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DIRECT PRODUCT AND WREATH PRODUCT OF TRANSFORMATION SEMIGROUPS

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ABSTRACT

In this paper direct product and wreath product of transformation semigroups have been defined, and associativity of both the products and distributivity of wreath product over direct product have been established.

Keywords: Transformation semigroup, Direct product, Wreath product

1. Introduction

Direct product and wreath product of transformation groups are well known (see [1,3,5]). We have generalized these products to transformation semigroups. We have proved that both direct product and wreath product are associative, and that wreath product is distributive over direct product.

2. Direct Product and Wreath Product

Definition 2.1

Let S be a semigroup and X a non-empty set. S will be called a **transformation semigroup** on X if there is a mapping ϕ: $S\times X\to X$, for which we write

$\phi(s, x) = s(x)$ and which satisfies the condition

$(s_1 s_2)(x) = s_1(s_2(x))$, for each $x\in X$ and for each $s_1, s_2 \in S$.

If S is a monoid, i.e., if S has an identify element 1, then the mapping ϕ is further assumed to satisfy $1(x) = x$, for each $x \in X$.

For every transformation semigroup S on X, there is a homomorphism $\psi : S \to E(X)$, the semigroup of all mappings $f : X \to X$, given by $\psi(s) = f$, where $f(x) = s(x)$. $E(X)$ is usually called the **full transformation semigroup** on X**.**

Let X_1 and X_2 be two non-empty disjoint sets and let S_1 and S_2 be transformation semigroups on X_1 and X_2 respectively.

Definition 2.2

The **direct product** of S_1 and S_2, written $S_1 \times S_2$, is defined as a transformation semigroup on $X_1 \cup X_2$, the elements of $S_1 \times S_2$ being the ordered pairs (s_1, s_2), $s_1 \in S_1$, $s_2 \in S_2$, with $(s_1, s_2)(x_1) = s_1(x_1)$,$(s_1, s_2)(x_2) = s_2(x_2)$, for each $x_1 \in X_1$, $x_2 \in X_2$. The multiplication in $S_1 \times S_2$ is component-wise. It is easily seen that $S_1 \times S_2$ is indeed a transformation semigroup. If S_1 and S_2 are finite, the number of elements of $S_1 \times S_2$ is obviously the product of the numbers of elements of S_1 and S_2.

Theorem 2.1

If S_1, S_2, S_3 are transformation semigroups on X_1, X_2, X_3, then $(S_1 \times S_2) \times S_3 \cong S_1 \times (S_2 \times S_3)$ is a transformation semigroup on $X_1 \cup X_2 \cup X_3$.

Proof

Obviously, the map $((s_1, s_2), s_3) \to (s_1, (s_2, s_3))$ is an isomorphism of semigroups $(S_1 \times S_2) \times S_3$ and $S_1 \times (S_2 \times S_3)$. To see that it is also so as transformation semigroups, we note as a typical case, $((s_1, s_2), s_3)(x_1) = (s_1, s_2)(x_1) = s_1(x_1)$, and also, $(s_1, (s_2, s_3))(x_1) = s_1(x_1)$.

Definition 2.3

The **wreath product** of S_1 with S_2, written $S_1 \varsigma S_2$ is the transformation semigroup on $X_1 \times X_2$ consisting of elements θ on $X_1 \times X_2$ which are given by $\theta : X_1 \times X_2 \to X_1 \times X_2$ such that $\theta(x_1, x_2) = (\mathrm{s}_{1,\mathrm{x}_2}(\mathrm{x}_1), \mathrm{s}_2(\mathrm{x}_2))$, with s_2 in S_2 and each $\mathrm{s}_{1,\mathrm{x}_2}$ in $S_1, \mathrm{s}_{1,\mathrm{x}_2}$, $\mathrm{s}_{1,\mathrm{x}_2}$ being an element of S_1 determined by x_2.

