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Characteristics of General Linear Group of Order 2 as Lie Group and Lie Algebra

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Abstract

The main target of this article is to study about Lie Groups, Lie Algebras. This article will enrich our knowledge about Algebraic properties of manifolds, how Lie Groups and Lie Algebras are working with their properties. Finally, we have discussed an example by showing all the properties of Lie Algebra,Lie Groups for a special Group and a Theorem has established.

Keywords: Lie Groups, Lie Algebras, Vector fields, Left Invariant, ψ -related vector fields, General Linear Groups, Manifolds.

I. Introduction

Lie Groups undoubtedly one of the significant special class of differentiable manifolds. A group where group operations are C^∞ is called the Lie Group.

In this article, we'll discuss about the foundations and formulations of Lie Groups. The central focus will be on the relationship between a Lie Group and its Algebra of the Left Invariant vector. We'll also prove a theorem, after verifying an example showing all the properties of Lie Groups and their Lie Algebras.The significance of this paper is the explanation of all properties of corresponding Groups.

II. Lie Groups

Definition 2.1 A differentiable Manifold N with group feature is defined as a Lie Group and the map

$$N \times N \to N \text{ defined by}$$

$$(\sigma,\tau) \mapsto \sigma\tau^{-1} \text{is } C^\infty .$$

Map $\tau \mapsto \tau^{-1}$ and $(\sigma,\tau) \mapsto \sigma\tau$ associated with $N \times N \to N$ are C^∞ because they are

compositions of

$$\tau \mapsto (e,\tau) \text{ and}$$

$$(\sigma,\tau) \mapsto (\sigma,\tau^{-1}) \mapsto \sigma\tau \text{ of } C^\infty \text{ maps.}$$

where, e denotes unity of N .

Example 2.2 A Lie group is a group of symmetries. In that sense we can say that the circle has a lot of symmetries and form a Lie group.

Example 2.3[1] General Linear Group $Gl(n,\mathbb{R})$ of all $n \times n$ matrices with real entries and non-zero determinant is a Lie group under the usual multiplsication of matrices.

III. Lie Algebras

Definition 3.1 Lie Algebra is a set of vectors $\mathfrak{g}$ accomplished by

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \text{ given by}$$

$$(i) [x,y] = -[y,x]$$

$$\text{(ii)} [[x,y],z] + [[y,z],x] + [[z,x],y] = 0 x,y,z \in \mathfrak{g}$$

Example 3.2[2] $Gl(n,\mathbb{R})$ form Lie Algebra when properties of Lie Bracket are satisfied. i.e.

$$[A,B] = AB - BA$$

where A,B are square matrices of order n .

Definition 3.3 Lie Algebra will be Abelian if all brackets are set equal to zero.

IV. Left Invariant

Lie algebra of N is like algebra of Left Invariant. Lie Algebra at any point is equivalent to Lie algebra as tangent space at identity of that Lie Group.

Definition 4.1[3] A form ω is said to be Left Invariant on N if

$$\delta l_\sigma \omega = \omega \text{ for every } \sigma \dot{o} N .$$

We denote differentiable $p-$ forms as

$$E_{l\ inv}^*(N) = \sum_{p=0}^{\dim N} E_{linv}^p(N).$$

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V. 9 Verifying all the Characteristics of Lie Groups and Its Algebras for the Group $Gl(2,\mathbb{C})$ [4]

Let us consider "general linear" group $Gl(n,\mathbb{C})$. If $n=2$, then $Gl(n,\mathbb{C})$ will be special group which is indicated by $Sl(2,\mathbb{C})$ where

$$Sl(2,\mathbb{C}) = \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \middle| ps - qr = 1 \right\} \quad \text{, where} \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathbb{C}^4.$$

(a) First we will prove that $Sl((2,\mathbb{C}),\circ)$ will be a group.

For convenience let $N = Sl(2,\mathbb{C})$.

Let us consider the map

$$\circ : N \times N \to N$$

Then N is closed, associative under usual matrix operation.

The neutral element of this group is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in N$.

$\forall \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in N$ there exists an element

$$\frac{1}{ps-qr}\begin{pmatrix} s & -q \\ -r & p \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}^{-1} \in N$$

for which $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \circ \begin{pmatrix} p & q \\ r & s \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Hence, N satisfies all the properties of being a group.

(b) Now we will show that (N,θ) is a topological space:

Let us define a topology $\theta_{\mathbb{C}}$ on $\mathbb{C}$ by virtue of the definition of open balls as

$$\mathfrak{B}_{r\in\mathbb{R}^+}(z) := \{y \in \mathbb{C} \,|\, |y - z| < r\}.$$

It will satisfy all the properties of being a topological space.

