed experimental design)

		α					
		0.3	0.6	0.9	1.5	10	50
	0.3	$(0.3,0.3)$	$(0.3,0.6)$	$(0.3,0.9)$	$(0.3,1.5)$	$(0.3,10)$	$(0.3,50)$
	0.6	$(0.6,0.3)$	$(0.6,0.6)$	$(0.6,0.9)$	$(0.6,1.5)$	$(0.6,10)$	$(0.6,50)$
	0.9	$(0.9,0.3)$	$(0.9,0.6)$	$(0.9,0.9)$	$(0.9,1.5)$	$(0.9,10)$	$(0.9,50)$
β	1.5	$(1.5,0.3)$	$(1.5,0.6)$	$(1.5,0.9)$	$(1.5,1.5)$	$(1.5,10)$	$(1.5,50)$
	10	$(10,0.3)$	$(10,0.6)$	$(10,0.9)$	$(10,1.5)$	$(10,10)$	$(10,50)$
	50	$(50,0.3)$	$(50,0.6)$	$(50,0.9)$	$(50,1.5)$	$(50,10)$	$(50,50)$

The rest of the paper is organized as follows. In section 2, a brief review of different estimation methods of parameters of Weibull distribution is presented. In section 3, we discuss the results of our simulation study while the final section shows the application of the findings of the simulation study where two real-life data sets are used.

II. Estimation Methods

Maximum Likelihood Estimation Method

The maximum likelihood estimation method (MLE) is one of the most popular robust techniques for estimating the parameters of a probability distribution. Let $X_1, X_2, \ldots, X_n$ be independently identically distributed $Weibull(\alpha,\beta)$, then the log likelihood function can be defined as

$$l(\alpha,\beta|x) = nlog\alpha - n\alpha log\beta + (\alpha - 1)\sum_{i=}^{n} logx_i - \sum_{i=}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha}.$$

The log likelihood function needs to be optimized to get the MLE of α and β and optimization is done iteratively using the "nlm" function in R.

Least Square Method

The least square method (LSM) is also known as Mean Rank Regression Method or Rank Regression Method. The cdf of the random variable $X \sim Weibull(\alpha,\beta)$, $F(x)$, can be expressed as a linear function of the form

$$Y = \psi_0 + \psi_1 X_1,$$

where $Y = \log(-\log(1-F(x)), X_1 = logX,$ $\psi_0 = -\alpha log\beta$ and $\psi_1 = \alpha.$

The least square estimators of ψ_0 and ψ_1 can be obtained by minimizing the following function with respect to ψ_0 and ψ_1

$$Q(\psi_0,\psi_1) = \sum_{i=1}^{n}(Y_i - \psi_0 - \psi_1 logx_{(i)})^2,$$

where $x_{(i)}$ be the i^{th} smallest observations of $x_1, x_2, \cdots x_n$. The estimators $\hat{\alpha}$ and $\hat{\beta}$ are

$$\hat{\alpha} = \hat{\psi}_1 \text{ and } \hat{\beta} = \exp\left[\frac{-\sum_{i=1}^{n}\log[-\log(1-\hat{F}_i)] - \hat{\alpha}\sum_{i=1}^{n}\log x_{(i)}}{n\hat{\alpha}}\right],$$

where the mean rank $\hat{F}(x_{(i)}) = \frac{1}{n+1}$ is the estimate of F, which can be derived from $\widehat{\psi_0}$ and $\widehat{\psi_1}$ after back transformation. The details about the estimators are available in Pobočíková and Sedliačková[6].

Weighted Least Square Metho

The weighted least square method works like the way the least square method works but it introduces weight for each data point. The weighted least square method minimizes

$$Q(\psi_0,\psi_1)=\sum_{i=1}^{n} w_i\left(Y_i-\psi_0-\psi_1 \log x_{(i)}\right)^2,$$

to estimates $\hat{\psi}_0$ and $\hat{\psi}_1$, where $w_i = \left[\left(1-\widehat{F}_i\right)\log\left(1-\widehat{F}_i\right)\right], i = 1, 2, \ldots, n,$ denote the weights proposed by

