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# Three-dimensional Manifolds as Slices of Infinite-dimensional Hilbert Manifolds: A Geometric Model of the Universe

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**Abstract**

This paper presents a geometric framework in which our observable three-dimensional universe is modeled as a smooth submanifold—specifically, a slice—embedded within an infinite-dimensional Hilbert manifold. Drawing on classical embedding theorems by Whitney, Nash, Kuiper, and Henderson, we reinterpret established results in differential topology through a novel lens that bridges geometry and physics. We demonstrate that any smooth 3-manifold can be realized as an isometric leaf in a smooth foliation of an infinite-dimensional manifold, and construct such foliations explicitly using smooth normal vector fields along a fixed embedding. We prove that the space of these foliations, parametrized by such fields, forms an infinite-dimensional Fréchet manifold—effectively a moduli space of parallel universes, each represented as a geometric slice. An explicit example using the 3-sphere $S^3$ embedded in the Hilbert space $\ell^2$ is developed, illustrating the theoretical construction in concrete terms. Diagrams and visualization accompany the model to clarify the geometric intuition and moduli variation. Our approach remains purely geometric, independent of physical field equations, yet conceptually resonates with brane world scenarios, emergent gravity, and infinite-dimensional quantum theories. This reinterpretation of classical geometry provides not only a rigorous mathematical result but also opens a pathway toward new models of dimensional emergence and foundational questions in cosmology and ontology. By positioning the universe as a geometric object embedded in an infinite-dimensional ambient structure, we offer a new direction for thinking about space, structure, and reality.

***Keywords***: Infinite-dimensional geometry, Hilbert manifold, Smooth foliation, Manifold embedding, Moduli space

## I. Introduction

The question of whether our observable universe is a fundamental entity or merely a lower-dimensional substructure embedded within a higher-dimensional reality has intrigued both physicists and philosophers for decades. In modern theoretical physics, this idea manifests in a variety of proposals that position our universe not as a self-contained totality, but as a lower-dimensional "brane" or hypersurface embedded in a higher-dimensional space.

One of the most influential frameworks in this regard is string theory, which posits that the fundamental constituents of reality are not point particles but one-dimensional strings whose vibrational modes give rise to the observed particle spectrum. These strings naturally exist in higher-dimensional spacetime, with extra spatial dimensions compactified or hidden from observation[1,2]. In M-theory, an extension of string theory, membranes (or "branes") of various dimensionalities appear, and our universe is sometimes modeled as a 3-brane within a higher-dimensional bulk[3].

Closely related are braneworld scenarios, such as those developed by Randall and Sundrum[4,5], in which gravity can propagate in extra dimensions, while standard model fields are confined to a lower-dimensional brane. These models have been used to explain hierarchies in fundamental forces and to explore testable deviations from Newtonian gravity.

Holographic dualities, particularly the AdS/CFT correspondence[20], offer a different but conceptually related view: they suggest that a gravitational theory in a bulk spacetime is equivalent to a conformal field theory on its boundary, implying that the bulk geometry itself may emerge from lower-dimensional quantum dynamics.

Philosophically, these approaches challenge traditional notions of space and dimensionality. If what we perceive as a $3 + 1$-dimensional universe is merely an emergent or embedded structure, then spatial dimensionality may not be an intrinsic property of the universe but a contextual one - dependent on the embedding or emergent mechanism[6,7,8].

In this paper, we explore a mathematically rigorous and geometrically motivated version of this idea: the hypothesis that our 3-dimensional world can be viewed as a *slice*, or embedded submanifold, within an infinite-dimensional manifold. Unlike traditional high-dimensional models which consider embedding into finite-dimensional ambient spaces, we focus on embeddings into manifolds modeled on infinite-dimensional separable Hilbert spaces. These spaces arise naturally in many areas of mathematical physics,

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including functional analysis, quantum mechanics, and field theory.

The mathematical study of embeddings of manifolds has a rich history. *The Whitney embeddingtheorem*[9,10] established that any smooth $n$-dimensional manifold can be embedded in $\mathbb{R}^{2n}$, providing one of the foundational results in differential topology. Later, *Nash's embedding theorem*[11] demonstrated that Riemannian manifolds can be isometrically embedded into finite-dimensional Euclidean spaces. In the infinite-dimensional context, *Kuiper*[12]showed that infinite-dimensional Hilbert spaces are remarkably flexible: every smooth Hilbert manifold is diffeomorphic to the Hilbert space itself. Further work by *Henderson*[13] proved that any finite-dimensional smooth manifold can be smoothly embedded in a separable infinite-dimensional Hilbert manifold.

These results imply, in particular, that our 3-dimensional spatial universe, modeled as a smooth manifold $\mathcal{M}^3$, can always be realized as a submanifold of some infinite-dimensional smooth manifold $\mathcal{M}^\infty$. While this embedding result is mathematically well established, its *geometric interpretation and physical significance* remain largely unexplored in the literature.