It follows from the definition that if S_1, S_2, X_1, X_2 are finite, then

$|S_1 \varsigma S_2| = |S_1|^{|X_2|} \times |S_2|$, where $|S_i|$ and $|X_2|$ denote the numbers of elements of S_i (i=1,2) and X_2 respectively.

3. Wreath Product as a Direct Product

An equivalent description of wreath product in terms of direct products is given below:

Theorem 3.1

If (S_1, X_1) and (S_2, X_2) are transformation semigroups, then

$(S_1 \varsigma S_2, X_1 \times X_2) \cong \left(\left(\underset{x_2 \in X_2}{\times} S_{1,x_2} \right) \times S_2, \left(\bigcup_{x_2 \in X_2} X_{1,x_2} \right) \cup X_2 \right)$, where each $x_2 \in X_2$, $S_{1,x_2} \cong S_1$ and $\left| X_{1,x_2} \right| = X_1$.

Proof

Let $\theta \in (S_1 \varsigma S_2, X_1 \times X_2)$. Then $\theta(x_1, x_2) = (\sigma_{1,x_2}(x_1), \sigma_2(x_2))$, for some $\sigma_2 \in S_2$ and $\sigma_{1,x_2} \in S_1$, where σ_{1,x_2} is in S_1 and depends on x_2.

Define $\Phi : (S_1 \varsigma S_2, X_1 \times X_2) \to \left(\left(\underset{x_2 \in X_2}{\times} S_{1,x_2} \right) \times S_2, \left(\underset{x_2 \in X_2}{\cup} X_{1,x_2} \right) \cup X_2 \right)$

by $(\Phi(\theta))(x_{1,x_2}) = \sigma_{1,x_2}(x_1)$, $(\Phi(\theta))(x_2) = \sigma_2(x_2)$.

Next, let $\overline{\theta} \in \left(\left(\underset{x_2 \in X_2}{\times} S_{1,x_2} \right) \times S_2, \left(\underset{x_2 \in X_2}{\cup} X_{1,x_2} \right) \cup X_2 \right)$. Then

$\overline{\theta}(x_{1,x_2}) = \overline{\sigma}_{1,x_2}(x_{1,x_2}) = \overline{\sigma}_2(x_2).$, for some

$\overline{\sigma}_{1,x_2} \in S_1, \overline{\sigma}_{1,x_2}$ depending on x_2 and $\overline{\sigma}_2 \in S_2$.

Define Ψ: $\left(\left(\underset{x_2 \in X_2}{\times} S_{1,x_2} \right) \times S_2, \left(\underset{x_2 \in X_2}{\cup} X_{1,x_2} \right) \cup X_2 \right) \to (S_1 \varsigma S_2, X_1 \times X_2)$ by

$(\Psi(\overline{\theta}))(x_1,x_2) = (\overline{\sigma}_{1,x_2}(x_1), \overline{\sigma}_2(x_2))$.

If $\theta' \in (S_1 \varsigma S_2, X_1 \times X_2)$ is given by

$\theta'(x_1,x_2) = (\sigma'_{1,x_2}(x_1), \sigma'_2(x_2))$, where $\sigma'_2 \in S_2$ and $\sigma'_{1,x_2} \in S_1$ and depends on x_2,

then $\varphi(\theta')(x_{1,x_2}) = \sigma'_{1,x_2}(x_1)$

$\varphi(\theta')(x_2) = \sigma'_2(x_2)$

and $(\theta\theta')(x_{1,x_2}) = ((\sigma_{1,x_2}\sigma'_{1,x_2})(x_1), (\sigma_2\sigma'_2)(x_2))$.

Hence $\varphi(\theta\theta')(x_{1,x_2}) = ((\sigma_{1,x_2}\sigma'_{1,x_2})(x_1)$

$\varphi(\theta\theta')(x_2) = (\sigma_2\sigma'_2)(x_2))$.