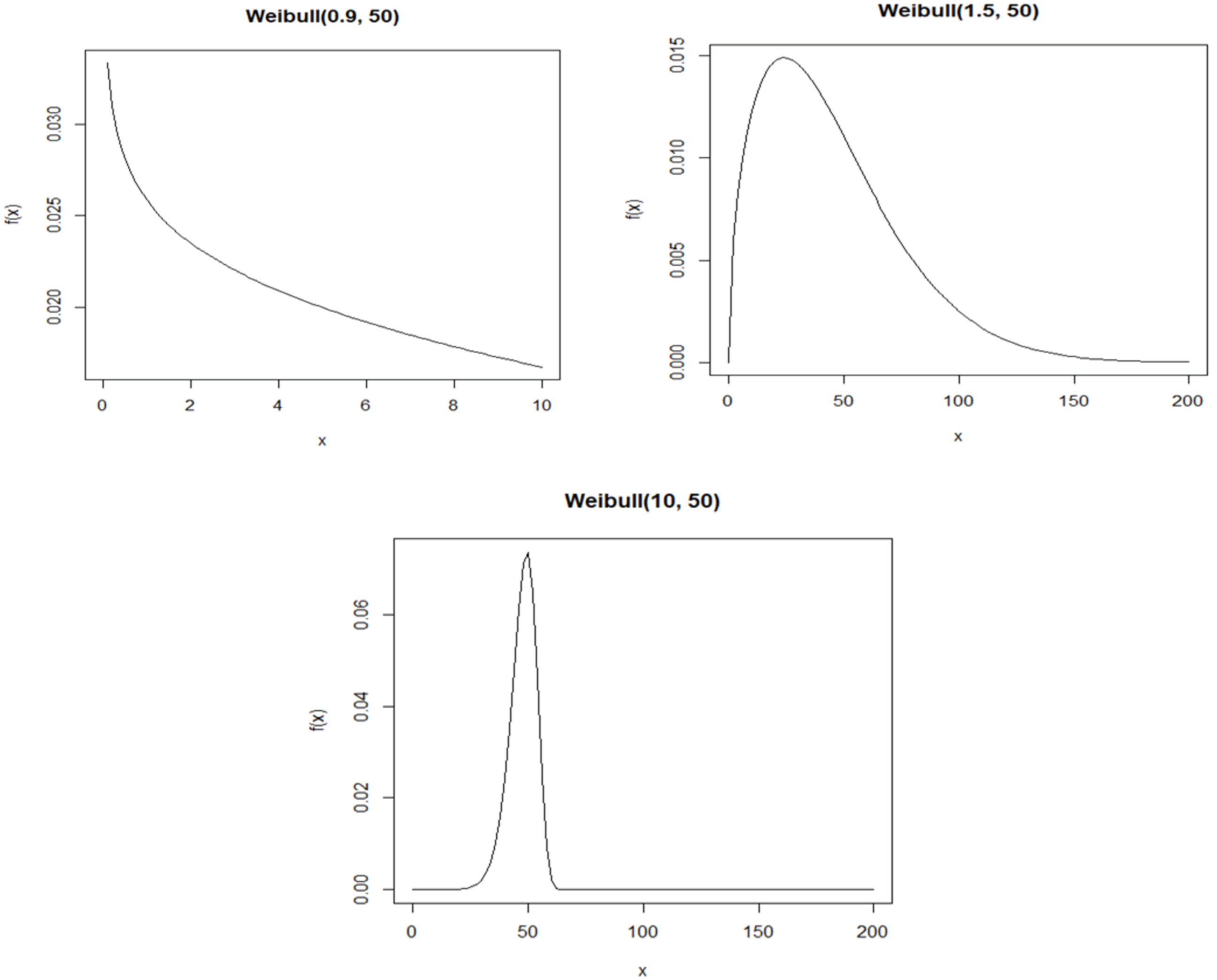


Fig. 1. Weibull density plots for different values of α and β (three different shapes).

Bergman [1]. The estimates $\hat{\alpha}$ and $\hat{\beta}$ of the parameter α and β are given by $\hat{\alpha}=\hat{\psi}_1$, $\hat{\beta}=$

$$\exp\left[\frac{-\sum_{i=1}^{n} w_i \log\left[-\log\left(1-\hat{F}_i\right)\right]-\hat{\alpha}\sum_{i=1}^{n} w_i \log x_{(i)}}{\hat{\alpha}\sum_{i=1}^{n} w_i}\right],$$

respectively and the details about the estimators are available in Pobočíková and Sedliačková[6].