Our work is motivated by the following guiding question: *Can we meaningfully model the physical universe as a geometric slice of an infinite-dimensional manifold, and if so, what structures or insights does this perspective offer?* To this end, we adopt the view that our universe is a 3-dimensional embedded submanifold-analogous to a hyperplane-residing within an infinite-dimensional ambient space. We explore this embedding not only as a mathematical construction, but as a potential *foundational framework* for thinking about the emergence of physical laws, dimensionality, and perhaps even parallel universes as neighboring slices in a foliated structure.

We emphasize that this approach does not rely on speculative physics such as strings, supersymmetry, or quantum gravity, but rather draws directly from the core principles of differential geometry and manifold theory, extended into infinite dimensions.

The remainder of this paper is organized as follows. In Section II, we establish the mathematical framework for embedding smooth 3-manifolds into infinite-dimensional Hilbert manifolds, introducing key definitions and recalling relevant embedding theorems. Section III offers a physical interpretation of these embeddings, proposing that our universe may be viewed as a geometric slice within an infinite-dimensional foliated structure. In Section IV, we explore the philosophical implications of this model, particularly regarding the ontology of space, the emergence of dimensionality, and the generative role of mathematical structures. Section V presents the main technical results: we construct explicit foliations of Hilbert manifolds by isometric copies of a given 3-manifold, and we prove that the space of such foliations forms a smooth infinite-dimensional Fréchet manifold. A concrete example involving the 3-sphere embedded in $\ell^2$ is also developed. Section VI concludes the paper by summarizing the main ideas and reinforcing the conceptual and geometric significance of the model.

## II. Mathematical Framework

We now formalize the mathematical setting in which our model operates. Our goal is to rigorously establish the possibility of embedding a smooth 3-dimensional manifold-representing a spatial model of the physical universe---into an infinite-dimensional smooth manifold. We then explore structural interpretations of such embeddings, including the analogy with hyperplanes and foliations.

*Preliminaries and Definitions*

***Hilbert Manifold***

Let $M^3$ be a connected, smooth 3-dimensional manifold. Let $\mathcal{H}$ denote a real, separable Hilbert space, and let $\mathcal{M}^\infty$ be a smooth infinite-dimensional manifold modeled on $\mathcal{H}$. A smooth manifold modeled on $\mathcal{H}$ is called a Hilbert manifold if each point admits a coordinate chart diffeomorphic to an open subset of $\mathcal{H}$, with smooth transition maps.

***Embedding***

A smooth map $f: M^3 \to \mathcal{M}^\infty$ is called an $embedding$ if it is an injective immersion and a homeomorphism onto its image, where the image is endowed with the subspace topology.

*Embedding Theorem in Infinite Dimensions*

The classical Whitney embedding theorem guarantees that any smooth $n-$dimensional manifold can be embedded in $\mathbb{R}^{2n}$[9]. Later, Nash extended this result to isometric embeddings of Riemannian manifolds[11]. In the context of infinite-dimensional geometry, these ideas were generalized significantly. In particular, the following theorem plays a central role:

***Theorem* 1**(Henderson,[13])

*Let $M^n$ be a smooth, paracompact, finite-dimensional manifold. Then there exists a smooth embedding $f: M^n \hookrightarrow \mathcal{H}$, where $\mathcal{H}$ is any separable infinite-dimensional Hilbert space.*

This result ensures that any 3-manifold $M^3$ can be realized as a submanifold of an infinite-dimensional Hilbert manifold, bypassing the dimension constraints present in finite-dimensional settings.

*Hyperplane Interpretation and Slicing Structure*

While in finite-dimensional Euclidean spaces a hyperplane is defined as a codimension-one affine subspace, in infinite-dimensional Hilbert spaces, the notion generalizes. A hyperplane may refer to a closed affine subspace defined by a continuous linear functional. In our model, however, we adopt a geometric analog: the 3-dimensional manifold $M^3$ is considered a *slice* of $\mathcal{M}^\infty$, i.e., a smooth embedded submanifold resembling a hyperplane in local structure.

This motivates the following conceptual structure: suppose $\mathcal{M}^\infty$ admits a smooth foliation

$$\mathcal{M}^\infty = \bigcup_{\alpha \in A} \Sigma_\alpha \tag{1}$$

where each $leaf$ $\Sigma_\alpha \subset \mathcal{M}^\infty$ is diffeomorphic to $M^3$. Then, our observed universe could correspond to a single leaf $\Sigma_{\alpha_0}$, while the ambient space contains many such slices, opening the possibility for a multiverse-like interpretation or a functional-theoretic perspective on dimensional emergence.

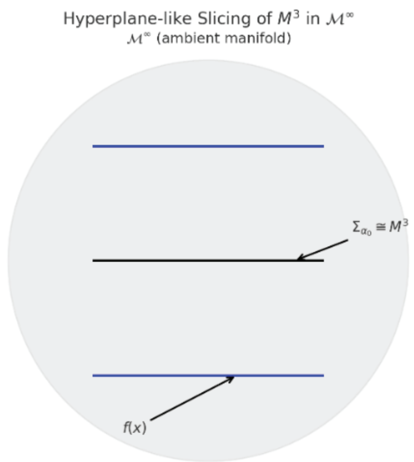


**Fig. 1.** Schematic illustration of hyperplane-like slices $\Sigma_\alpha$ embedded in the infinite-dimensional manifold $\mathcal{M}^\infty$. The central slice $\Sigma_{\alpha_0} \cong M^3$ represents the universe as we observe it.