(c) We need to show that $((N,\mathbb{C}),\circ)$ is a topological manifold:

Let us construct the chart

$$u := \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in N \middle| p \neq 0 \right\},$$

$$x : u \to \mathbb{C}^3$$

with $x\left(\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right) = (p,q,r)$. This is continuous.

$$x : u \to x(u) \subseteq \mathbb{C}^3,$$

$$x^{-1} : x(u) \to u \text{ with}$$

$$x^{-1}(p,q,r) = \begin{pmatrix} p & q \\ r & \frac{1+qr}{p} \end{pmatrix}.$$ This is also continuous.

So,the above map is a homeomorphism. Hence, (u,x) is a chart of N.

Let us construct another chart as

$$v := \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in N \middle| s \neq 0 \right\}$$

$$y : v \to \mathbb{C}^3$$

with $y\left(\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right) = (q,r,s)$. This is continuous.

$$y : v \to y(v) \subseteq \mathbb{C}^3 \text{ and}$$

$$y^{-1} : y(v) \to v \text{ with}$$

$$y^{-1}(q,r,s) = \begin{pmatrix} \frac{1+qr}{s} & q \\ r & s \end{pmatrix}.$$ This is also continuous.

Hence, the map $y : v \to y(v)$ is homeomorphism.

So (v,y) is alsoa chart of N.

We'll find third chart as

$$w := \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in N \middle| q \neq 0 \right\}$$

$$z : w \to \mathbb{C}^3$$

with $z\left(\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right) = (p,q,s)$. This is continuous.

$$z : w \to z(w) \subseteq \mathbb{C}^3 \text{ and}$$

$$z^{-1} : z(w) \to w \text{ with}$$

$$z^{-1}(p,q,s) = \begin{pmatrix} p & q \\ \frac{ps-1}{q} & s \end{pmatrix}.$$ This is also continuous.

So, $z : w \to z(w)$ is a homeomorphism i.e. (w,z) is a chart of N.

Now $u \cup w = N$ which implies $((N,\mathbb{C}),\theta)$ is 3 -dimensional complex manifold.

(d) Now we'll show that $(N,\mathcal{A})$ is a C^∞-differentiable manifold:

We have seen

$\{(u,x), p \neq 0 \, and \, (w,z), q \neq 0\} = \mathcal{A}_{top.} = topological \, atlas$. Now we'll show that $\mathcal{A}_{top.}$ is C^∞-compatible.

Consider the transition map

$$z \circ x^{-1} : x(u \cap w) \to z(u \cap w),$$

$$x^{-1}(p,q,r) = \begin{pmatrix} p & q \\ r & \dfrac{1+qr}{p} \end{pmatrix}$$

$$z\left(\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right) = (p,q,s)$$

So, $z \circ x^{-1} : (p,q,r) = (p,q,\frac{1+qr}{p})$. Hence, $z \circ x^{-1}$ is C^∞-differentiable.

Similarly, $x \circ z^{-1} : z(u \cap w) \to x(u \cap w)$ is C^∞-differentiable.

Hence, $(N,\mathcal{A})$ is a C^∞-differentiable manifold.

(e) Now we've to show that N be a Lie group.

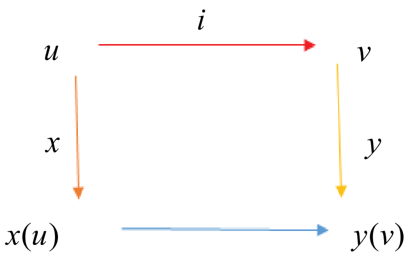
Let us consider the map $\mu : N \times N \to N$. We have to check that

$$\mu\left(\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} p_1 & q_1 \\ r_1 & s_1 \end{pmatrix}\right) = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \circ \begin{pmatrix} p_1 & q_1 \\ r_1 & s_1 \end{pmatrix} \text{ and}$$

$$i : N \to N \text{ with}$$

$$i\left(\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right) = \frac{1}{ps-qr}\begin{pmatrix} s & -q \\ -r & p \end{pmatrix} \text{ are smooth.}$$

Multiplication property is obvious under the usual matrix multiplication.



Now construct $y \circ i \circ x^{-1}$ considering the above commutative diagram.

So that $$y \circ i \circ x^{-1}(p,q,r) = (y \circ i)\left(\begin{pmatrix} p & q \\ r & \dfrac{1+qr}{p} \end{pmatrix}\right)$$

$$= y\left(\frac{1}{1+qr-qr}\begin{pmatrix} \dfrac{1+qr}{p} & -q \\ -r & p \end{pmatrix}\right)$$

$= (-q,-r,p)$ which implies i is differentiable.

