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DETERMINATION OF THE HOMOLOGY AND THE COHOMOLOGY OF A FEW GROUPS OF ISOMETRIES OF THE HYPERBOLIC PLANE

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ABSTRACT

In this paper we determine the homology and the cohomology groups of two properly discontinuous groups of isometries of the hyperbolic plane having non-compact orbit spaces and the fundamental group of a graph of groups with a finite vertex groups and no trivial edges by extending Lyndon's partial free resolution for finitely presented groups. For the first two groups, we obtain partial extensions and the corresponding homology. We also compute the corresponding cohomology groups for one of these groups. Finally we obtain homology and cohomology in all dimensions for the last of the above mentioned groups by constructing a full resolution for this group.

Keywords: Group presentation, Metacyclic group, Heisenberg group, Free resolution Huebschmann perturbation method, Homology, Cohomology.

1. Introduction

In this paper we consider the homology and cohomology of groups of isometries of the hyperbolic plane H^2.

We recollect that a group G acts on a space X properly discontinuously if for any compact subset C of X { $g \in G : gC \cap C \neq \Phi$ } is finite.

McCullugh and Zimmermann [19] obtained two algebraic characterizations of properly discontinuous groups of isometries of the hyperbolic plane having non compact orbit spaces. One of these characterizations provides a presentation for each such group as is given by the following theorem (see McCullugh and Zimmermann [19]), Theorem 4.1, p.282).

Theorem 1.1

A group *H* is a properly discontinuous group of isometries on the hyperbolic plane having non-compact orbit space if and only if *H* is a (countable) free product of cyclic groups of the form

(i) $\left\langle x_1, x_2, \ldots, : x_1^2, x_2^2, \ldots, (x_1x_2)^{n_1}, (x_2x_3)^{n_2}, \ldots \right\rangle$

where $n_i > 1$ and the number of generators is finite or infinite,

(ii) $\langle \ldots, x_{-1}, x_0, x_1, x_2, \ldots : \ldots, x_{-1}^2, x_0^2, x_1^2, \ldots (x_{-1}x_0)^{n_{-1}}, (x_0x_1)^{n_0}, (x_1x_2)^{n_1}, \ldots \rangle$

and

(iii) $\langle x_1, \ldots, x_r, x_{r+1} : x_1^2, \ldots, x_r^2, (x_1x_2)^{n_1}, \ldots, (x_{r-1}x_r)^{n_{r-1}}, x_r x_{r+1} x_1 x_{r+1}^{-1} \rangle$. .

These presentations were originally obtained by Macbeath and Hoare [8] using geometrical arguments. The main theorem of McCullough and Zimmermann ([19], p.275-276) guarantees that the group H is the fundamental group of a non compact 2-orbifold.

Here we extend Lyndon's 3-term partial resolutions [7] to 6-term partial resolution for the following groups E and G, with the use of Fox`s free partial derivatives. The technique of extension has been elaborately described in [18] and this technique has already been used in [1], [10], … , [17].

(a) E, which is (iii) of Theorem 1.1,

(b) G, which is (i) of Theorem 1.1.

We calculate the homology and cohomology of the group E and the corresponding homology of G up to dimension 4.

The above resolutions can be further extended to full resolutions by similar procedure and the homology and the cohomology calculated.

We also determine the homology and the cohomology of the group K with presentation

$\langle x_1, x_2 : x_1^{2r} = x_2^{2r} = 1, x_1^2 = x_2^2 \rangle$. here r is a positive integer.

This group occurs as a subgroup of the extension X given by $1 \to F \to X \to \mathbf{Z}_n \to 1$, where F is a free group of rank at least 2. X is the fundamental group of a graph of groups with finite vertex groups and no trivial edges. It plays an important role in the proof of Theorem 5.1 of McCullough, Miller and Zimmermann ([19], p. 285).

2. (i). Group E of Hyperbolic Isometries

Here E is given by $E = \dfrac{F}{R}$, where F is the free group generated by, say, $x_1, x_2, \ldots, x_s, x_{s+1}$ and R is the normal subgroup of F generated by $r_1, r_2, \ldots, r_{2s}$, where

$$r_1 = x_1^2, \ldots, r_s = x_s^2, r_{s+1} = (x_1x_2)^{n_1}, \ldots, r_{2s-1} = (x_{s-1}x_s)^{n_{s-1}}, r_{2s} = x_s x_{s+1} x_1 x_{s+1}^{-1},\ s \geq 1 \text{ and } n_i > 1.$$

Let $\pi : \mathbf{Z}F \to \mathbf{Z}E$ be the homomorphism induced by the canonical homomorphism of F onto E with R as the kernel. Let $\pi(x_i) = h_i, i = 1, \cdots, s+1$.

Theorem 2.1

The following is a free 6-term $\mathbf{Z}\,E$-resolution of $\mathbf{Z}$.

$$Y_4 \xrightarrow{d_4} Y_3 \xrightarrow{d_3} Y_2 \xrightarrow{d_2} Y_1 \xrightarrow{d_1} Y_0 \xrightarrow{d_0} \mathbf{Z}E \xrightarrow{\in} \mathbf{Z} \to 0$$

where

Y_0 is a right **Z** E-module free on $\alpha_1,\dots,\alpha_{s+1}$

Y_1 ,, ,, ,, ,, ,, ,, $\beta_1,\dots,\beta_{2s}$

Y_2 ,, ,, ,, ,, ,, ,, $\delta_1,\dots,\delta_{3s-1}$

Y_3 ,, ,, ,, ,, ,, ,, $\lambda_1,\dots,\lambda_{4s-2}$

Y_4 ,, ,, ,, ,, ,, ,, $\mu_1,\dots,\mu_{5s-3}$

and $\varepsilon(g) = 1 \in \mathbf{Z}$, for $g \in E$,

$d_0(\alpha_i) = h_i - 1, \quad i = 1,2,\dots.,s+1,$

$d_1(\beta_i) = \alpha_i(h_i + 1), \quad i = 1,2,\dots,s,$

$d_1(\beta_{s+i}) = \alpha_i h_{i+1}[(h_i h_{i+1})^{n_i-1} + \dots + 1)] + \alpha_{i+1}[(h_i h_{i+1})^{n_i-1} + \dots + 1)], \quad i = 1,2,\dots,s-1,$