*Metric and Geometric Considerations*

We may equip $M^3$ with a Riemannian or Lorentzian metric $g$, and the ambient manifold $\mathcal{M}^\infty$ with a compatible infinite-dimensional metric structure. The embedding,

$f: M^3 \to \mathcal{M}^\infty$ may then be analyzed for whether it is isometric, conformal, or more generally geometric in nature. These structural constraints will play a role in determining what physical features of the embedded manifold are preserved or altered by the embedding.

## III. Physical Interpretation

While the embedding of finite-dimensional manifolds into infinite-dimensional Hilbert manifolds is a well-established mathematical result, the physical interpretation of such embeddings is far less explored. In this section, we develop a conceptual framework in which our 3-dimensional universe is modeled as a geometric *slice* within an infinite-dimensional ambient space. This interpretation is not merely topological or differential in nature, but aims to offer a pathway to understanding the emergence of physical laws, dimensions, and structure from more abstract geometric foundations.

*The Universe as a Geometric Slice*

We consider the possibility that our observable universe corresponds to a specific embedded submanifold $M^3 \subset \mathcal{M}^\infty$, where $\mathcal{M}^\infty$ is a smooth infinite-dimensional Hilbert manifold modeled on a separable Hilbert space $\mathcal{H}$. This embedding is realized through a smooth injective immersion $f: M^3 \hookrightarrow \mathcal{M}^\infty$, which is assumed to be topologically and geometrically regular, so that $f(M^3)$ retains the differential structure of $M^3$ as a slice in $\mathcal{M}^\infty$.

Locally, this embedded image may be understood as a hyperplane-like slice-in the sense that its tangent space at any point is a finite-dimensional subspace of the tangent space of the ambient manifold. While hyperplanes in finite-dimensional geometry are typically defined by the level sets of linear functionals, we adopt a more general interpretation: the slice $M^3$ represents a codimension-infinite submanifold, realized by holding fixed a specific foliation parameter in a global product-like structure on $\mathcal{M}^\infty$, such as $\mathcal{M}^\infty \cong R \times M^3$ or more abstractly $\mathcal{M}^\infty = \bigcup_\alpha \Sigma_\alpha$.

This construction invites us to reinterpret the physical universe not as the totality of the geometric space, but as a selected leaf - a section of a far richer structure. The higher-dimensional manifold $\mathcal{M}^\infty$ may carry geometric and topological features that are not directly accessible from within the slice $M^3$, but that influence the geometry of the embedding and thus constrain the physics observed within it.

This idea resonates with various paradigms in modern physics. In string theory, our universe is modeled as a three-dimensional brane embedded in a higher-dimensional bulk spacetime, where gravity and other fields can propagate into the extra dimensions. Similarly, the AdS/CFT correspondence considers a conformal field theory living on the boundary of a higher-dimensional anti-de Sitter space, where physical laws in the lower-dimensional boundary encode the dynamics of the bulk.

However, our model differs fundamentally in that it does not rely on specific physical field theories or action principles. Instead, we propose that the embedding itself and the infinite-dimensional geometric context it implies - serves as a foundational structure. The embedding,

$f: M^3 \hookrightarrow \mathcal{M}^\infty$ is not merely a technical artifact, but rather a statement about the ontological position of the universe within a potentially infinite-dimensional geometric reality.

In this view, the properties of spacetime and the physical laws we observe may emerge as effective descriptions constrained by the geometry of the slice and its interaction (or lack thereof) with the ambient manifold. If $\mathcal{M}^\infty$ admits a foliation into similarly embedded submanifolds, each diffeomorphic to $M^3$, then the universe may be one among many - not in a speculative cosmological sense, but as a consequence of the mathematical structure of infinite-dimensional topology.

*Foliation and Multiverse Analogy*

We now consider the case where the ambient infinite-dimensional manifold $\mathcal{M}^\infty$ admits a smooth foliation $\mathcal{F}$ whose leaves are 3-dimensional submanifolds $\Sigma_\alpha$, each diffeomorphic to a fixed compact manifold $M^3$. That is, there exists a decomposition:

$$\mathcal{M}^\infty = \bigcup_{\alpha \in A} \Sigma_\alpha, \quad \text{with } \Sigma_\alpha \cong M^3 \text{ for all } \alpha,$$

such that each $\Sigma_\alpha$ is a connected, injectively immersed, 3-dimensional submanifold of $\mathcal{M}^\infty$, and the foliation satisfies the usual local triviality condition: around every point in $\mathcal{M}^\infty$, there exists a chart diffeomorphic to $R^3 \times R^\infty$ in which the leaves are locally of the form $R^3 \times \{\text{const}\}$.

