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MODIFIED INVERSES OF CENTRALIZERS OF SEMIPRIME RINGS

Md Rezaul Islam[1,*] and Satrajit Kumar Saha[2]
[1]Dhaka Cantt. Girls' Public School and College, Dhaka
[2]Department of Mathematics, Jahangirnagar University, Savar, Dhaka-1342
[*]Corresponding author: rezaadhimoni@gmail.com

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ABSTRACT

The main purpose of this paper is to investigate the properties of centralizers of semiprime ring *R* as well as to establish necessary and sufficient conditions for a centralizer *T* of a semiprime ring *R* to have a commuting modified inverse.

Keywords: Semiprime ring, left (right) centralizer, centralizer, modified inverse.

1. Introduction

Modified inverses of operators on various algebraic structures have been an active area of research since the last fifty-two years due to their usefulness in various fields of mathematics, statistics and engineering. Moreover, the theory of centralizers, also called multipliers, of Banach algebras and C^*–algebras is well established. Recently, some authors have studied centralizers in the general framework of semiprime rings ([2-6]). We have tried to investigate some further properties of centralizers of semiprime rings as well as to establish necessary and sufficient conditions for a centralizer of a semiprime ring to have a commuting modified inverse. We are inspired by the fundamental work of Vukman ([3-5]) and Zalar ([6]).

First, we recall some preliminaries and fix our notations. A ring R is said to be semiprime if $aRa = (0)$ implies $a = 0$; it is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$. By Zalar [6], an additive mapping $T: R \to R$ is called a left (right) centralizer if $T(xy) = T(x)y(T(xy) = xT(y))$holds for all $x, y \in R$. If $a \in R$, then $La(x) = ax$ and $Ra(x) = xa, (x \in R)$ define a left centralizer and a right centralizer of R, respectively. An additive mapping $T: R \to R$ is called a centralizer if $T(xy) = T(x)y = xT(y)$holds for all $x,y \in R$. We denote the set of all centralizers of R by $M(R)$. We introduce the notion of modified inverses of additive mappings of rings and prove that a centralizer T of a semiprime ring R has a commuting modified inverse if and only if it is orthogonal [Theorem 2.3].

2. Results Regarding Modified Inverses of Centralizers

At first some of the propositions regarding centralizers will be discussed. Orthogonal complement and modified inverses of centralizers will be defined. With the help of these ideas main results will be proved i.e. if T be a centralizer of a semiprime ring R then T has a commuting m-inverse $S \in M(R)$ if and only if T is orthogonal.

Proposition 2.1. [1]. Let R be a semiprime ring and $T: R \to R$ a mapping satisfying $T(x)y = xT(y)$ for all $x, y \in R$. Then T is a centralizer.

Proof. We need to show that T is additive and $T(xy) = T(x)y = xT(y)$ for all $x, y \in R$. So, let $x, y, z \in R$. Then

$$
\begin{aligned}
(T(x + y) - T(x) - T(y))z &= T(x + y)z - T(x)z - T(y)z \\
&= (x+y)T(z) - xT(z) - yT(z) = ((x+y) - x - y)T(z) \\
&= 0
\end{aligned}
$$

and hence $T(x+y)-T(x)-T(y) = 0$ by the semiprimeness of R.

So, $T(x+y) = T(x)+T(y)$ for all $x, y \in R$.

To prove the second property, consider

$$
\begin{aligned}
(T(xy)-T(x)y)z &= T(xy)z-T(x)yz \\
&=(xy)T(z) - (xT(y))z \\
&= (xy)T(z) - x(T(y)z) \\
&= (xy)T(z) - (xy)T(z) \\
&= 0.
\end{aligned}
$$

Again, by the semiprimeness of R, we get $T(xy) = T(x)y$ for all $x, y \in R$.