Median Ranks Regression Method

The median ranks regression method works exactly the same way like the LSM works but the estimate of cdf(F) is measured by median rank instead of mean rank. The median ranks can be found by solving the cumulative binomial equation

$$\sum_{k=i}^{n}\binom{n}{k}(F_i)^k(1-F_i)^{(n-k)}=0.5$$

for F_i, where n is the sample size and i is the ordered number. Benard[7] suggested $\hat{F}_i=\frac{i-0.3}{n+0.4}$ as an approximation for estimating the median ranks which is fast but less efficient.

Shapes of Weibull Hazard Density

Three different types of hazard functions, namely increasing, decreasing and constant hazards function can be seen for different combinations of α and β values considered in Table 1.

The hazard functions decay exponentially irrespective of all β values when $\alpha<1$ (not shown here for the sake of visibility). Different types of increasing hazard functions, produced for $\alpha > 1$ and $\beta > 0$ are shown in Figure 2. The characteristic of the first type of increasing hazard functions is roughly linear with a positive slope. More specifically, the hazard functions look like the graph of $y = \sqrt{x}$, and this can be seen for $\alpha = 1.5$ and $\beta > 0$. The second type of increasing hazard functions can be seen for $\alpha = 10$ and $\beta > 0$. The main characteristic of this kind of hazard functions is that they start from zero and then gradually increase to infinity. The third type of increasing hazard functions are parallel to $Y-$axis at a particular point, and apart from this point the value of hazard function is zero everywhere (can be seen for $\alpha = 50$ and $\beta > 0$). One of the main contributions of this paper is to explore these three types of hazard functions in details and suggest a suitable method for estimating the parameters of Weibull distribution in such cases. Figure 2 presents different shapes of hazard function of Weibull distribution for different combination of α and β values. For the sake of visibility, all the graphs of hazard functions for different combination of α and β values are not considered here.

III. Simulation Study

This section presents all the results produced in the simulation study along with the discussion. The RMSEs is used to measure the performance of each estimation method for the simulated data sets.