This structure naturally leads to a "multiverse-like" interpretation. Each leaf $\Sigma_\alpha$ can be regarded as a distinct realization of a 3-dimensional geometric universe embedded within the same infinite-dimensional geometric framework. While the individual leaves are locally identical in topology and dimension, they may differ in curvature, embedding data, or even in induced geometric structures, such as metrics or connections inherited from the ambient manifold. As such, each leaf could encode distinct physical configurations or laws, constrained by the local geometry of $\Sigma_\alpha$ and its placement within $\mathcal{M}^\infty$.

This leads to a profound conceptual shift: the multiverse is not a speculative cosmological hypothesis, but rather a mathematical consequence of the global structure of $\mathcal{M}^\infty$. The foliation structure does not require probabilistic or metaphysical assumptions. Instead, it provides a rigorous differential-topological mechanism by which multiple universes can coexist as distinct geometric "sheets" in a common ambient manifold.

Importantly, the physical interpretation of these slices depends not just on their local geometry, but on how they are embedded in $\mathcal{M}^\infty$, including their relative positioning, the smooth vector fields that define their normal directions, and any global constraints or symmetries present in the foliation. This framework invites new perspectives on the variability of physical laws, the concept of cosmic neighborhoods, and the role of higher-dimensional geometry in shaping observed phenomena.

While we do not suggest that this foliation corresponds directly to a physical multiverse in the traditional cosmological sense, the analogy is conceptually and mathematically rich. It shows how a rigorous mathematical structure - the foliation of an infinite-dimensional manifold – naturally supportsa multiverse-like configuration, grounded entirely in smooth geometry and topology.

*Remark 1(Mathematical Structure of the Foliation)*

The foliation $\mathcal{F} = \{\Sigma_\alpha\}$ described above can be understood, at least formally, as arising from an integrable distribution of 3-dimensional tangent planes within the tangent bundle $T\mathcal{M}^\infty$. In the classical finite-dimensional setting, this corresponds to the Frobenius integrability condition; in the infinite-dimensional setting, similar results have been studied under functional-analytic constraints in Hilbert and Banach manifolds.

Further, if each leaf $\Sigma_\alpha$ is endowed with an induced metric ($g_\alpha$ from the ambient geometry, one can consider smooth variation of geometric invariants (e.g., scalar curvature, Ricci curvature, or topological class) as a function of the foliation parameter $\alpha$. This opens the door to the study of leafwise geometry and moduli theory within infinite-dimensional foliated spaces, as developed in the works of Haefliger[14], Godbillon[15], and Moore-Schochet[16].

*Emergent Dimensionality and Physical Laws*

A key implication of viewing our universe as a geometric slice $\Sigma_{\alpha_0} \subset \mathcal{M}^\infty$ is that the observed dimensionality of space may be a derived, rather than fundamental, property. That is, three-dimensionality may emerge as a consequence of the structure of the foliation and the nature of the embedding, rather than being hardwired into the fabric of the ambient reality.

In this framework, $\Sigma_{\alpha_0}$ appears as a finite-dimensional "cross-section" of an infinite-dimensional manifold, and the effective physics observed on this slice is shaped by the embedding map $f$, the geometry induced from $\mathcal{M}^\infty$, and the local configuration of the surrounding foliation. Dimensionality here is contextual: it reflects the structure of the leaf $\Sigma_{\alpha_0}$, but not necessarily that of the ambient manifold, which may contain geometric or topological degrees of freedom inaccessible from within the slice.

Mathematically, the effective physical content on $\Sigma_{\alpha_0}$ can be derived by considering the restriction of global geometric structures on $\mathcal{M}^\infty$-such as a curvature tensor $R$, connection $\nabla$, or metric $g$ to the image of the embedding $f(M^3)$. The pullback $f^*g$ induces a local geometry on $M^3$,while the normal bundle and second fundamental form encode how this slice is curved or twisted within the ambient manifold.

In this sense, differential operators such as the Laplacian $\Delta$, the Dirac operator $D$, or the d'Alembertian $\blacksquare$, which play fundamental roles in physical theories, may arise as induced or projected structures from the geometry of $\mathcal{M}^\infty$. These operators then govern the dynamics of fields restricted to the slice.

This perspective resonates with theories of *emergent gravity*[17], in which gravitational dynamics arise from entropic or statistical principles, and with *holographic dualities* such as AdS/CFT[20], where the physics on a lower-dimensional boundary encodes a higher-dimensional bulk. In *quantum field theory*, fields are commonly modeled as sections over infinite-dimensional configuration spaces[18,19], making the idea of slice-based emergence geometrically natural.

By grounding this interpretation in rigorous manifold theory, we aim to reposition geometry not merely as the background upon which physics unfolds, but as a *generator* of dimensional structure and physical law. The embedding of $(M^3)$into$(\mathcal{M}^\infty)$ is not just a topological curiosity-it is a geometric act that induces, filters, and possibly determines what kinds of physics can exist on the slice.

*Remark 2 (Geometric Origin of Physical Operators)*

Let$\nabla$ be a connection on $\mathcal{M}^\infty$ with curvature tensor $R$. Then the restriction $R|_{T\Sigma_{\alpha_0}}$ governs intrinsic dynamics on the slice, while mixed components of R involving normal directions relate to extrinsic constraints. Field equations such as the Einstein $equation\ G = T$, when formulated in the ambient space, may reduce to effective versions on $\Sigma_{\alpha_0}$ via projection, pullback, or constraint implementation.