Also, $(\varphi(\theta)\varphi(\theta'))(x_{1,x_2}) = ((\sigma_{1,x_2}\sigma'_{1,x_2})(x_1)$

$(\varphi(\theta)\varphi(\theta'))(x_2) = ((\sigma_2\sigma'_2)(x_2)$

$\therefore \Phi(\theta\theta') = \Phi\Psi(\theta)\Phi(\theta')$.

Thus φ is a homomorphism.

If $\overline{\theta}' \in \left(\left(\underset{x_2 \in X_2}{\times} S_{1,x_2} \right) \times S_2, \left(\underset{x_2 \in X_2}{\cup} X_{1,x_2} \right) \cup X_2 \right)$ is given by

$\overline{\theta}'(x_{1,x_2}) = \sigma'_{1,x_2}(x_{1,x_2}) = \sigma'_2(x_2)$

then $\varphi(\overline{\theta}')(x_1, x_2) = (\overline{\sigma}'_{1,x_2}(x_1), \overline{\sigma}'_2(x_2))$.

Also, $(\overline{\theta}\overline{\theta}')(x_{1,x_2}) = (\overline{\sigma}_{1,x_2}\overline{\sigma}'_{1,x_2})(x_{1,x_2})$

$(\overline{\theta}\overline{\theta}')(x_2) = (\overline{\sigma}_2\overline{\sigma}'_2)(x_2)$

Hence $(\psi(\theta\theta'))(x_1, x_2) = ((\overline{\sigma_{1,x_2}}\,\overline{\sigma'_{1,x_2}}(x_{1,x_2}), \overline{\sigma_2}\,\overline{\sigma'_2}(x_2))$ and

$(\psi(\theta)\psi(\theta'))(x_1, x_2) = ((\overline{\sigma_{1,x_2}}\,\overline{\sigma'_{1,x_2}}(x_1), \overline{\sigma_2}\,\overline{\sigma'_2}(x_2))$ so that $\therefore \Psi(\theta\theta') = \Psi(\theta)\Psi(\theta')$ i.e., Ψ is a homomorphism .

Now

$$(\varphi\psi)(\overline{\theta})(x_{1,x_2}) = \varphi(\psi(\overline{\theta}))(x_{1.x_2}) = \overline{\sigma_{1,x_2}}(x_1),$$

$$and\ ((\varphi\psi)(\overline{\theta})(x_2) = \varphi(\psi(\overline{\theta}))(x_2) = \overline{\sigma_2}(x_2))$$

$\therefore (\Phi\Psi)(\theta) = \theta.$

$$\therefore \varphi\psi = \left(\left(\underset{x_2 \in X_2}{\times} S_{1,x_2}\right) \times S_2, \left(\bigcup_{x_2 \in X_2} X_{1,x_2}\right) \cup X_2\right).$$

Also, $(\Psi\Phi)(\theta)(x_1, x_2) = \Psi(\Phi(\theta))(x_1, x_2) = (\sigma_{1,x_2}(x_1), \sigma_2(x_2)) = \theta(x_1, x_2)$.

$\therefore \Psi\Phi(\theta) = \theta$, and so $\therefore \Psi\Phi = 1_{(S_1 \varsigma S_2, X_1 \times X_{21})}$.

Thus Φ and Ψ are inverses of each other. Therefore, both Φ and Ψ are isomorphisms.

The following remarks are very significant and useful.

Remarks

(i) If (S_1, X_1) and (S_2, X_2) are transformation semigroups with $S_2 = \{1_{x_2}\}$, then $(S_1 \times S_2, X_1 \cup X_2)$ and $(S_1 \varsigma S_2, X_1 \times X_2)$ may be identified with (S_1, X_1) and $\left(\left(\prod_{x_2 \in X_2} S_{1,x_2}\right), \bigcup_{x_2 \in X_2} X_{1,x_2}\right)$ respectively, ignoring the trivial action of S_2 on X_2. Here, each $S_{1,x_2} \cong S_1$ and each X_{1,x_2} is in 1-1 correspondence with X_1 with $x_{1,x_2} \leftrightarrow x_1$, and $s_{1,x_2}(x_{1,x_2}) = s_1(x_1)$. Thus, in this case, $S_1 \times S_2 \cong S_1$ and $S_1 \varsigma S_2 \cong \left(\prod_{x_2 \in X} S_{1,x_2}\right)$ (direct product) as semigroups.