$d_1(\beta_{2s}) = \alpha_1 h_{s+1}^{-1} + \alpha_s h_s^{-1} + \alpha_{s+1}(h_1 h_{s+1}^{-1} - h_{s+1}^{-1})$

$d_2(\delta_i) = \beta_i(h_i - 1), \; i = 1,2,\dots,s.$

$d_2(\delta_{s+i}) = -(\beta_i + \beta_{i+1})[(h_i h_{i+1})^{n_i-1} + \dots + 1)] + \beta_{s+i}(h_{i+1} + 1), \quad i = 1,2,\dots,s-1,$

$d_2(\delta_{2s}) = -\beta_1 h_1^{-1} h_{s+1}^{-1} - \beta_s h_s^{-1} + \beta_{2s}(h_s + 1)$

$d_2(\delta_{2s+i}) = \beta_{s+i}(h_i h_{i+1} - 1), \; i = 1,2,\dots,s-1.$

$d_3(\lambda_i) = \delta_i(h_i + 1), \quad i = 1,2,\dots,s,$

$d_3(\lambda_{s+i}) = (\delta_i + \delta_{i+1})[(h_i h_{i+1})^{n_i-1} + \dots + 1)] + \delta_{s+i}(h_{i+1} - 1), \quad i = 1,2,\dots,s-1,$

$d_3(\lambda_{2s}) = \delta_1 h_1^{-1} h_{s+1} + \delta_s h_s^{-1} + \delta_{2s}(h_s^{-1})$

$d_3(\lambda_{2s+i}) = \delta_{2s+i}[(h_i h_{i+1})^{n_i-1} + \dots + 1)], \quad i = 1,2,\dots,s-1,$

$d_3(\lambda_{3s-1+i}) = \delta_{s+i}(h_i h_{i+1} - 1) - \delta_{2s+i}[(h_i h_{i+1})^{n_i-2} + \dots + 1)](h_{i+1} + 1), \quad i = 1,2,\dots,s-1,$

$d_4(\mu_i) = \lambda_i(h_i - 1), \quad i = 1,2,\dots.,s,$

$d_4(\mu_{s+i}) = -(\lambda_i + \lambda_{i+1})[(h_i h_{i+1})^{n_i-1} + \dots + 1)] + \lambda_{s+i}(h_{i+1} + 1), \quad i = 1,2,\dots,s-1,$

$d_4(\mu_{2s}) = -\lambda_1 h_1^{-1} h_{s+1} - \lambda_s h_s^{-1} + \lambda_{2s}(h_s + 1)$

$d_4(\mu_{2s+i}) = \lambda_{2s+i}(h_i h_{i+1} - 1), \; i = 1,2,\dots,s-1.$

$d_4(\mu_{3s-1+i}) = \lambda_{3s-1+i}[(h_i h_{i+1})^{n_i-1} + \dots + 1] + \lambda_{2s+i}[(h_i h_{i+1})^{n_i-2} + \dots + 1)](h_{i+1} + 1), \quad i = 1,2,\dots,s-1,$

$d_4(\mu_{4s-2+i}) = \lambda_{s+i}(h_i h_{i+1} - 1) - \lambda_{3s-1+i}[(h_i h_{i+1})^{n_i-2} + \dots + 1)](h_{i+1} - 1), \quad i = 1,2,\dots,s-1,$

Let *A* be a left *ZE*-module, then the homology groups *H_n(E, A)* are given by the homology of the complex:

$$A^{5s-3} \overset{\bar{d}_4}{\rightarrow} A^{4s-2} \overset{\bar{d}_3}{\rightarrow} A^{3s-1} \overset{\bar{d}_2}{\rightarrow} A^{2s} \overset{\bar{d}_1}{\rightarrow} A^{s+1} \overset{\bar{d}_0}{\rightarrow} A \rightarrow 0$$

where A^k stands for the direct sum of k isomorphic copies of A and the homomorphisms $\bar{d}_0, \bar{d}_1, \bar{d}_2, \bar{d}_3, \bar{d}_4$ are induced by d_0, d_1, d_2, d_3, d_4 respectively and are given by

$$\bar{d}_0(a_1,...,a_{s+1}) = (h_1 - 1)a_1 + ... + (h_{s+1} - 1)a_{s+1},$$

$$\begin{aligned}\bar{d}_1(a_1,...,a_{2s}) = ((h_1+1)a_1 + h_2\left[(h_1h_2)^{n_1-1} + ... + 1\right]a_{s+1} + ... + h_{s+1}^{-1}a_{2s},\ (h_2+1)a_2 + \\ \left[(h_1h_2)^{n_1-1} + ... + 1\right]a_{s+1} + h_3\left[(h_2h_3)^{n_2-1} + ... + 1\right]a_{s+2},...,\ (h_s+1)a_s + \\ \left[(h_{s-1}h_s)^{n_{s-1}-1} + ... + 1\right]a_{2s-1} + ... + h_s^{-1}a_{2s},\ (h_1h_{s+1}^{-1} - h_{s+1}^{-1})a_{2s}),\end{aligned}$$

$$\begin{aligned}\bar{d}_2(a_1,...,a_{3s-1}) = ((h_1-1)a_1 - \left[(h_1h_2)^{n_1-1} + ... + 1\right]a_{s+1} - h_1^{-1}h_{s+1}^{-1}a_{2s},\ (h_2-1)a_2 - \\ \left[(h_1h_2)^{n_1-1} + ... + 1\right]a_{s+1} - \left[(h_2h_3)^{n_2-1} + ... + 1\right]a_{s+2},...,\ (h_2+1)a_{s+1} + \\ (h_1h_2-1)a_{2s+1},...,(h_s+1)a_{2s-1} + (h_{s-1}h_s^{-1})a_{3s-1},(h_s+1)a_{2s},\end{aligned}$$