Scenario 1 ($\alpha < \beta$, *lower triangle elements of Table 1)*

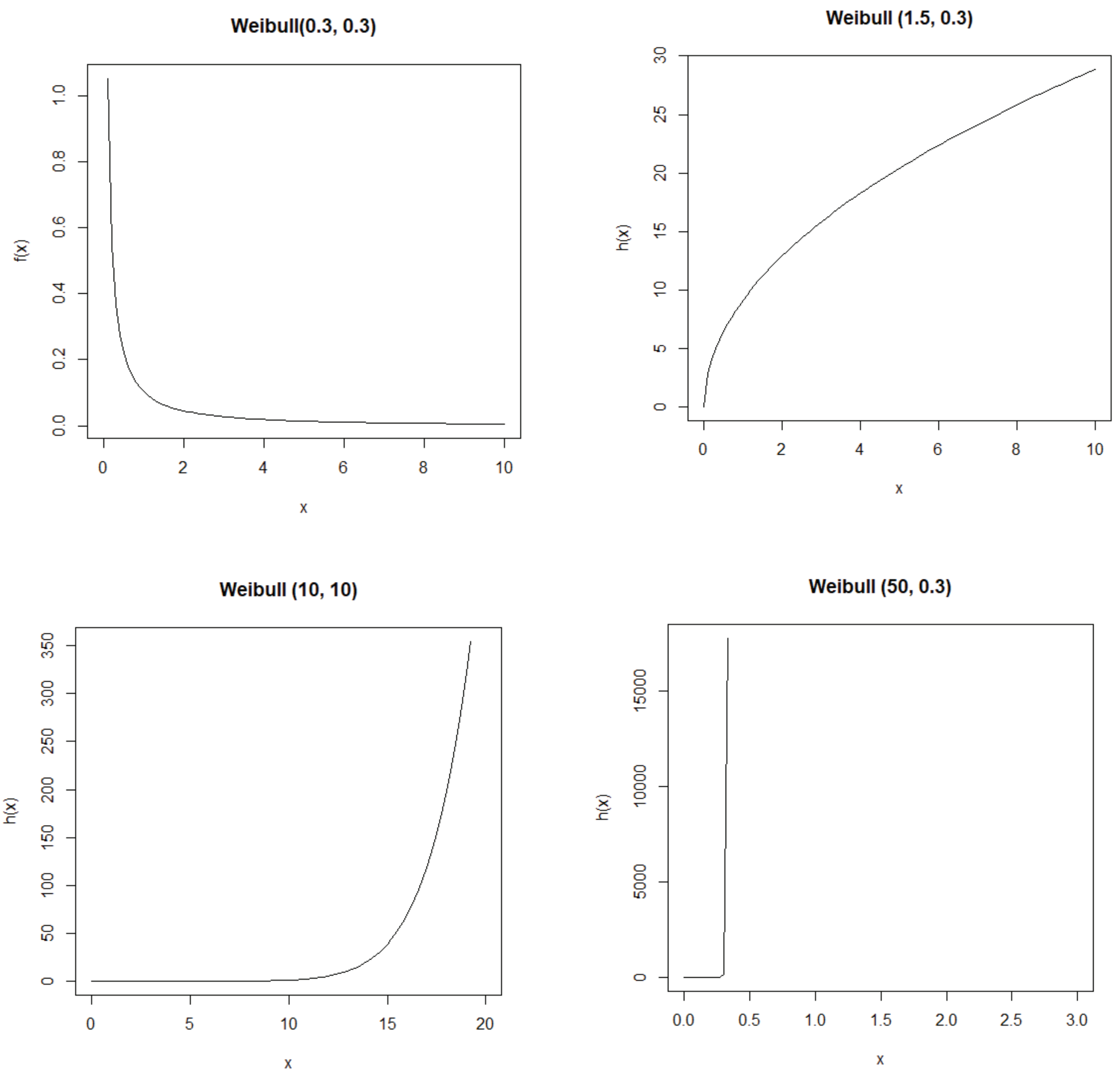


Fig. 2. Four different types of hazard function (shapes) of Weibull density for different values α and β values considered in Table 1.