(ii) If $S_1 = \{1_{x_1}\}$, then both $(S_1 \times S_2, X_1 \cup X_2)$ and $(S_1 \varsigma S_2, X_1 \times X_2)$ may be identified with (S_2, X_2) since $s_{1,x_2} = s_{1,x'_2} = 1_{x_2}$, for each pair of elements $x_2, x'_2 \in X_2$.

(iii) If S and S' are transformation semigroups on the same set X, then $(S_1 \varsigma S', X \times X)$ may be identified with $\left(\left(\prod_{x \in X} S_x\right) \times S', \bigcup_{x \in X} X_x \cup X\right)$. As semigroups, $S \varsigma S' \cong \left(\prod_{x \in X} S_x\right) \times S'$.

If, in particular, $S = S'$ and X is finite with $|X| = n$, then $S \varsigma S \cong \underleftrightarrow{S \times S \times S \times \cdots\cdots \times S \times S}$ (n + 1 copies).

4. Associativity of Wreath Products

Theorem 4.1

If (S_1, X_1), (S_2, X_2), (S_3, X_3) are three transformation semigroups, then $((S_1 \varsigma S_2) \varsigma S_3, (X_1 \times X_2) \times X_3) \cong (S_1 \varsigma (S_2 \varsigma S_3), (X_1 \times (X_2 \times X_3))$.

Proof

Define $\varphi : (S_1 \varsigma S_2) \varsigma S_3 \rightarrow S_1 \varsigma (S_2 \varsigma S_3)$ and

$\psi : S_1 \varsigma (S_2 \varsigma S_3) \rightarrow (S_1 \varsigma S_2) \varsigma S_3$ as follows:

If $\theta \in ((S_1 \varsigma S_2) \varsigma S_3)$ is given by $\theta((x_1,x_2),x_3) = (\alpha_{12,x_3}, \sigma_3(x_3))$ where

$\alpha_{12,x_3} \in S_1 \varsigma S_2$, depends on x_3 and is defined by

$\alpha_{12,x_3}(x_1, x_2) = (\sigma_{1,x_{2,x_3}}(x_1), \sigma_{2,x_3}(x_2))$ so that

$\theta((x_1, x_2), x_3) = (((\sigma_{1,x_{2,x_3}}(x_1), \sigma_{2,x_3}(x_2)), \sigma_3(x_3))$, then $\varphi(\theta)$ is given by

$\varphi(\theta)(x_1, (x_2, x_3)) = (((\sigma_{1,x_2,x_3}(x_1), \sigma_{2,x_3}(x_2)), \sigma_3(x_3))$.

Also, if $\bar{\theta} \in S_1 \varsigma (S_2 \varsigma S_3)$ is given by

$$\bar{\theta}(x_1,(x_2, x_3)) = (\bar{\sigma}_{1,(x_2,x_3)}(x_1), \bar{\sigma}_{23}(x_2, x_3)) = (\bar{\sigma}_{1,x_{2,x_3}}(x_1), (\bar{\sigma}_{2,x_3}(x_2), \bar{\sigma}_3(x_3))),$$

then $\psi(\theta) = ((\bar{\sigma}_{1,x_{2,,x_3}}(x_1), \bar{\sigma}_{2,x_3}(x_2), \bar{\sigma}_3(x_3))$.