$$\begin{aligned}\bar{d}_3(a_1,...,a_{4s-2}) = ((h_1+1)a_1 + \left[(h_1h_2)^{n_1-1} + ... + 1\right]a_{s+1} + h_1^{-1}h_{s+1}^{-1}a_{2s},\ (h_2+1)a_2 + \\ \left[(h_1h_2)^{n_1-1} + ... + 1\right]a_{s+1} + \left[(h_2h_3)^{n_2-1} + ... + 1\right]a_{s+2},...,\ (h_2-1)a_{s+1} + \\ (h_1h_2-1)\alpha_{3s-1+1},(h_s-1)a_{2s},\left[(h_1h_2)^{n_1-1} + ... + 1\right]a_{2s+1} \\ -\left[(h_1h_2)^{n_1-2} + ... + 1\right](h_2+1)a_{3s-1+1},...,\left[(h_{s-1}h_s)^{n_{s-1}-1} + ... + 1\right]a_{3s-1} - \\ \left[(h_{s-1}h_s)^{n_{s-1}-2} + ... + 1\right](h_s+1)a_{4s-2}),\end{aligned}$$

$$\begin{aligned}\bar{d}_4(a_1,...,a_{4s-2}) = ((h_1-1)a_1 - \left[(h_1h_2)^{n_1-1} + ... + 1\right]a_{s+1} - h_{s+1}^{-1}a_{2s},\ (h_2+1)a_2 - \\ \left[(h_1h_2)^{n_1-1} + ... + 1\right]a_{s+1} - \left[(h_2h_3)^{n_2-1} + ... + 1\right]a_{s+2},...,\ (h_2+1)a_{s+1} + (h_1h_2-1)a_{4s-2+1},..., \\ (h_s+1)a_{2s},(h_1h_2-1)a_{2s+1} + \left[(h_1h_2)^{n_1-2} + ... + 1\right](h_2+1)a_{3s-1+1},...,\left[(h_1h_2)^{n_1-1} + ... + 1\right]a_{3s-1+1} - \\ \left[(h_1h_2)^{n_1-2} + ... + 1\right](h_2-1)a_{4s-2+1},...,\left[(h_{s-1}h_s)^{n_{s-1}-1} + ... + 1\right]a_{4s-2} - \\ \left[(h_{s-1}h_s)^{n_{s-1}-2} + ... + 1\right](h_s-1)a_{5s-3}),\end{aligned}$$

Hence the integral homology groups of E are

$$H_0(E,\mathbf{Z}) \cong \mathbf{Z}.$$

$$H_1(E,\mathbf{Z}) \cong \frac{\{(a_1,...,a_{s+1}) : a_i \in \mathbf{Z}\}}{\{(2a_1 + n_1a_{s+1} + a_{2s}, 2a_2 + n_1a_{s+1} + n_2a_{s+2},...,2a_s + n_{s-1}a_{2s-1} + a_{2s},0)\}}$$

$$\cong \frac{\langle x_1 \rangle \oplus \langle x_2 \rangle \oplus ... \oplus \langle x_{s+1} \rangle}{\langle 2x_1,...,2x_s, n_1(x_1+x_2), n_2(x_1+x_2),..., n_{s-1}x_1, x_1+x_s \rangle},$$

$\cong \langle x_1, x_2, ..., x_{s+1} | 2x_1 = 0, ..., 2x_s = 0, n_1(x_1 + x_2) = 0, n_2(x_2 + x_3) = 0, ..., n_{s-1}x_s = 0, x_1 + x_s = 0 \rangle$

$\cong \mathbf{Z}_\infty(x_{s+1}) \oplus \mathbf{Z}_2(x_{i_1}) \oplus ... \mathbf{Z}_2(x_{i_r})$, r is the number of even n_i's, $2 \le i \le s-1$.

$H_2(E, \mathbf{Z}) =$

$$\frac{\{(a_1, ..., a_{2s}) : 2a_1 + n_1 a_{s+1} + a_{2s} = 2a_2 + n_1 a_{s+1} + n_2 a_{s+2} = 0, ..., 0 = 2a_s + n_{s-1} a_{2s-1} + a_{2s}; a_i \in \mathbf{Z}\}}{\{(-n_1 a_{s+1} - a_{2s}, -n_1 a_{s+1} - n_1 a_{s+1} - n_2 a_{s+2}, ..., -n_{s-11} a_{2s-1} - a_{2s}, 2a_{s+1}, ..., 2a_{2s-1}, 2a_{2s}) | a_i \in \mathbf{Z}\}} \cong 0$$

$$H_3(E, \mathbf{Z}) = \frac{\begin{array}{r}\{(a_1, ..., a_{2s}, a_{2s+1}, ... a_{3s-1}) : -n_1 a_{s+1} = a_{2s}, n_1 a_{s+1} = -n_2 a_{s+2}, ..., n_{s-1} a_{2s-1} = -a_{2s},\\ 2a_{s+1} = 0, ..., 2a_{2s+1} = 0, a_{2s} = 0; a_i \in \mathbf{Z}\}\end{array}}{\begin{array}{l}\{(2a_1 + n_1 a_{s+1} + a_{2s}, 2a_2 + n_1 a_{s+1} + n_2 a_{s+2}, ..., 2a_s + n_{s-1} a_{2s-1} + a_{2s},\\ 0, ..., 0, n_1 a_{2s+1} - 2(n_1 - 1) a_{3s}, ..., n_{s-1} a_{3s-1}) - 2(n_{s-1} - 1) a_{4s-2}; a_i \in \mathbf{Z}\}\end{array}}$$