Similarly, $T(xy) = xT(y)$ for all $x, y \in R$. So, T is a centralizer. □

Remark 2.1. [1]

a. It is easy to verify that if T is a centralizer of a ring R, then $Ker(T)$ and $R(T)$ are ideals of R where $R(T)$ denoted the range of T.

b. Let $S, T \in M(R)$, the set of all centralizers of a ring R. Define $(S + T)(x) = S(x) + T(x)$ and $(ST)(x) = S(T(x))$, $x \in R$. Then $M(R)$ is a ring with identity. It is easy to verify that if R is semiprime, then $M(R)$ is commutative.

Proposition 2.2. [1]. Let T be a centralizer of a semiprime ring R. Then $Ker(T) = Ker(T^2)$.

Proof. Obviously $Ker(T) \subseteq Ker(T^2)$. Let $u \in Ker(T^2)$. Then $T^2(u) = 0$. Let r$\in$R, then $T(u)rT(u) = T(u)(r(T(u)) = T(u)T(ru) = T(uT(ru)) = T(T(uru)) = T^2(uru) = T^2(u)ru = 0$.

Since R is semiprime, we have T(u) = 0. Thus u $\in$ Ker (T), which implies $Ker(T^2) \subseteq K(T)$. Hence $Ker(T) = Ker(T^2)$.□

Theorem 2.1. [1]. Let T be a centralizer of a semiprime ring R. Then the following are equivalent:

(a) $R(T) = R(T^2)$,

(b) $R = Ker(T) \oplus R(T)$.

Proof. Assume that (a) holds. Since T is a centralizer, we have by Remark 2.1(a) that $Ker(T)$ and $R(T)$ are ideals of R. We first show that $Ker(T) \cap R(T) = (0)$.

Let $y \in Ker(T) \cap R(T)$. Then $T(y) = 0$ and $y = T(q)$ for some $q \in R$. Hence

$0 = T(y) = T^2(q)$. Thus $q \in Ker(T^2)$.

By Proposition 2.2, $Ker(T) = Ker(T^2)$. Thus $q \in Ker(T)$ which implies $T(q) = 0$. Hence $y = 0$, which implies $Ker(T) \cap R(T) = (0)$.

Now assume that $z \in R$. Then $T(z) \in R(T)$, which by (a) implies $T(z) \in R(T^2)$. Thus $T(z) = T^2(v)$ for some $v \in R$, and so $T(z - T(v)) = 0$. Hence $z - T(v) \in Ker(T)$. We write $z = (z - T(v)) + T(v)$. This proves that $R = Ker(T) \oplus R(T)$.

Conversely, assume that (b) holds. We prove that $R(T) = R(T^2)$. Obviously $R(T^2) \subseteq R(T)$. Let $y \in R(T)$. Then $y = T(x_0)$ for some $x_0 \in R$. By assumption, $x_0 = x_1 + x_2$ where $x_1 \in Ker(T)$ and $x_2 \in R(T)$. Thus

$y = T(x_0) = T(x_1 + x_2) = T(x_1) + T(x_2) = T(x_2)$.

Since $x_2 \in R(T)$, therefore $x_2 = T(z_0)$ for some $z_0 \in R$. Hence $y = T(x_2) = T(T(z_0)) = T^2(z_0)$, which implies $y \in R(T^2)$. Thus $R(T^2) \subseteq R(T)$. Hence $R(T) = R(T^2)$. □

Definition 2.1. Let M is a subset of a ring R. We define the orthogonal complement of M to be the set $M^{\perp} = \{x \in R : xy = yx = 0 \quad for\ all\ y \in M\}$.

Proposition 2.3. [1]. Let R be a semiprime ring and $T: R \to R$ a centralizer of R. Then $Ker(T) = R(T)^{\perp}$.

Proof. $Ker(T) \subseteq R(T)^{\perp}$ follows from the preceding remarks. Conversely, let $z \in R(T)^{\perp}$. Then

$zT(x) = T(z)x = 0$ for all $x \in R$ and hence by the semiprimeness of R, we get $T(z) = 0$; that is, $z \in Ker(T)$. Therefore, $R(T)^{\perp} \subseteq Ker(T)$ and hence $Ker(T) = R(T)^{\perp}$. □

Definition 2.2. [1]. A mapping T of a ring R is defined to be orthogonal if $R = R(T) \oplus R(T)^{\perp}$.