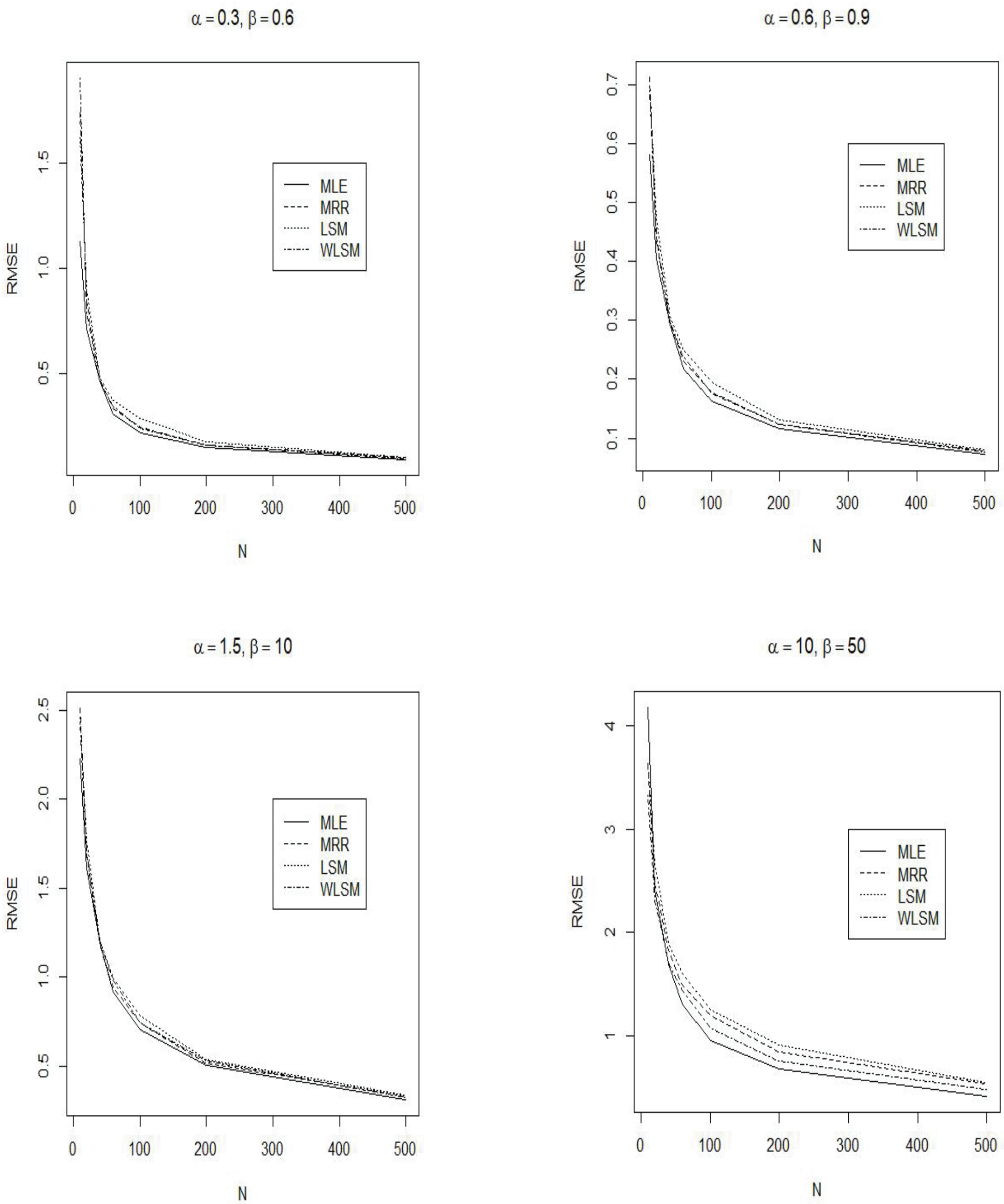


Fig. 3. RMSE plots for different combinations of the parameter under scenario 1.

Under this scenario, there are 15 combinations for different values of α and β. However, we have presented the results only for 4 combinations in Figure 3 for the sake of visibility. From the simulation study, it is observed that the MLE produces the lowest RMSE when $\alpha < 1$ and $\alpha \ll \beta$ (hazard function decreases exponentially).

Furthermore, for roughly linear hazard function with a positive slope, the MLE also produces the lowest RMSE which is seen for the combinations $\alpha = 1.5$, $\beta = 10$ and $\alpha = 1.5$, $\beta = 50$. Finally, for gradually increasing hazard function the WLSM produces the lowest RMSE until $n \le 40$ but after that the MLE produces the lowest RMSE.

Scenario 2 ($\alpha = \beta$*, diagonal elements of Table 1)*

All the simulated data sets (6 combinations) considered under this scenario are created by considering α and β values are equal. From the simulation study, it is observed that when the hazard function decays exponentially and $\alpha = \beta \to 1$, for small sample sizes ($n < 20$) the performance of WLSM is the best but for the large sample sizes ($n > 20$) the MLE performs better than any other method. We have seen very similar patterns of performance for other types of hazard functions of Weibull distribution. However, the larger the value of the shape parameter, the larger the value of n for which the WLSM produces the least RMSE. Again for the sake of visibility we have provided the results only for 2 combinations in Figure 4.

Scenario 3 ($\alpha > \beta$*, upper triangle elements of Table 1)*

From the simulation study conducted under scenario 3, it is observed that when hazard function decays exponentially and both α and β are less than 1 with $\alpha > \beta\left(\alpha, \beta < 1, \alpha > \beta\right)$ the performance of WLSM is the best for small sample sizes ($n < 20$) but for the large sample sizes ($n > 20$) it is the MLE. A similar pattern can also be seen in the simulation study for roughly linear hazard function and gradually increasing hazard function $\left(\alpha > \beta, \alpha > 1\right)$. For the hazard function parallel to Y axis $\left(\alpha \gg \beta\right)$, the performance of WLSM is the best irrespective of all sample sizes while MLE method is numerically unstable in this case shown in Figure 5.