Similarly, using the atlas defined as $N \times N \to N$ by virtue of the differentiable $\mathcal{A}$ as N and also N is differentiable.

Hence $((N,\mathbb{C}),\circ)$ is a 3-dimensional imaginary Lie group.

(f) Suppose Lie algebra of N is $L(N)$. We want to show that N satisfies all the properties of Lie Algebras.

Let $L(N) = \left\{ X \in \Gamma(TN) \left[l_{r_1^*}(X_{s_1}) \right]_{l_{r_1} s_1 \circ r_1 s_1} = X_{r_1 s_1} \right\}$,

$TN =$ vector field on N.

Let $l_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} : N \to N$ with

$$l_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}}\begin{pmatrix} p_1 & q_1 \\ r_1 & s_1 \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \circ \begin{pmatrix} p_1 & q_1 \\ r_1 & s_1 \end{pmatrix}.$$

We can again write that $T_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} N \cong L(N) \subseteq \Gamma(TN)$ where

$$L(N) = [\circ,\circ] : T_{id}N \times T_{id}N \to T_{id}N.$$

We have to determine $[\circ,\circ]$ using chart (u,x). Since $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in u$ for $i = 1,2,3$ we have

$$\left[l_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} \cdot \left(\frac{\partial}{\partial x^i}\right)_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \right] f = \left(\frac{\partial}{\partial x^i}\right)_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \left(f \circ l_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} \right), \quad f \in C^\infty(N)$$

$$= \partial_i \left(f \circ l_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} \circ x^{-1} \right) \Bigg|_{x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$= \partial_i \left((f \circ x^{-1}) \circ \left(x \circ l_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} \circ x^{-1} \right) \right) \Bigg|_{x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \circ \partial_i \left(x^m \circ l_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} \circ x^{-1} \right) \Bigg|_{x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

Now

$$\partial_m (f \circ x^{-1}) \Bigg| \left({}_{x \circ l_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}}} \circ x^{-1} \right) \left(x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = \left(\frac{\partial}{\partial x^m} \right)_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} f$$

and

$$\left(x^m \circ l_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} \circ x^{-1}\right)(p_1, q_1, r_1) = \left(x^m \circ l_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}}\right)\begin{pmatrix} p_1 & q_1 \\ r_1 & \frac{1+q_1 r_1}{p_1} \end{pmatrix}$$

$$= x^m \begin{pmatrix} pp_1 + qr_1 & pq_1 + \frac{qr + q_1 r_1}{p_1} \\ rp_1 + sr_1 & rq_1 + \frac{s(1+q_1 r_1)}{p_1} \end{pmatrix}$$

$$= \left(pp_1 + qr_1, pq_1 + \frac{qr + q_1 r_1}{p_1}, rp_1 + sr\right)$$

Now

$$\partial_i \left(x^m \circ l_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} \circ x^{-1}\right)(p_1, q_1, r_1)\Big|_{x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} = \partial_i \left(x^m \circ l_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} \circ x^{-1}\right)(p_1, q_1, r_1)\Big|_{(1,0,0)}$$

$$= \begin{bmatrix} p & 0 & q \\ -\frac{q(1+q_1 r_1}{p_1^2} & p + \frac{qr_1}{p_1} & \frac{qq_1}{p_1} \\ sr & 0 & s \end{bmatrix}_{|(1,0,0)}$$

$$= \begin{bmatrix} p & 0 & q \\ -q & p & 0 \\ r & 0 & s \end{bmatrix}_i^m$$

So, the value of

$$\partial_i \left(\left(f \circ x^{-1}\right) \circ \left(x \circ l_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} \circ x^{-1} \right)\right)\Big|_{x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \circ \partial_i \left(x^m \circ l_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} \circ x^{-1}\right)\Big|_{x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

will be

$$\begin{bmatrix} p & 0 & q \\ -q & p & 0 \\ r & 0 & s \end{bmatrix}_i^m \left(\frac{\partial}{\partial x^m}\right)_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} f.$$

$$\therefore \; l_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}^*} \left(\left(\frac{\partial}{\partial x^i}\right)_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \right) = \begin{bmatrix} p & 0 & q \\ -q & p & 0 \\ r & 0 & s \end{bmatrix}_i^m \left(\frac{\partial}{\partial x^m}\right)_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}}$$