If we write x_i for (0,…,0,1, 0,…,0), where 1 is in the i-th position, then

$$H_3(E, \mathbf{Z}) \cong \langle x_1, x_2, ..., x_s, x_{2s+1}, ..., x_{3s-1} : 2x_1 = 0, ..., 2x_s = 0, n_1(x_1 + x_2) = 0, n_{s-1}x_s = 0, x_1 + x_s = 0,$$
$$n_1 x_{2s+1} = 0, -2(n_1 - 1) x_{2s+1} = 0, ..., 2(n_{s-1} - 1) x_{3s-1} = 0 \rangle$$

$\cong \mathbf{Z}_2(x_{i_1}) \oplus ... \oplus \mathbf{Z}_2(x_{i_r}) \oplus \mathbf{Z}_2(x_{j_1}) \oplus ... \oplus \mathbf{Z}_2(x_{j_r})$, where r is the number of even n_i,

$2 \le i_1, ..., i_r \le s$ and $2s + 1 \le j_1, ..., j_r \le 3s - 1$.

$H_4(E, \mathbf{Z}) \cong 0.$

Let *A* be a right *ZE*-module, then the cohomology groups $H^n(E, A)$ are given by the homology of the complex:

$$A^{5s-3} \overset{d_4^*}{\leftarrow} A^{4s-2} \overset{d_3^*}{\leftarrow} A^{3s-1} \overset{d_2^*}{\leftarrow} A^{2s} \overset{d_1^*}{\leftarrow} A^{s+1} \overset{d_0^*}{\leftarrow} A \leftarrow 0$$

where the homomorphisms $d_0^*, d_1^*, d_2^*, d_3^*, d_4^*$ are induced by d_0, d_1, d_2, d_3, d_4 respectively and are given by

$d_0^*(a) = (a(h_1 - 1), ..., a(h_{s+1} - 1)),$

$$d_1^*(a_1, ..., a_{s+1}) = (a_1(h_1 + 1), ..., a_s(h_s + 1), a_1 h_2 \left[(h_1 h_2)^{n_1 - 1} + ... + 1\right] + a_2 \left[(h_1 h_2)^{n_1 - 1} + ... + 1\right], ...,$$
$$a_{s-1} h_s \left[(h_{s-1} h_s)^{n_{s-1} - 1} + ... + 1\right] + a_s \left[(h_{s-1} h_s)^{n_{s-1} - 1} + ... + 1\right], a_1 h_{s+1}^{-1} + a_s h_s^{-1} +$$
$$a_{s+1}(h_1 h_{s+1}^{-1} - h_{s+1}^{-1})),$$

$$d_2^*(a_1, ..., a_{2s}) = (a_1(h_1 - 1), ..., a_s(h_s - 1), -(a_1 + a_2)\left[(h_1 h_2)^{n_1 - 1} + ... + 1\right] + a_{s+1}(h_2 + 1), ..., -$$
$$(a_{s-1} + a_s)\left[(h_{s-1} h_s)^{n_{s-1} - 1} + ... + 1\right] + a_{2s-1}(h_s + 1), -a_1 h_1^{-1} h_{s+1}^{-1} - a_s h_s^{-1} + a_{2s}(h_s + 1),$$
$$a_{s+1}(h_1 h_2 - 1), ..., a_{2s-1}(h_{s-1} h_s - 1)),$$

$$d_3^*(a_1, ..., a_{3s-1}) = (a_1(h_1 + 1), ..., a_s(h_s + 1), (a_1 + a_2)\left[(h_1 h_2)^{n_1 - 1} + ... + 1\right] + a_{s+1}(h_2 - 1), ...,$$

$$(a_{s-1}+a_s)\left[(h_{s-1}h_s)^{n_{s-1}-1}+\ldots+1\right]+a_{2s-1}(h_s-1), a_1h_1^{-1}h_{s+1}^{-1}+a_sh_s^{-1}+a_{2s}\,(h_s-1),$$
$$a_{2s+1}\left[(h_1h_2)^{n_1-1}+\ldots+1\right],\ldots,a_{s+1}(h_1h_2-1)-a_{2s+1}\left[(h_1h_2)^{n_1-2}+\ldots+1\right](h_2+1),$$
$$a_{2s-1}(h_{s-1}h_s-1)-a_{3s-1}\left[(h_{s-1}h_s)^{n_{s-1}-2}+\ldots+1\right](h_s+1)),$$

$$d_4^*(a_1,\ldots,a_{4s-2})=(a_1(h_1-1),\ldots,a_s(h_s-1),-(a_1+a_2)\left[(h_1h_2)^{n_1-1}+\ldots+1\right]+a_{s+1}(h_2+1),\ldots,-$$
$$(a_{s-1}+a_s)\left[(h_{s-1}h_s)^{n_{s-1}-1}+\ldots+1\right]+a_{2s-1}(h_s+1),-a_1h_{s+1}^{-1}-a_sh_s^{-1}+a_{2s}\,(h_s+1),$$
$$a_{2s+1}(h_1h_2-1),\ldots,a_{3s}\left[(h_1h_2)^{n_1-1}+\ldots+1\right]+a_{2s+1}\left[(h_1h_2)^{n_1-2}+\ldots+1\right](h_2+1),\ldots,$$
$$a_{4s-2}\left[(h_{s-1}h_s)^{n_{s-1}-1}+\ldots+1\right]+a_{3s-1}\left[(h_{s-1}h_s)^{n_{s-1}-2}+\ldots+1\right](h_s+1),a_{s+1}(h_1h_2-1)-$$
$$a_{3s}[(h_1h_2)^{n_1-2}+\ldots+1](h_2-1),\ldots,a_{2s-1}(h_{s-1}h_s-1)-a_{4s-1}\left[(h_{s-1}h_s)^{n_{s-1}-1}+\ldots+1\right](h_s-1)),$$