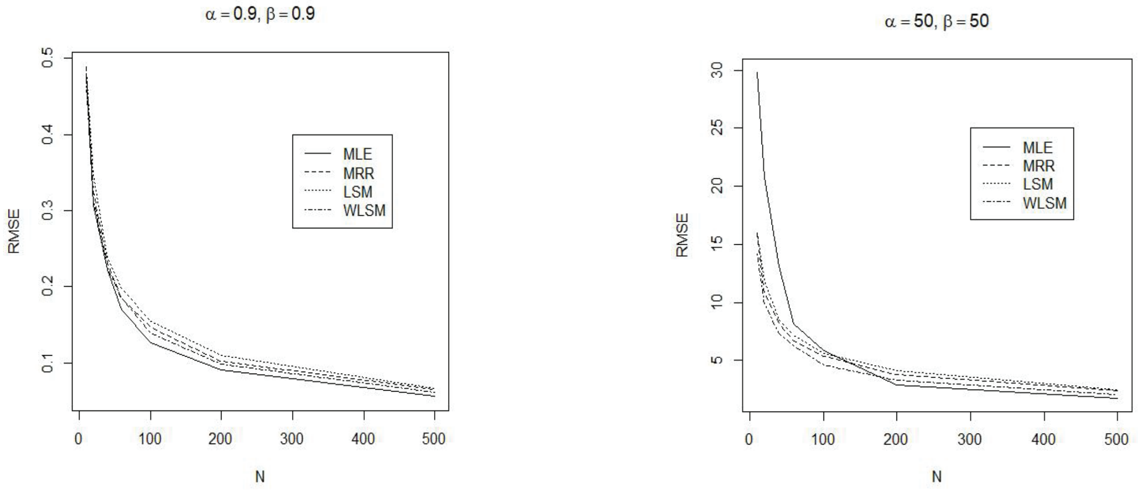


Fig. 4. RMSE plots for different combinations of the parameter under scenario 2.

IV. Application to Airborne and Ball Bearing Data

This section shows the utility of our simulation study by analyzing two data sets using the knowledge obtained from the simulation study. The first data set represents the repair times (in hours) for 46 failures of an airborne communications receiver which is available in Lawless[8]. The Weibull distribution can be used to model this data as histogram of repair times is right-skewed shown in Figure 6. We need to have an estimate of α and β to get an idea about the type of hazard function, and the estimate of α and β can be obtained by using any method (say MRR). The MLE, MRR, LSM and WLSM produce estimates of α and β for repair times data which are 0.8985 and 3.3913, 1.0420 and 3.3058, 1.0029 and 3.3320, and 0.8521 and 3.0169, respectively.