So,

$$l_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}^*} \left(\left(\frac{\partial}{\partial x^1}\right)_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \right) = p\left(\frac{\partial}{\partial x^1}\right)_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} - q\left(\frac{\partial}{\partial x^2}\right)_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} + r\left(\frac{\partial}{\partial x^3}\right)_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}},$$

,

$$l_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}^*} \left(\left(\frac{\partial}{\partial x^2}\right)_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \right) = p\left(\frac{\partial}{\partial x^2}\right)_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}}$$

$$l_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}^*} \left(\left(\frac{\partial}{\partial x^3}\right)_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \right) = q\left(\frac{\partial}{\partial x^1}\right)_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} + s\left(\frac{\partial}{\partial x^3}\right)_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}}.$$

Now we'll calculate

$$\left[\left(\frac{\partial}{\partial x^1}\right)_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \left(\frac{\partial}{\partial x^2}\right)_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}\right] f = \left[l_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}^*} \left(\frac{\partial}{\partial x^1}\right)_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, l_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}^*} \left(\frac{\partial}{\partial x^2}\right)_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \right] f$$

$$= \left[p\left(\frac{\partial}{\partial x^1}\right)_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} - q\left(\frac{\partial}{\partial x^2}\right)_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} + r\left(\frac{\partial}{\partial x^3}\right)_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}}, p\left(\frac{\partial}{\partial x^2}\right)_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} \right] f$$

Let

$$p\left(\frac{\partial}{\partial x^1}\right)_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} - q\left(\frac{\partial}{\partial x^2}\right)_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} + r\left(\frac{\partial}{\partial x^3}\right)_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} = X$$

and $p\left(\frac{\partial}{\partial x^2}\right)_{\begin{pmatrix} p & q \\ r & s \end{pmatrix}} = Y$

So, $\left[\left(\frac{\partial}{\partial x^1}\right)_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \left(\frac{\partial}{\partial x^2}\right)_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}\right] f = X(Yf) - Y(Xf)$

$$= a\left(\frac{\partial}{\partial x^1}\right)_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + b\left(\frac{\partial}{\partial x^2}\right)_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + c\left(\frac{\partial}{\partial x^3}\right)_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$= C_{12}^m \begin{pmatrix} a \\ b \\ c \end{pmatrix} =$ structure constant of N.

Hence, N satisfies all properties of Lie Algebras.

Theorem 1 Suppose N and H be two Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. also let $\psi : N \rightarrow H$ is a homomorphism. Then

(i) $\mathcal{X}$ and $d\psi(\mathcal{X})$ are ψ -related for every $\mathcal{X}$ò$\mathfrak{g}$

(ii) $d\psi : \mathfrak{g} \rightarrow \mathfrak{h}$ will be homomorphism.

Proof. Suppose that $\tilde{\mathcal{X}} = d\psi(\mathcal{X})$. Then $\tilde{\mathcal{X}}$ and $\mathcal{X}$ are ψ -related. Since ψ is a homomorphism, we have

$$l_{\psi(\sigma)} \circ \psi = \psi \circ l_{\sigma} .$$

So, we have

$$\tilde{\mathcal{X}}(\psi(\sigma)) = dl_{\psi(\sigma)} \tilde{\mathcal{X}}(e)$$

$$= dl_{\psi(\sigma)} d\psi(\mathcal{X}(e))$$

$$= d(l_{\psi(\sigma)} \circ \psi)\mathcal{X}(e)$$

$$= d(\psi \circ l_{\sigma})\mathcal{X}(e)$$

$$= d\psi(\mathcal{X}(\sigma))$$

which proves (i).

Now $\mathcal{X}, \mathcal{Y}$ò$\mathfrak{g}$. Then $\left[\widetilde{\mathcal{X},\mathcal{Y}}\right] = \left[\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}\right]$. But $[\mathcal{X}, \mathcal{Y}]$ is ψ -related to the left invariant vector field $\left[\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}\right]$.

Specifically,

$$\left[\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}\right](e) = d\psi([\mathcal{X}, \mathcal{Y}](e)) \cdot$$

But $\left[\widetilde{\mathcal{X},\mathcal{Y}}\right]$ is singular left invariant set of vectors on H which has value $d\psi([\mathcal{X}, \mathcal{Y}](e))$ at identity which proves (ii).

VI. Conclusion

Throughout this paper, some important and primary level definitions, examples and one theorem which are unignorable are discussed. Finally, in section V, the example has been proved which is related toLie Group and Lie Algebra of Manifold $Sl(2, \mathbb{C})$. This type of representation of group as well as manifold will be beneficial for our further research in the corresponding field.

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