Combining **Theorem 2.1** and **Proposition 2.3**, we get the following theorem.

Theorem 2.2. [1]. A centralizer T of a semiprime ring R is orthogonal if and only if $R(T) = R(T^2)$.

The following corollary is obvious.

Corollary 2.1. [1]. An idempotent centralizer of a semiprime ring is orthogonal.

Definition 2.3.[1]. A mapping $T: R \to R$ of a ring R into itself is said to have a modified inverse if there is a mapping $S: R \to R$ such that $STS = S$ and $TST = T$. In this case S is said to be a modified inverse of T or S is an m-inverse of T.

Proposition 2.4. Let $T: R \to R$ be an additive mapping with S as an m-inverse. Then the following hold:

(a) TS and ST are idempotents.

(b) $R(TS) = R(T)$, and $Ker(ST) = Ker(T)$.

Proof. (a) $(TS)(TS) = (TST)S = TS$. So TS is an idempotent. Similarly, ST is an idempotent.

(b) We show that $R(TS) = R(T)$. Now,

$T(x) = (TST)(x) = (TS)(T(x)),(x \in R)$. So $R(T) \subseteq R(TS)$.

Also, $(TS)(x) = T(S(x)) \in R(T)$ implies $R(TS) \subseteq R(T)$.

Therefore, $R(TS) = R(T)$.

We now show that $Ker(T) = Ker(ST)$. Let $z \in Ker (T)$. Then $(ST)(z) = S(T(z)) = 0$. So, $z \in Ker (ST)$ and hence $Ker(T) \subseteq Ker(TS)$. Conversely, let $z \in Ker (ST)$. Then $(ST)(z) = 0$ and hence $T(ST(z)) = T(0) = 0$. Therefore, $z \in Ker (T)$ which implies that $Ker (ST) \subseteq Ker (T)$. Thus $Ker (T) = Ker (ST)$.□

Remark 2.2. It is well known that an m-inverse S of a mapping $T: R \to R$ is not unique. But there is at most one m-inverse which commutes with T. If S and S' are m-inverses of T, both commuting with T, then $TS' = TSTS' = STS'T = ST$, and hence $S' = S'TS' = S'ST = S'TS = TS'S = STS = S$.

Proposition 2.5. Let $T: R \to R$ be a centralizer of a semiprime ring R such that $T = fh$, where $f, h \in M(R)$, f is invertible and h is an idempotent. Then T has a commuting m-inverse in $M(R)$.

Proof. By assumption, $T = fh$. So,

$Tf-1T = fhf-1fh = fh^2 = fh = T$; that is,

$Tf-1T = T$. Put $S = f^{-1}Tf^{-1}$.Then it is easy to verify that S is am-inverse of T. Moreover, it commutes with T, by Remark 2.1(b).

We now prove the following theorem which gives necessary and sufficient conditions for a centralizer T of R to have a commuting m-inverse $S \in M(R)$.

Theorem 2.3. Let T be a centralizer of a semiprime ring R. Then T has a commuting m-inverse $S \in M(R)$ if and only if T is orthogonal.

Proof. Let T be orthogonal. Then, by Theorem 2.2, $R(T) = R(T^2)$. Thus $T(R) = T(T(R))$, which implies that $T_0 = T / R(T)$, the restriction of T on $R(T)$,is surjective. Also by hypothesis, $R = R(T)^{\perp} \oplus R(T) = Ker(T) \oplus R(T)$.

We define $B : R \to R$ by $B(x) = T_0(u) + v = T(u)+v$ where $x \in R$ and $x = u + v$, $u \in R(T)$ and $v \in Ker(T)$. It is easy to verify that B is additive. We now prove that B is bijective. Let $x \in R$ and $x = u + v$, $u \in R(T)$ and $v \in Ker(T)$. Since $u \in R(T)$,there exists $y \in R(T)$ such that $T_0(y) = T(y) = u$. Thus $B(y + v) = T(y) + v = u + v = x$. This shows that B is surjective.