If θ' and $\bar{\theta}$ are defined similarly with the σ's and σ's replacing by σ's and $\bar{\sigma}$'s then $\theta\theta'$ and $\theta'\theta$ are given by

$$(\theta\theta')((x_1, x_2), x_3) = \theta(((\sigma'_{1,x_{2,x_3}}(x_1), \sigma'_{2,x_3}(x_2)), \sigma'_3(x_3))$$

$$= (((\sigma_{1,x_2x_3}\sigma'_{1,x_{2,x_3}})(x_1), (\sigma_{2,x_3}\sigma'_{2,x_3})(x_2)), (\sigma_3\sigma'_3)(x_3))$$

and $\overline{(\theta\ \bar{\theta}')}((x_1, x_2), x_3)) = \bar{\theta}(((\overline{\sigma'}_{1,x_{2,x_3}}(x_1), \overline{(\sigma'}_{2,x_3}(x_2), \overline{\sigma'}_3(x_3))$

$$= ((\bar{\sigma}_{1,x_2x_3}\overline{\sigma'}_{1,x_{2,x_3}}(x_1), \overline{(\sigma}_{2,x_3}\overline{\sigma'}_{2,x_3}(x_2), \bar{\sigma}_3\overline{\sigma'}_3(x_3))$$

It is clear that $\varphi(\theta\theta') = \varphi(\theta)\varphi(\theta')$ and $\psi(\theta\ \bar{\theta}') = \psi(\theta)\psi(\theta')$, i.e, φ and ψ are homomorphisms. Also it is evident from the definitions of φ and ψ that they are inverses of each other. Hence both φ and ψ are isomorphisms.

5. Distributivity of Wreath Products over Direct Products

The following isomorphism theorem may be viewed as showing that wreath product of the stated type is distributive over as a direct product that arises in a natural manner.

Theorem 5.1

Let (S_1, X_1), (S_2, X_2) and (S_3, X_3) be three transformation semigroups. Then

$(S_1 \varsigma (S_2 \times S_3), X_1 \times (X_2 \cup X_3)) \cong ((S_1 \varsigma S_2) \times (S_1 \varsigma S_3), (X_1 \times X_2) \cup (X_1 \times X_3))$.

Proof

Define

$\varphi : (S_1 \varsigma (S_2 \times S_3), X_1 \times (X_2 \cup X_3)) \to ((S_1 \varsigma S_2) \times (S_1 \varsigma S_3), (X_1 \times X_2) \cup (X_1 \times X_3))$

by $\varphi(\theta) = (\theta', \theta'')$ (1)

where, if $\theta(x_1, x_2) = (\sigma_{1,x_2}(x_1), (\sigma_2, \sigma_3)(x_2)) = (\sigma_{1,x_2}(x_1), \sigma_2(x_2))$, (2)

and $\theta(x_1, x_3) = (\sigma_{1,x_3}(x_1), (\sigma_2, \sigma_3)(x_3)) = (\sigma_{1,x_3}(x_1), \sigma_3(x_3))$, (3)

then $(\theta', \theta'')(x_1, x_2) = \theta'(x_1, x_2) = (\sigma_{1,x_2}(x_1), \sigma_2(x_2))$ (4)

$(\theta', \theta'')(x_1, x_3) = \theta''(x_1, x_3) = (\sigma_{1,x_3}(x_1), \sigma_3(x_3))$ (5)

Also define

$\psi : (S_1 \varsigma S_2) \times (S_1 \varsigma S_3), (X_1 \times X_2) \cup (X_1 \times X_3)) \to (S_1 \varsigma (S_2 \times S_3), X_1 \times (X_2 \cup X_3))$

by $\psi(\theta, \theta') = \theta''$ where if $(\theta, \theta')(x_1, x_2) = \theta'(x_1, x_2) = (\sigma_{1,x_2}(x_1), \sigma_2(x_2))$,