## IV. Philosophical Implications

The mathematical framework developed in this work offers a precise and rigorous setting in which our observable universe is reinterpreted as a geometric slice-specifically, a finite-dimensional submanifold-embedded within an infinite-dimensional ambient space. While grounded in differential topology and infinite-dimensional geometry, this formulation raises deep philosophical questions about the nature of physical reality, the emergence of dimensionality, and the role of mathematics in describing, or perhaps constituting, the universe.

*Is the Ambient Manifold Physically Real?*

A central ontological question is whether the ambient manifold $\mathcal{M}^\infty$ should be regarded as physically real, or merely as a mathematical construct used to model observed phenomena. In standard physics, higher-dimensional embeddings often serve as auxiliary framework (as in *Kaluza-Klein theory or string theory*), with no obligation to be empirically accessible. However, in the present model, the ambient structure plays a generative role: it determines the geometry and potential dynamics of the slice $\Sigma_{\alpha_0}$that corresponds to our universe. If physical laws emerge from restrictions of global structures on $\mathcal{M}^\infty$, then the ontology of the ambient space may be at least as relevant as that of the slice itself.

This aligns with philosophical views that treat mathematical structures as ontologically substantial, such as Tegmark's Mathematical Universe Hypothesis [21], or with structural realist interpretations of physics, where entities are secondary to the relations and structures in which they are embedded[22].

*Dimensionality and Emergence*

From the perspective of a being confined to a slice $\Sigma_{\alpha_0}$, the dimensionality of the universe appears as a fixed, empirical fact. Yet in this framework, dimensionality is local and relative: it emerges from the embedding structure and foliation, not from the intrinsic architecture of reality. This echoes philosophical discussions on the relational and emergent nature of space, and challenges ontologies that treat dimensionality as an absolute category of being.

Furthermore, the inability to directly perceive or measure the ambient infinite-dimensional manifold may not diminish its relevance. As with Kantian noumena, or unobservable structures in quantum field theory, the unobserved may still be foundational. The slice provides the empirical interface; the ambient manifold provides the structural context.

*Mathematics as Description or Genesis?*

One of the most profound questions prompted by this model is whether mathematics merely describes the structure of physical reality or actively generates it. In traditional Platonism, mathematical structures are timeless and real but external to the physical world. In contrast, the present framework suggests that **mathematical structure may be causally or ontologically prior to physical law**. The embedding $f: M^3 \hookrightarrow \mathcal{M}^\infty$ is not merely a notational convenience, but a move that determines what kind of physics can exist.

This resonates with Wigner's question about the "unreasonable effectiveness of mathematics in the natural sciences"[23], but reframes it: mathematics is effective because reality is mathematical. It also echoes the views of Butterfield and Isham[24]who explore the idea that space, time, and matter may emerge from deeper algebraic and topological structures, perhaps best understood not as "things" but as interrelated mathematical relations.

*Concluding Thought*

The interpretation of our universe as a slice in an infinite-dimensional geometric reality is more than a formal exercise. It suggests that **ontology is geometry**, that dimensionality is contingent, and that the mathematical structure of space may be both the language and the substance of the physical world. While speculative in scope, this framework is rigorous in its construction and thus opens a conceptual window between geometry and metaphysics, between physics and philosophy.

## V. Structured Embeddings and Infinite Dimensional Foliations

We now extend the previous discussion by formalizing a class of embeddings of smooth 3-manifolds into infinite-dimensional Hilbert manifolds that admit foliations by isometric copies of the same manifold. This structure introduces not only a geometric embedding, but a repetition of the embedded geometry across a foliation of the ambient manifold. Our aim is to provide both existence results and a structural framework for this type of configuration.

*Setup and Definitions*

Let $M^3$ be a compact, connected, oriented Riemannian 3-manifold with metric $g$ . Let $\mathcal{H}$ denote a separable infinite-dimensional Hilbert space, and let $\mathcal{M}^\infty$ be a smooth Hilbert manifold modeled on $\mathcal{H}$.

We recall the following definitions, adapted from standard sources on foliation theory and infinite-dimensional differential geometry[14,15,16,25]:

*Definition 1(Foliation)*

A $foliation$ $\mathcal{F}$ of a smooth manifold $\mathcal{M}$ is a decomposition of $\mathcal{M}$ into a disjoint union of connected, injectively immersed submanifolds (called $leaves$) such that $\mathcal{M}$ admits an atlas of charts $\{(U_i, \phi_i)\}$ where each chart maps $U_i \subset \mathcal{M}$ diffeomorphically onto an open set in $R^k \times R^{\infty-k}$, and such that leaves locally correspond to slices of the form $(R^k \times \{\text{const}\})$[14,16].