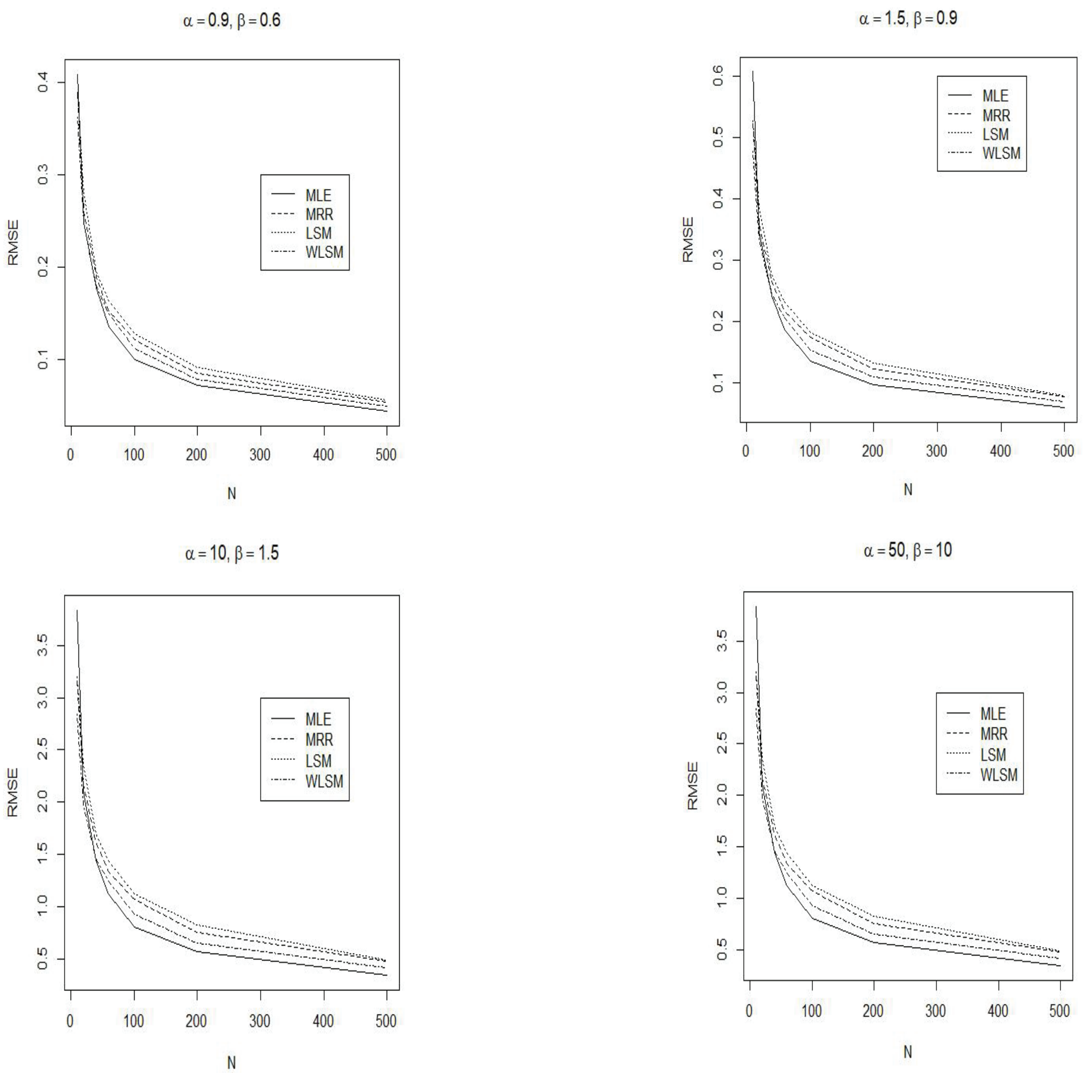


Fig. 5. RMSE plots for different combinations of the parameter under scenario 3.

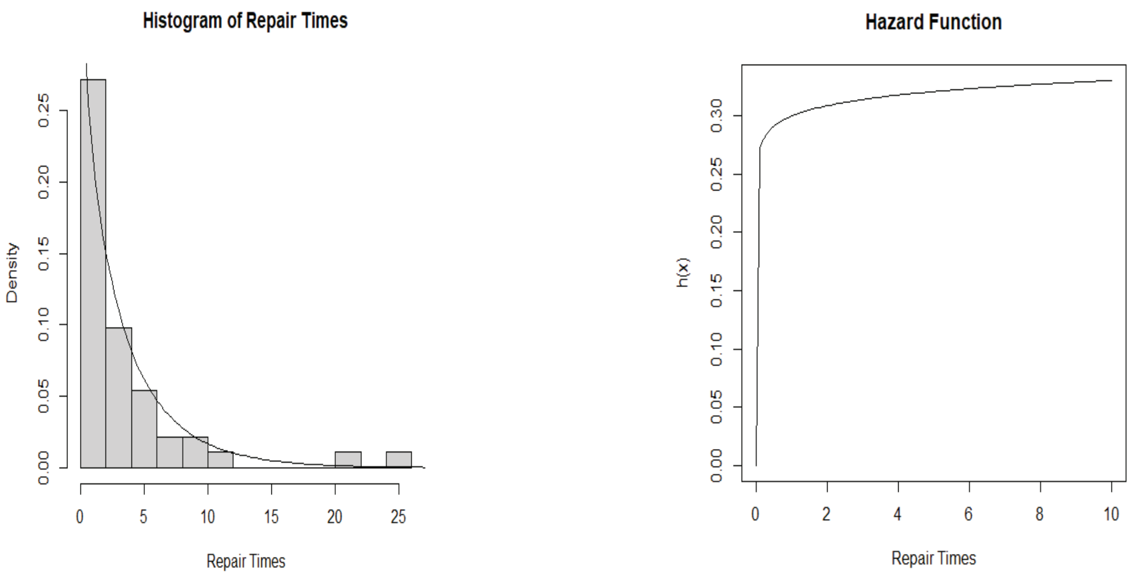


Fig. 6. Histogram of repair times data and the corresponding Weibull hazard function.

From the estimates of α and β values, we can see that $\hat{\alpha}$ is close to $1, \hat{\alpha} < \hat{\beta}$ and the hazard function roughly linear which is shown in Figure 6.Under these circumstances, our simulation study suggests that the MLE is an efficient estimation method (shown in Figure 3) which is consistent with the findings of Kolmogorov-Smirnov (KS) test (minimum distance) presented in Table 2.