Therefore, the integral cohomology groups of E are

$H^0(E,\mathbf{Z})\cong\mathbf{Z}.$

$H^1(E,\mathbf{Z})\cong\mathbf{Z}\,.$

$H^2(E,\mathbf{Z})\cong 0$

$H^3(E,\mathbf{Z})$

$$\cong\frac{\{(a_1,\ldots,a_{3s-1})\left|\begin{array}{l}2a_1=0\ldots=2a_s,(a_1+a_2)n_1=0,\ldots,a_1+a_s=0,n_1a_{2s+1}=0=n_{s-1}a_{3s-1},\\-2(n_1-1)a_{2s+1}-2(n_{s-1}-1)a_{3s-1}=0;a_i\in\mathbf{Z}\}\end{array}\right.}{\{(0,\ldots,0,-(a_1+a_2)n_1+2a_{s+1},\ldots,-(a_{s-1}+a_s)n_{s-1}+2a_{2s-1}-a_1-a_s+2a_{2s},0,\ldots,0\}}$$

$$\cong\frac{\{(0,\ldots,0,a_{s+1},a_{s+2},\ldots,a_{2s},0,\ldots,0)|a_i\in\mathbf{Z}\}}{\{(0,\ldots,0,2a_{s+1}-(a_1+a_2)n_1,\ldots,2a_{2s-1}-(a_{s-1}+a_s)n_{s-1},-a_1-a_s+2a_{2s},0,\ldots,0\}}$$

$$\cong\frac{\{(a_{s+1},a_{s+2},\ldots,a_{2s})|a_i\in\mathbf{Z}\}}{\{(2a_{s+1}-(a_1+a_2)n_1,\ldots,2a_{2s-1}-(a_{s-1}+a_s)n_{s-1},-a_1-a_s+2a_{2s}\}},$$

if we write $x_i=(0,\ldots,0,1,0,\ldots,0)$, where 1 is in the i-th position, then

$$\cong\frac{\langle x_{s+1}\rangle\oplus\langle x_{s+2}\rangle\oplus\ldots\oplus\langle x_{2s}\rangle}{\langle n_1x_{s+1}+x_{2s},,n_1x_{s+1}+n_2x_{s+2},\ldots,n_{s-2}x_{2s-2}+n_{s-1}x_{2s-1}+x_{2s},2x_{s+2},\ldots,2x_{2s}\rangle}$$

$$H^3(E,\mathbf{Z})\cong\langle x_{s+1},\ldots,x_{2s}|n_1x_{s+1}+x_{2s}=0,n_1x_{s+1}+n_2x_{s+2}=0,\ldots,(n_s-1)x_{2s-1}=0,2x_{s+1}=0,\ldots,2x_{2s}=0\rangle$$

$H^3(E,\mathbf{Z})\cong\underbrace{\mathbf{Z}_2\oplus\ldots\oplus\mathbf{Z}_2}_{k-1}$, if $k>0$,

$\cong\mathbf{Z}_2$, if k=0,

where k is the number of even n_i's.

$H^4(E,\mathbf{Z})\cong 0.$

(ii) The Group *K*

Here the group *K* is given by $K = \dfrac{F}{R}$, where *F* is the free group generated by x_1, x_2 (say), and *R* is the normal subgroup generated by r_1, r_2, r_3 where

$$r_1 = x_1^2\ , r_2 = x_2^{2r},\ \ r_3 = x_1^2 x_2^{-2}$$

Let $\pi : \mathbf{Z}F \to \mathbf{Z}K$ be the homomorphism induced by the canonical homomorphism of *F* onto *K* with kennel *R*. Let $\pi(x_1) = h_1,\ \ \pi(x_2) = h_2$.

Theorem 2.2

The following is a free $\mathbf{Z}$ *K*-resolution of $\mathbf{Z}$:

$$...Y_1 \xrightarrow{d_3} Y_1 \xrightarrow{d_2} Y_1 \xrightarrow{d_3} Y_1 \xrightarrow{d_2} Y_1 \xrightarrow{d_1} Y_0 \xrightarrow{d_0} \mathbf{Z}K \xrightarrow{\varepsilon} \mathbf{Z} \to 0$$

where Y_0 is a right $\mathbf{Z}$ *K*-module free on α_1, α_2

Y_1 ,, ,, ,, ,, ,, $\beta_1, \beta_2, \beta_3$

and $\varepsilon,\ d_0,\ d_1,\ d_2,\ d_3$ are the $\mathbf{Z}$ *K*-homomorphisms and given by

$\varepsilon(g) = 1 \in \mathbf{Z},$ for all $g \in K,$

$d_0(\alpha_i) = h_i - 1,\quad i = 1,2$.

$d_1(\beta_i) = \alpha_i(h_i^{2r-1} + ... + 1),\ \ i = 1,2.$

$d_1(\beta_3) = \alpha_1(h_1 + 1)h_2^{-2} - \alpha_2 h_2^{-2}(h_2 + 1),$

$d_2(\beta_i) = \beta_i(h_i - 1),\ i = 1,2.$

$d_2(\beta_3) = -\beta_1 + \beta_2 + \beta_3(h_1^{2(r-1)} + ... + h_1^2 + 1),$

$d_3(\beta_i) = \beta_i(h_i^{2r-1} + ... + h_i + 1),\ \ i = 1,2.$

$d_3(\beta_3) = \beta_1(h_1 + 1) - \beta_2(h_2 + 1) + \beta_3(h_1^2 - 1),$

Let *A* be a *ZK*-module. The homology groups *$H_n(K,A)$* are given by the homology of the complex:

$$... \xrightarrow{\bar{d}_3} A^3 \xrightarrow{\bar{d}_2} A^3 \xrightarrow{\bar{d}_3} A^3 \xrightarrow{\bar{d}_2} A^3 \xrightarrow{\bar{d}_1} A^2 \xrightarrow{\bar{d}_0} A \to 0$$

where A^n stands for the direct sum of *n* isomorphic copies of *A* and the homomorphisms $\bar{d}_0, \bar{d}_1, \bar{d}_2, \bar{d}_3, \bar{d}_4$ are induced by d_0, d_1, d_2, d_3, d_4 respectively and are given by