*Definition 2* (Leaf)

Given a foliation $\mathcal{F}$ on a manifold $\mathcal{M}$, a *leaf* $\Sigma \in \mathcal{F}$ is a connected, injectively immersed submanifold of $\mathcal{M}$ such that $\mathcal{M}$ is locally diffeomorphic to $R^k \times R^{\infty-k}$, and $\Sigma$ locally corresponds to $(R^k \times \{\text{const}\})$[15,16].

*Definition 3 (Codimension)*

The *codimension* of a foliation $\mathcal{F}$ of a manifold $\mathcal{M}$ is defined as the difference between the dimension of $\mathcal{M}$ and the dimension of each leaf. For a foliation with $k$-dimensional leaves in an $n-$dimensional manifold, the codimension is $n - k$. In the case of Hilbert manifolds, this may be infinite[16,25].

*Definition 4 (Regular Foliation)*

A foliation $\mathcal{F}$ on $\mathcal{M}$ is said to be *regular* if all leaves have the same dimension and the foliation is defined by a globally integrable smooth distribution $\mathcal{D} \subset T\mathcal{M}$. This means there exists a smooth sub bundle such that $T_p\Sigma = \mathcal{D}_p$ for each $p \in \Sigma$, for all leaves $\Sigma$[16,25].

### *Main Result*

We now state and prove the central result of this section.

*Theorem 2 (Existence of Isometric Foliation by Copies of a 3-Manifold)*

*Let $(M^3, g)$ be a compact Riemannian 3-manifold. Then there exists a smooth Hilbert manifold $\mathcal{M}^\infty$ modeled on a separable Hilbert space $\mathcal{H}$, and a smooth foliation $\mathcal{F} = \{\Sigma_\alpha\}_{\alpha \in R}$ of $\mathcal{M}^\infty$, such that for every $\alpha \in R$, the leaf $\Sigma_\alpha$ is isometrically diffeomorphic to $(M^3, g)$.*

***Proof:***

Let $(M^3, g)$ be a compact Riemannian 3-manifold. Since $M^3$ is compact and smooth, there exists a smooth embedding $(f_0: M^3 \hookrightarrow \mathcal{H})$ into a separable infinite-dimensional Hilbert space $\mathcal{H}$, by the result of Henderson [13].

Let us construct a one-parameter family of embeddings $f_\alpha: M^3 \to \mathcal{H}$ defined by

$$f_\alpha(x) = f_0(x) + \alpha \cdot v(x) \tag{2}$$

where $\alpha \in R$, and $v: M^3 \to \mathcal{H}$ is a smooth non-vanishing vector field along the image of $f_0$ such that:

1. $v(x) \perp df_0(T_x M^3) x \in M^3 \;\; \forall \;\; x \in M^3$
2. $|v(x)| = c$ for some constant $c > 0$ we may choose $c = 1$ without loss of generality.

These conditions ensure that each embedded image $f_\alpha(M^3)$ is a parallel copy of the original embedding, displaced in the direction orthogonal to the tangent space of $f_0(M^3)$. The vector field $v$ is smooth and non-vanishing because such vector fields exist globally on compact manifolds embedded in Hilbert spaces, given the triviality of the normal bundle in infinite dimensions.

Now define:

$$\mathcal{M}^\infty := \bigcup_{\alpha \in R} f_\alpha(M^3) \subset \mathcal{H} \tag{3}$$

This set inherits the structure of a smooth Hilbert manifold modeled on $R \times M^3$, and the parameter α provides a global foliation coordinate. Each slice $\Sigma_\alpha := f_\alpha(M^3)$ is diffeomorphic to $M^3$ via $x \mapsto f_\alpha(x)$, and the family $\{\Sigma_\alpha\}_{\alpha \in R}$ defines a smooth foliation of $\mathcal{M}^\infty$.

To show that each embedding $f_\alpha$ is isometric, consider the induced metric $f_\alpha^*\langle\cdot,\cdot\rangle_{\mathcal{H}}$. Since the displacement vector $\alpha v(x)$ lies in the normal direction and is constant in α, it does not affect the pullback of the inner product along the tangent directions. Thus, for any tangent vectors $X, Y \in T_xM^3$,

$$\langle df_\alpha(X), df_\alpha(Y)\rangle = \langle df_0(X), df_0(Y)\rangle = g(X, Y)$$

Therefore, each $f_\alpha$ is an isometric embedding, and each leaf $\Sigma_\alpha$ is isometrically diffeomorphic to $(M^3, g)$.

Hence, $\mathcal{M}^\infty$ is foliated by isometric copies of $M^3$, parameterized smoothly by $\alpha \in R$.

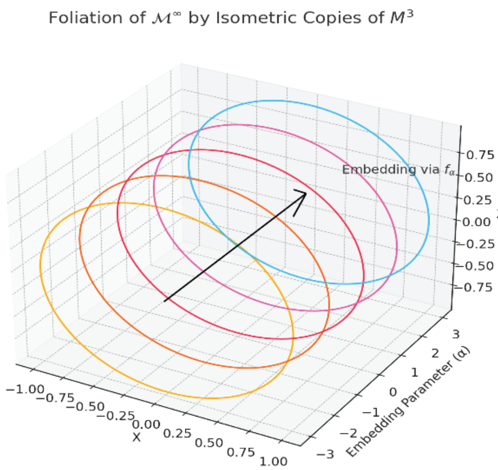


**Fig 2.** Foliation of $\mathcal{M}^\infty$ by isometric copies of the 3-manifold $M^3$. Each slice $\Sigma_\alpha$ corresponds to a distinct embedded copy.