We now show that B is injective. Let $z \in Ker(B)$. By hypothesis, there exist $p \in R(T)$ and $q \in Ker(T)$ such that $z = p+q$, which implies $0 = B(z) = B(p+q) = T(p)+q$. Thus $q = -T(p) \in R(T)$.Since $Ker(T) \cap R(T) = (0)$, we have $q = 0$. So $T(p) = 0$, which implies $p \in Ker(T)$.Thus $p \in R(T) \cap Ker(T) = (0)$, which implies that $p = 0$. Thus $x = p + q = 0$, which implies that B is injective. Hence B is bijective, and therefore, invertible.

We now show that B is a centralizer of R. Let $x, y \in R$. Then $x = u + v$, $y = s+t$, where $u, s \in R(T)$ and $v, t \in Ker(T) = R(T)\perp$. So $B(x)y = (T(u)+v)(s+t) = T(u)s + T(u)t + vs + vt = T(u)s + vt$. That is, $B(x)y = T(u)s + vt$(1)

Similarly,

$$xB(y) = (u+v)(T(s)+t) = uT(s)+ut+vT(s)+vt = uT(s)+vt.$$

That is,

$$xB(y) = uT(s) + t \quad (2)$$

From (1) and (2), we get $B(x)y - xB(y) = T(u)s - uT(s)$.

Now

$T(T(u)s - uT(s)) = T(u)T(s) - T(u)T(s) = 0$. Thus $T(u)s - uT(s) \in Ker(T)$.

Also, $T(u)s - uT(s) \in R(T)$ because $R(T)$ is an ideal. Since $Ker(T) \cap R(T) = (0)$,we get

$T(u)s - uT(s) = 0$. This implies that $B(x)y = xB(y)$ for all $x, y \in R$. Hence, by Proposition 2.1, B is a centralizer.

Further, we define $P: R \to R$ by $P(x) = u$, where $x = u + v$, $u \in R(T)$, $v \in R(T)^{\perp} = Ker(T)$. It is easy to verify that P is an idempotent centralizer of R and $T = BP$. Thus, by Proposition 2.5, T has a commuting m-inverse $S = B^{-1}TB^{-1}$.

Conversely, let T have a commuting m-inverse $S \in M(R)$. So $TS = ST$. Also, TS and ST are centralizers and are idempotent by Proposition 2. 4. Moreover, TS is orthogonal by Corollary 2.1.

Thus

$R = Ker(TS) \oplus R(TS) = Ker(ST) \oplus R(TS)$.

But $Ker(ST) = Ker(T)$ and $R(TS) = R(T)$ by Proposition 2.4. Thus, $R = Ker(T) \oplus R(T)$.Moreover, by Proposition 2. 3, $Ker(T) = R(T)^{\perp}$. Thus $R = R(T)^{\perp} \oplus R(T)$.

Hence T is orthogonal.□

3. Conclusion

The centralizers of semiprime rings and their properties were the main focus of this paper. Proposition 2.4, Proposition 2.5 and Theorem 2.3 are our main results.

REFERENCES

[1] Chaudhry, M. A and Samman, M. S,*Generalized inverses of centralizers of semiprime rings',* Aequationes Math. 71 (2006) 246-252.

[2] J. Vukman and I. Kosi-Ulbl, *On centralizers of semiprime rings,* Aequationes Math. 66(2003), 277-283.

[3] J. Vukman, *Centralizers on semiprime rings,* Comment. Math. Univ. Carolinae 42 (2001), 237-245.

[4] J. Vukman, *An identity related to centralizers in semiprime rings,* Comment. Math. Univ. Carolinae 40 (1999), 447–456.

[5] J. Vukman, *Centralizers on semiprime rings,* Comment. Math. Univ. Carolinae 38 (1997), 231-240.

[6] B. Zalar, *On centralizers of semiprime rings,* Comment. Math. Univ. Carolinae 34 (1991), 609-614.