$\bar{d}_0(a_1, a_2) = (h_1 - 1)a_1 + (h_2 - 1)a_2,$

$\bar{d}_1(a_1, a_2, a_3) = ((h_1^{2r-1} + ... + 1)a_1 + (h_1 + 1)h_2^{-2}a_3, (h_2^{2r-1} + ... + 1)a_2 - h_2^{-2}\ (h_2 + 1)a_3),$

$$\bar{d}_2(a_1,a_2,a_3)=((h_1-1)a_1-a_3,(h_1+1)h_2^{-2}a_3,(h_2^{2r-1}+...+1)a_2-h_2^{-2}\ (h_2+1)a_3),$$

$$\bar{d}_3(a_1,a_2,a_3)=((h_1^{2r-1}+...+1)a_1+(h_1+1)a_2+a_3,(h_2^{2r-1}+...+1)a_2-(h_2+1)a_3),(h_1^2-1)a_3),$$

for some $a_1,a_2,a_3\in A$.

The integral homology and cohomology groups are

$H_0(K,\mathbf{Z})\cong\mathbf{Z}.$

$H_1(K,\mathbf{Z})\cong\mathbf{Z}_2\oplus\mathbf{Z}_{2r}.$

$H_2(K,\mathbf{Z})\cong 0.$

$H_3(K,\mathbf{Z})\cong\mathbf{Z}_2\oplus\mathbf{Z}_{2r}.$

$H_4(K,\mathbf{Z})\cong 0.$

Hence $H_{2n-1}(K,\mathbf{Z})\cong\mathbf{Z}_2\oplus\mathbf{Z}_{2r}$ and

$H_{2n}(K,\mathbf{Z})\cong 0$, for $n\geq 1$.

$H^0(K,\mathbf{Z})\cong 0.$

$$H^1(K,\mathbf{Z})\cong\frac{\{(a_1,a_2)2ra_1=0,2ra_2=0,a_1=a_2\}}{\{(0,0)\}}$$

$$\cong 0.$$

$$H^2(K,\mathbf{Z})\cong\frac{\{(a_1,a_2,a_3)|ra_3-a_1+a_2=0,a_i\in\mathbf{Z}\}}{\{(2ra_1,2ra_2,2a_1-2a_2),a_i\in\mathbf{Z}\}}.$$

$$\cong\frac{\{(a_2+ra_3,a_2,a_3)|a_i\in\mathbf{Z}\}}{\{(2ra_1,2ra_2,2a_1-2a_2)|a_i\in\mathbf{Z}\}}.$$

$$\cong\frac{\langle(r,0,1)\rangle\oplus\langle(1,1,0)\rangle}{\langle 2(r,0,1)+2(o,r,-1)\rangle}$$

$$\cong\frac{\langle x\rangle\oplus\langle y\rangle}{\langle 2y,2(rx-y)\rangle},$$

writing $x = (1,1,0)$ and $y = (r, 0,1)$.

$$\cong\mathbf{Z}_2\oplus\mathbf{Z}_{2r}$$

$$H^3(K,\mathbf{Z})\cong\frac{\{(a_1,a_2,a_3)|2ra_1=0=2ra_2,a_1=a_2;a_i\in\mathbf{Z}\}}{\{(0,0,ra_3-a_1+a_2);a_i\in\mathbf{Z}\}}$$

$$\cong\frac{\{(0,0,a_3);a_3\in\mathbf{Z}\}}{\{(0,0,ra_3-a_1+a_2);a_i\in\mathbf{Z}\}}$$

$$\cong \frac{\langle (0,0,1) \rangle}{\langle (0,0,r),(0,0,-1),(0,0,1) \rangle}$$

$$\cong \langle x | rx = 0, -x = 0, x = 0 \rangle$$

$$\cong 0,$$

$$H^4(K,\mathbf{Z}) \cong \frac{\{(a_1,a_2,a_3) | ra_3 - a_1 + a_2 = 0, a_i \in \mathbf{Z}\}}{\{(2ra_1, 2ra_2, 2a_1 - 2a_2)\}}.$$

$$\cong H^2(K,\mathbf{Z}) \cong \mathbf{Z},$$

Hence $H^{2n+1}(K,\mathbf{Z}) \cong \{0\}, \quad n \geq 0,$

$$H^{2n}(K,\mathbf{Z}) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_{2r}, \quad n \geq 1.$$

(iii) The Group *G* of Hyperbolic Isometries with Infinite Generators and Relations

The group *G* is given by $G = \dfrac{F}{R}$, where *F* is the free group generated by, say, $x_1, x_2, x_3, \ldots.$ and *R* is the normal subgroup of *F* generated by $r_1, r_2, r_3, \ldots.$ where

$$r_{2k-1} = x_k^2,\ r_{2k} = (x_k x_{k+1})^{n_k}, \quad k = 1,2,3,\ldots$$

Although the number of generators and relation of *G* are infinite, our method of construction of the free resolution as described in [18] is still valid for *G*, since (i) each generator of the group occurs only in the finite number of relations, (ii) each arbitrary group-ring element corresponding to a free generator of the solution module occurs as a co-efficient in the value of a finite number of unknowns at each stage of solving the set of the relevant linear equations. Here we have obtained a 6-term partial resolution which yields the corresponding homology immediately. However the cohomology can't be determined from this resolution since the free module of the resolution are infinitely generated, and in general,

$Hom(\sum_{i \in I} A_i, B)$ is not isomorphic to $\sum_{i \in I} Hom(A_i, B)$, when I is infinite.

Let $\pi : \mathbf{Z}F \to \mathbf{Z}G$ be the homomorphism induced by the canonical homomorphism of *F* onto *G* with *R* as the kernel, let $\pi(x_i) = h_i$.