$(\theta, \theta')(x_1, x_3) = \theta'(x_1, x_3) = (\sigma_{1,x_3}(x_1), \sigma_3(x_3))$, (6)

then $\theta''(x_1, x_2) = (\sigma_{1,x_2}(x_1), (\sigma_2, \sigma_3)(x_2) = (\sigma_{1,x_2}(x_1), \sigma_2(x_2))$ (7)

$\theta''(x_1, x_3) = (\sigma_{1,x_3}(x_1), (\sigma_2, \sigma_3)(x_3) = (\sigma_{1,x_3}(x_1), \sigma_3(x_3))$. (8)

It follows from (1) - (8) that

$\varphi\psi = 1_{((S_1 \varsigma S_2) \times (S_1 \varsigma S_3), (X_1 \times X_2) \cup (X_1 \times X_3))}$

and $\psi\varphi = 1_{(S_1 \varsigma (S_2 \times S_3), X_1 \times (X_2 \cup X_3))}$

Thus both φ and ψ are 1-1 and onto.

Now, let θ, $\bar{\theta} \in (S_1 \varsigma (S_2 \times S_3), X_1 \times (X_2 \cup X_3))$ by given by

$\theta(x_1, x_2) = \sigma_{1,x_2}(x_1), \sigma_2(x_2))$

$\theta(x_1, x_3) = \sigma_{1,x_3}(x_1), \sigma_3(x_3))$ where $\sigma_2 \in S_2$, $\sigma_3 \in S_3$, and σ_{1,x_2}, $\sigma_{1,x_3} \in S_1$

the former being determined by x_2 and the latter by x_3 and

$\bar{\theta}(x_1, x_2) = \overline{\sigma_{1,x_2}}(x_1), \sigma_2(x_2))$, $\bar{\theta}(x_1, x_2) = \overline{\sigma_{1,x_3}}(x_1), \sigma_2(x_3))$

where $\bar{\sigma}_2 \in S_2$, $\bar{\sigma}_3 \in S_3$, and $\overline{\sigma_{1,x_2}}$, $\overline{\sigma_{1,x_3}} \in S_1$, the former being determined by x_2 and the latter by x_3.

Then $(\theta\ \bar{\theta})(x_1, x_2) = (\sigma_{1,x_2}\overline{\sigma_{1,x_2}}(x_1), \sigma_2\bar{\sigma}_2(x_2))$,

$$(\theta\ \bar{\theta})\ (x_1, x_3) = (\sigma_{1,x_3}\bar{\sigma}_1, x_3^{(x_1)}, \sigma_3\bar{\sigma}_3(x_3)).$$

Since, $\varphi(\theta) = (\theta'_1, \theta'_2)$ and $\varphi(\bar{\theta}) = (\bar{\theta}'_1, \bar{\theta}'_2)$,

where $(\theta'_1, \theta'_2)(x_1, x_2) = (\sigma_{1,x_2}(x_1), \sigma_2(x_2))$,

$(\theta'_1, \theta'_2)\ (x_1, x_3) = (\sigma_{1,x_2}(x_1), \sigma_3(x_3))$,

and $(\overline{\theta'}_1, \overline{\theta'}_2)\ (x_1, x_2) = (\bar{\sigma}_{1,x_2}(x_1), \bar{\sigma}_2(x_2))$

$(\overline{\theta'}_1, \overline{\theta'}_2)(x_1, x_3) = (\bar{\sigma}_{1,x_3}(x_1), \bar{\sigma}_3(x_3))$.

We have $\varphi(\theta\ \bar{\theta}) = \varphi(\theta)\ \varphi(\bar{\theta})$.

Hence φ is a homomorphism. Therefore φ is an isomorphism. Thus

$(S_1 \varsubsetneq (S_2 \times S_3), X_1 \times (X_2 \cup X_3)) \cong ((S_1 \varsubsetneq S_2) \times (S_1 \varsubsetneq S_3), (X_1 \times X_2) \cup (X_1 \times X_3))$.

Application of direct product and wreath product of transformation groups and transformation semigroups appear in [5,6].

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