***Corollary* 1**

The ambient Hilbert manifold $\mathcal{M}^\infty$ constructed above admits an infinite number of disjoint, isometrically embedded copies of $(M^3, g)$, forming a smooth foliation parameterized by $R$.

***Remark 3***

This construction may be viewed as an abstract generalization of brane-world models, where each leaf corresponds to a separate universe. Unlike in string theory, however, this construction relies only on differential topology and functional geometry, without invoking any specific field equations or energy conditions.

***Remark 4***

The choice of $R$ as the parameter space is arbitrary; more general parameter spaces can be used depending on the regularity and global properties of $v$. This opens the door to analyzing the moduli of such foliations.

*Moduli of Foliations Induced by Normal Fields*

We now explore the space of foliations constructed via parallel embeddings along normal vector fields. Let $(\mathcal{E}(M^3, \mathcal{H}))$ denote the space of smooth embeddings of a compact manifold $M^3$ into a Hilbert space $\mathcal{H}$. Fix a reference embedding $f_0 \in \mathcal{E}(M^3, \mathcal{H})$, and define a space of smooth normal vector fields:

$$\mathcal{V} := \{v \in C^\infty(M^3, \mathcal{H}) \mid v(x) \perp df_0(T_xM^3), \ |v(x)| = 1, \forall\, x \in M^3\}$$

Each $v \in \mathcal{V}$ gives rise to a 1-parameter family of embeddings:

$$f_\alpha(x) = f_0(x) + \alpha v(x), \quad \alpha \in R \tag{4}$$

and thus defines a foliation $\mathcal{F}_v = \{f_\alpha(M^3)\}_{\alpha \in R}$ of an open subset of $\mathcal{H}$.

*Theorem 3 (Moduli of Parallel Foliations)*

*Let $(M^3, g)$ be a compact Riemannian manifold and $f_0 : M^3 \hookrightarrow \mathcal{H}$ a fixed embedding. Then the space of foliations $\{\mathcal{F}_v\}$ induced by parallel translations along unit normal vector fields $v \in \mathcal{V}$ is a smooth infinite-dimensional Fréchet manifold, modeled on a closed subspace of $C^\infty(M^3, \mathcal{H})$.*

***Proof:*** Let $f_0: M^3 \hookrightarrow \mathcal{H}$ be a fixed smooth embedding. The space $C^\infty(M^3, \mathcal{H})$, the set of all smooth maps from $M^3$ to $\mathcal{H}$, is a Fréchet space when endowed with the $C^\infty$ −topology (i.e., convergence in all derivatives uniformly on compact sets). Since $M^3$ is compact, this topology is metrizable and complete.

Define the subspace:

$$\mathcal{V} := \{v \in C^\infty(M^3, \mathcal{H}) \mid v(x) \perp df_0(T_xM^3), \ |v(x)| = 1\}$$

We now show that $\mathcal{V}$ is a smooth infinite-dimensional submanifold of $C^\infty(M^3, \mathcal{H})$. To do so, consider the constraint map defined in Equation (5):

$$\Phi: C^\infty(M^3, \mathcal{H}) \to C^\infty(M^3, R^{k+1}),$$

$$\Phi(v)(x) = (\langle v(x), df_0(e_1(x))\rangle, , . \langle v(x), df_0(e_k(x))\rangle, |v(x)|^2 - 1) \quad (5)$$

where $\{e_1(x), \dots, e_k(x)\}$ is a smooth local orthonormal frame for $T_x M^3$. This map encodes the orthogonality conditions and the unit norm condition.

Each component of $\Phi$ is smooth (since the inner product and norm are smooth in $\mathcal{H}$), and $\Phi^{-1}(0) = \mathcal{V}$. We aim to apply the **implicit function theorem** in Fréchet spaces. For this, we must check that the derivative $D\Phi_v$ at any $v \in \mathcal{V}$ is surjective and admits a continuous right inverse.

*The linearization:*

$D\Phi_v: C^\infty(M^3, \mathcal{H}) \to C^\infty(M^3, \mathbb{R}^{k+1})$ *is given by:*

$$(D\Phi_v)(w)(x) = (\langle w(x), df_0(e_i(x))\rangle, 2\langle w(x), v(x)\rangle) \quad (6)$$

This is a surjective bundle map (pointwise surjective at each $x \in M^3$), and since all maps involved are smooth and linear, and $v(x) \neq 0$, the inverse function theorem (Hamilton's version) applies.

Therefore, $\mathcal{V} = \Phi^{-1}(0) \subset C^\infty(M^3, \mathcal{H})$ is a smooth infinite-dimensional Fréchet submanifold.