Theorem 2.3

The following is a free 6-term partial $\mathbf{Z}$ *G*-resolution of $\mathbf{Z}$:

$$Y_4 \xrightarrow{d_4} Y_3 \xrightarrow{d_3} Y_2 \xrightarrow{d_2} Y_1 \xrightarrow{d_1} Y_0 \xrightarrow{d_0} \mathbf{Z}G \xrightarrow{\varepsilon} \mathbf{Z} \to 0$$

where Y_0 is a right $\mathbf{Z}$ *G*-module free on $\alpha_1, \alpha_2, \alpha_3, \ldots$

Y_1 ,, ,, ,, ,, ,, $\beta_1, \beta_2, \beta_3, \ldots$

Y_2 ,, ,, ,, ,, ,, $\delta_1, \delta_2, \delta_3, ...$

Y_3 ,, ,, ,, ,, ,, $\lambda_1, \lambda_2, \lambda_3, ...$

Y_4 ,, ,, ,, ,, ,, $\mu_1, \mu_2, \mu_3, ...$

and ε, d_0, d_1, d_2, d_3, d_4 are the **Z** G-homomorphisms and given by

$\varepsilon(g) = 0$, for all $g \in G$,

$d_0(\alpha_i) = h_i - 1, \quad i = 1,2,3, ...$.

$d_1(\beta_{2i-1}) = \alpha_i(h_i + 1)$ and

$d_1(\beta_{2i}) = \alpha_i h_{i+1}[(h_i h_{i+1})^{n_1 - 1} + ... + 1] + \alpha_{i+1}[(h_i h_{i+1})^{n_i - 1} + ... + 1], \quad i = 1,2,3, ...$.

$d_2(\delta_{4i-3}) = \beta_{2i-1}(h_i - 1)$,

$d_2(\delta_{2i}) = \beta_{2i}(h_i h_{i+1} - 1)$ and

$d_2(\delta_{4i-1}) = -(\beta_{2i-1} + \beta_{2i+1})[(h_i h_{i+1})^{n_i - 1} + ... + 1] + \beta_{2i}(h_{i+1} + 1)$, for $i = 1,2,3, ...$.

$d_3(\lambda_{8i-7}) = \delta_{4i-3}(h_i + 1)$,

$d_3(\lambda_{2i}) = \delta_{4i-1}(h_i h_{i+1} - 1) - \delta_{2i}[(h_i h_{i+1})^{n_i - 2} + ... + 1](h_{i+1} + 1)$,

$d_3(\lambda_{4i-1}) = \delta_{2i}[(h_i h_{i+1})^{n_i - 1} + ... + 1]$ and

$d_3(\lambda_{8i-3}) = (\delta_{4i-3} + \delta_{4i+1})[(h_i h_{i+1})^{n_i - 1} + ... + 1] + \delta_{4i-1}(h_{i+1} - 1)$, for $i = 1,2,3, ...$.

$d_4(\mu_{16i-15}) = \lambda_{8i-7}(h_i - 1)$,

$d_4(\mu_{2i}) = \lambda_{8i-3}(h_i h_{i+1} - 1) - \lambda_{2i}[(h_i h_{i+1})^{n_i - 2} + ... + 1](h_{i+1} + 1)$,

$d_4(\mu_{4i-1}) = \lambda_{2i}[(h_i h_{i+1})^{n_i - 1} + \ldots + 1] + \lambda_{2i-1}[(h_i h_{i+1})^{n_i - 2} + ... + 1](h_{i+1} + 1)$,

$d_4(\mu_{8i-3}) = \lambda_{4i-1}(h_i h_{i+1} - 1)$ and

$d_4(\mu_{16i-7}) = \lambda_{8i-3}(h_{i+1} + 1) - (\lambda_{8i-7} + \lambda_{8i-1})[(h_i h_i)^{n_i - 1} + ... + 1]$, for $i = 1, 2, 3, \ldots$.

Homology Groups of *G*

Let A be a left **Z** G-module. The homology groups $H_n(G, A)$ are given by the homology of the complex:

$$A^\infty \xrightarrow{\bar{d}_4} A^\infty \xrightarrow{\bar{d}_3} A^\infty \xrightarrow{\bar{d}_2} A^\infty \xrightarrow{\bar{d}_1} A^\infty \xrightarrow{\bar{d}_0} A^\infty \rightarrow 0$$

where A^∞ stands for the direct sum of countably infinite copies of A and $\bar{d}_i$ are induced by d_i , for $i = 0, 1, 2, 3, 4$ and are given by

$$\bar{d}_0(a_1, a_2, a_3, ...) = (h_1 - 1)a_1 + (h_2 - 1)a_2 + (h_3 - 1)a_3 + ...,$$