Table 2. KS test results for repair times data

Methods	Distance	p-value
MLE	0.1204	0.5170
MRR	0.1447	0.2902
LSM	0.1382	0.3435
WLSM	0.1509	0.2455

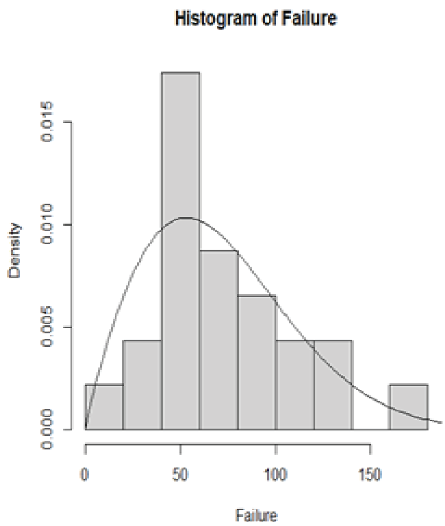
H_0 :repair times data came from Weibull (α, β)

The second data set also taken from Lawless[8] which represents the number of million revolutions before failure for each of the 23 ball bearings in their life test.The histogram of revolutions data shown in Figure 7 suggests that the Weibull distribution can be used to model this data set. Like earlier, the estimates of α and β under the MLE, MRR, LSM and WLSM are 2.1029 and 81.8933, 2.1818 and 81.5947, 2.0430 and 82.2113, and 1.8741 and 79.8487, respectively. Furthermore, for these estimated parameter values the corresponding Weibull hazard function is roughly linear shown in Figure 7. From the estimates of α and β values, we can see that $\hat{\alpha}$ is greater than $1, \hat{\alpha} \ll \hat{\beta}$ and the hazard is roughly linear (shown in Figure 7). Under these circumstances, our simulation study suggests that the WLSM is an efficient estimation method (shown in Figure 3) which is consistent with the findings of Kolmogorov-Smirnov (KS) test (minimum distance) presented in Table 3.

Table 3. KS test results for revolutions data

Methods	Distance	p-value
MLE	0.1513	0.6685
MRR	0.1532	0.6525
LSM	0.1504	0.6754
WLSM	0.1240	0.8714

H_0 :revolutions data came from Weibull (α, β)



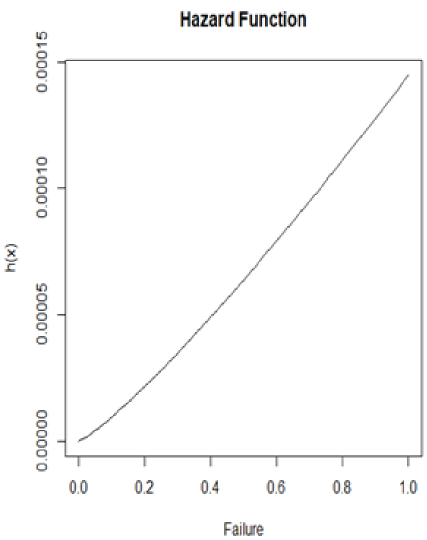


Fig. 7. Histogram of revolutions data and the corresponding Weibull hazard function.

V. Conclusion

In this paper, we have compared the performance of the MLE, the MRR, the LSM and the WLSM for estimating parameters of the Weibull distribution through a simulation study based on RMSE. From our simulation study, we found that for exponentially decaying and roughly linear hazard functions with $\alpha \ll \beta$ (shape parameter significantly smaller than scale parameter) the performanceof the MLE is the best irrespective of all sample sizes. For the same scenario i.e. $\alpha \ll \beta$, the WLSM and the MLE produce the lowest RMSE for sample sizes $n \le 40$ and $n > 40$, respectively, for gradually increasing hazard function. A very similar performance of WLSM and MLE can be seen when $\alpha = \beta \to 1$ irrespective of all types of hazard functions. Furthermore, when $\alpha \gg \beta$ and for the exponentially decays, roughly linear and gradually increasing hazard functions the WLSM and the MLE outperform other methods for sample sizes $n \leq 40$ and $n > 40$, respectively. On the other hand, the WLSM outperforms other methods for all sample sizes when the hazard function is parallel to Y-axis. However, the MLE does not work due to the effect of large shape and considerably small scale parameters in this situation. Findings of our simulation study can be used to choose an efficient method for estimating Weibull parameters which is consistent with the findings of Kolmogorov-Smirnov (KS) test. As a future work, one can extend this simulation study to find an efficient estimation method for Weibull distribution when there is censoring in the data.

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