Now, each $v \in \mathcal{V}$ defines a foliation:

$$\mathcal{F}_v := \{f_\alpha(M^3) \mid f_\alpha(x) = f_0(x) + \alpha v(x), \ \alpha \in R\} \quad (7)$$

Since the dependence on $v$ is smooth, and the map $v \mapsto f_\alpha$ is smooth in $C^\infty$-topology (with fixed $f_0$, the space of such foliations inherits a smooth structure from $\mathcal{V}$.

Hence, the moduli space of such foliations is a smooth Fréchet manifold modeled on $\mathcal{V}$.

***Remark 5***

Two foliations $\mathcal{F}_{v_1}, \mathcal{F}_{v_2}$ may be considered equivalent if there exists a diffeomorphism of $\mathcal{H}$ mapping one leaf structure to the other. Quotienting by such equivalences leads to a stratified moduli space, potentially with rich geometric features.

*Explicit Example: Embedding and Foliation of* $S^3$

We now present a concrete example of the abstract construction outlined above, using the standard 3-sphere $S^3$. Consider $S^3 \subset R^4$ defined by

$$S^3 = \{(x_1, x_2, x_3, x_4) \in R^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$

We construct an embedding $f_0: S^3 \hookrightarrow \mathcal{H}$ into the infinite-dimensional Hilbert space $\mathcal{H} = \ell^2(N)$ via a Fourier-type expansion:

$$f_0(x) = (x_1, x_2, x_3, x_4, 0, 0, \dots)$$

This is a smooth and injective map, and $f_0(S^3) \subset \mathcal{H}$ is an isometric image of $S^3$ under the standard inner product restricted to the first four coordinates.

Next, define a smooth vector field $v: S^3 \to \mathcal{H}$ by

$$v(x) = (0,0,0,0, \cos(2\pi x_1), \cos(2\pi x_2), \cos(2\pi x_3), \cos(2\pi x_4), 0, \dots)$$

normalized such that $|v(x)| = 1$ and $v(x) \perp df_0(T_x S^3)$ for all $x \in S^3$. This construction ensures that $v$ is a smooth normal vector field.

For each $\alpha \in R$, define

$$f_\alpha(x) = f_0(x) + \alpha v(x).$$

The image $\Sigma_\alpha := f_\alpha(S^3)$ is again diffeomorphic to $S^3$, and the family $\{\Sigma_\alpha\}_{\alpha \in R}$ defines a smooth foliation of an open subset of $\mathcal{H}$ into isometric copies of the 3-sphere. This provides a tangible realization of the abstract foliation framework, grounded in familiar geometric objects.

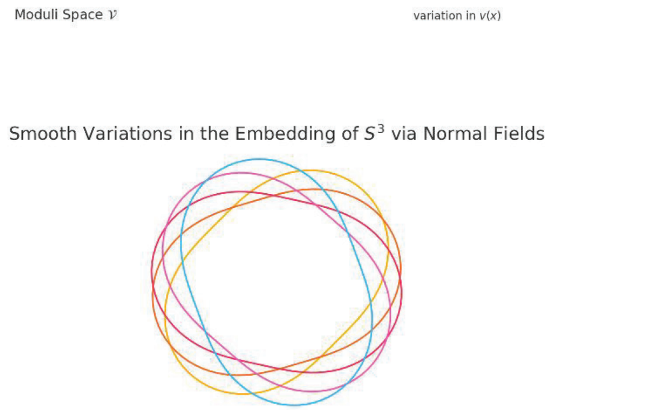


**Fig. 3.** Visualization of variations in the embedding of $S^3$ into $\mathcal{H}$ using different normal vector fields $v(x)$. Each curve represents a distinct foliation leaf $\Sigma_\alpha$, and the vertical deviation reflects changes in the moduli space $\mathcal{V}$.

## VI. Conclusion

In this paper, we have proposed a geometric and conceptual framework in which our observable 3-dimensional universe is modeled as a smooth submanifold-a slice-embedded within an infinite-dimensional Hilbert manifold. Building on classical results in differential topology and infinite-dimensional geometry, we constructed explicit embeddings of 3-manifolds into infinite-dimensional spaces and introduced a foliation structure where each leaf is isometrically diffeomorphic to a fixed 3-manifold $M^3$.

We proved that such a foliation can be generated through smooth variation of a normal vector field along a fixed embedding, and demonstrated that the space of all such foliations-a moduli space of geometric configurations-forms an infinite-dimensional Fréchet manifold. An explicit example involving the 3-sphere $S^3$ embedded in $\ell^2$ was presented, along with visualizations illustrating variation within this moduli space.

This perspective not only enriches the mathematical theory of embeddings and foliations in infinite-dimensional manifolds, but also opens the door to new physical and philosophical interpretations. The idea that physical space may emerge as a slice of a more complex ambient geometry resonates with theories of emergent spacetime, brane worlds, and infinite-dimensional quantum frameworks. While the model remains geometric and kinematic in nature, it provides a fertile ground for introducing dynamics, variational principles, and possibly field-theoretic structures over the foliation.

## VII. Declarations

**Conflict of interest**: The authors declare no conflict of interest.

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