$$\bar{d}_1(a_1,a_2,a_3,...)=((h_1+1)a_1+h_2\left[(h_1h_2)^{n_1-1}+...+1\right]a_2,\left[(h_1h_2)^{n_1-1}+...+1\right]a_2$$
$$+(h_2+1)a_3+h_3\left[(h_2h_3)^{n_2-1}+...+1\right]a_4,\left[(h_2h_3)^{n_2-1}+...+1\right]a_4+$$
$$(h_3+1)a_5+h_4\left[(h_3h_4)^{n_3-1}+...+1\right]a_6,\ldots),$$
$$\bar{d}_2(a_1,a_2,a_3,...)=((h_1-1)a_1-\left[(h_1h_2)^{n_1-1}+...+1\right]a_3,(h_2+1)a_3+(h_1h_2-1)a_2,$$
$$-\left[(h_1h_2)^{n_1-1}+...+1\right]a_3-\left[(h_2h_3)^{n_2-1}+...+1\right]a_7+(h_2-1)a_5,$$
$$(h_2h_3-1)a_4+(h_3+1)a_7,(h_3-1)a_9-[(h_2h_3)^{n_2-1}+...+1]a_7-$$
$$[(h_3h_4)^{n_3-1}+...+1]a_{11},...),$$
$$\bar{d}_3(a_1,a_2,a_3,a_4,...)=((h_1+1)a_1+\left[(h_1h_2)^{n_1-1}+...+1\right]a_5,-[(h_1h_2)^{n_1-2}+...+1](h_2+1)a_2+$$
$$\left[(h_1h_2)^{n_1-1}+...+1\right]a_3,(h_1h_2-1)a_2+(h_2-1)a_5,$$
$$\left[(h_2h_3)^{n_2-1}+...+1\right](h_3+1)a_4+\left[(h_2h_3)^{n_2-1}+...+1\right]a_7,$$
$$\left[(h_2h_3)^{n_2-2}+...+1\right]a_{13}+\left[(h_1h_2)^{n_1-1}+...+1\right]a_5+(h_2+1)a_9,-$$
$$\left[(h_3h_4)^{n_3-2}+...+1\right](h_4+1)a_6+\left[(h_3h_4)^{n_3-1}+...+1\right]a_{11},(h_2h_3-1)a_4+(h_3-1)a_{13},...)$$
$$\bar{d}_4(a_1,a_2,a_3,a_4,...)=((h_1-1)a_1-\left[(h_1h_2)^{n_1-1}+...+1\right]a_9,-[(h_1h_2)^{n_1-2}+...+1](h_2-1)a_2$$
$$+\left[(h_1h_2)^{n_1-1}+...+1\right]a_3,\left[(h_1h_2)^{n_1-2}+...+1\right](h_2+1)a_3+(h_1h_2-1)a_5,-$$
$$\left[(h_2h_3)^{n_2-2}+...+1\right](h_3-1)a_4+\left[(h_2h_3)^{n_2-1}+...+1\right]a_7,(h_1h_2-1)a_2+$$
$$(h_2+1)a_9,-\left[(h_3h_4)^{n_3-2}+...+1\right](h_4-1)a_6+\left[(h_3h_4)^{n_3-1}+...+1\right]a_{11},$$
$$\left[(h_2h_3)^{n_2-2}+...+1\right](h_3+1)a_7+(h_2h_3-1)a_{13},-\left[(h_4h_5)^{n_4-2}+...+1\right](h_5-1)a_8+$$
$$\left[(h_4h_5)^{n_4-1}+...+1\right]a_{15},(h_2-1)a_{17}+[(h_1h_2)^{n_1-1}+...+1)a_9-\left[(h_2h_3)^{n_2-1}+...+1\right]a_{25},...),$$

for all $a_i \in A, i=1,\ 2,\ 3,\ 4,\ \ldots$.

The integral homology groups of G are

$$H_0(G,\mathbf{Z})\cong\mathbf{Z},$$

$$H_1(G,\mathbf{Z})\cong\frac{\{(a_1,a_2,a_3,a_4,...)|a_i\in\mathbf{Z}\}}{\{(2a_1+n_1a_2,n_1a_2+2a_3+n_2a_4+2a_5+n_3a_6,...)|a_i\in\mathbf{Z}\}}$$
$$\cong\frac{\{a_1(1,0,0,...)+a_2(0,1,0,...)+a_3(0,0,1,0,...)+a_4(0,0,0,1,0,...),...|a_i\in\mathbf{Z}\}}{\{a_1(2,0,0,...)+a_2(n_1,n_1,0,...)+a_3(0,2,0,...)+a_4(0,n_2,n_2,0,...)+a_5(0,0,2,0,...)\}}$$
$$\cong\frac{\langle x_1\rangle\oplus\langle x_2\rangle\oplus\langle x_3\rangle...}{\langle 2x_1,n_1(x_1+x_2),2x_2,n_2(x_2+x_3),2x_3,...\rangle},$$

writing $x_i = (0,...,0,1,0,0,...)$, where 1 is in the *i*-th position,

$$\cong \left\langle x_1, x_2, x_3, x_4,... \middle| 2x_1 = 0, 2x_2 = 0, 2x_3 = 0, ..., n_1(x_1 + x_2) = 0, n_2(x_2 + x_3) = 0,... \right\rangle$$

Thus, $H_1(G,\mathbf{Z}) \cong \underbrace{\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus ...}_{k}$, k is the number of even n_i.

$H_2(G,\mathbf{Z}) \cong 0$.

$$H_3(G,\mathbf{Z}) \cong \frac{\{(a_1,a_2,a_3,a_4,...) | -n_1a_3 = 0, 2a_3 = -n_1a_3 - n_2a_7 = 2a_7 = 0 = -n_2a_7 - n_3a_{11}; a_i \in \mathbf{Z}\}}{\{(2a_1 + n_1a_5, -2(n_1-1)a_2 + n_1a_3, 0, -2(n_2-1)a_4 + n_2a_7,...) | a_i \in \mathbf{Z}\}}$$

$$\cong \frac{\langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \langle x_3 \rangle \oplus \langle x_4 \rangle \oplus ...}{\langle 2x_1, n_1x_1, 2(n_1-1)x_2, n_1x_2, 2x_2, 2(n_2-1)x_3, n_2x_3,... \rangle}$$

where x_i = (0,…,0,1,0,…,0) with 1 is an the *i*-th position,

$$\cong \left\langle x_1, x_2, x_3, x_4,... \middle| 2x_1 = 0, n_1x_1 = 0, 2(n_1-1)x_2 = 0, n_1x_2 = 0, 2(n_2-1)x_3 = 0, n_2x_3 = 0... \right\rangle.$$

Hence $H_3(G,\mathbf{Z}) \cong Z_2 \oplus \sum_i \oplus \mathbf{Z}_{h_{i+1}}$,or $\sum_i \oplus \mathbf{Z}_{h_{i+1}}$,

where *i* runs over all those integers $i \geq 2$ for which n_i is even, and h_{i+1} is the h.c.f. of $2(n_i - 1)$ and n_i , according as n_i is even or odd.

$H_4(G,\mathbf{Z}) \cong 0.$

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