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NUMERICAL APPROXIMATION OF FREDHOLM INTEGRAL EQUATION (FIE) OF 2ND KIND USING GALERKIN AND COLLOCATION METHODS

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ABSTRACT

In this research work, Galerkin and collocation methods have been introduced for approximating the solution of FIE of 2nd kind using LH (product of Laguerre and Hermite) polynomials which are considered as basis functions. Also, a comparison has been done between the solutions of Galerkin and collocation method with the exact solution. Both of these methods show the outcome in terms of the approximate polynomial which is a linear combination of basis functions. Results reveal that performance of collocation method is better than Galerkin method. Moreover, five different polynomials such as Legendre, Laguerre, Hermite, Chebyshev 1st kind and Bernstein are also considered as a basis functions. And it is found that all these approximate solutions converge to same polynomial solution and then a comparison has been made with the exact solution. In addition, five different set of collocation points are also being considered and then the approximate results are compared with the exact analytical solution. It is observed that collocation method performed well compared to Galerkin method.

Keywords: Galerkin; Collocation; LH polynomials; Fredholm Integral Equation; Bernstein.

1. Introduction

There has been a wide variety of methods to solve different types of Fredholm integral equations. Different versions of Wavelet, Collocation, Galerkin, Decomposition, Quadrature, Determinant and Monte Carlo methods are developed and studied by large number of researchers. Some of these methods find the approximate solution of Fredholm integral equation in terms of polynomials. Different well known polynomials have been used for approximating the solution in those methods. Chakrabarti and Martha [1] were found the solution by means of a polynomial using Quadrature method. This method is also being used by both Panda and Martha *et el.* [2] and Bhattacharya and Mondal [3] and they approximated the solution in terms of linear combination of Lagrange's and Bernstein polynomials respectively. All of them have found excellent agreement in approximate solution with exact solution. Also, Mohamed and Taher [4] compared approximated solution which is obtained from Collocation method with the exact solution. In this research, they have used Chebyshev and Legendre polynomials. They have reported that Chebyshev polynomial is better than Legendre polynomial. Moreover, Domingo [5] discussed several versions of

Collocation techniques for solving FIEs of 2^{nd} kind and found that Chebyshev method gives best result than the other methods in terms of closeness to the exact solution. Again, Bellour *et el*. [6] introduced two collocation methods based on natural cubic spline interpolation and cubic spline quasi-interpolation. Ikebe [7] discussed Galerkin method to approximate the solution of Fredholm Integral Equations of 2^{nd} kind. Discrete Galerkin and iterated discrete Galerkin method for FIE of 2^{nd} kind are presented by Joe [8] along with error analysis and claimed that both discrete Galerkin methods and their exact counterparts have same order of convergence under some given conditions. Hendi and Albugami [9] used both Galerkin and Collocation method for numerical solution of system of Fredholm-Volterra integral equations of 2^{nd} kind where kernel is continuous with respect to position and time. They used monomials in both methods as the basis to approximate the solution. Shirin and Islam [10] presented Galerkin method with Bernstein polynomials as a basis functions in order to get the approximate solution of FIE of 2^{nd} kind. They have found very good approximations even for less number of polynomials. Rabbani and Maleknejad [11] used Alpert multi-wavelet as the basis in Petrov-Galerkin method to approximate the solution of discrete FIE of 2^{nd} kind. In this method FIE is first converted into a system of linear equations. Later, Akhavan and Maleknejad [12] improved the Petrov-Galerkin elements using Chebyshev polynomials which eliminates some restrictions and improves accuracy from the previous method. Recently, Rostami and Maleknejad [13] used Galerkin method with Franklin wavelet as the basis for numerical approximation of two-dimensional FIE. The main privilege of this method is simplicity and exponential decay which are inherited from the properties of Franklin wavelet.

To the extent of our understanding, no researchers yet have considered product of different polynomials to predict the numerical approximation of FIE of 2nd kind. Therefore, in this research work, product of Hermite and Laguerre (LH) polynomials are considered as basis functions for Galerkin method. Then the outcomes are compared with the solutions obtained from the collocation method and exact solution. Moreover, the research is also carried out to see the effect of different polynomials for the solution of FIE of 2^{nd} kind using Galerkin method. In this investigation, different polynomials such as Bernstein, Chebyshev first kind, Legendre, Hermite and Laguerre polynomials are used. In addition, five different set of collocation points are considered for above mentioned polynomials and then Collocation method is used in order to see the variation between the results.

2. Introduction of Polynomials

Among the classical orthogonal polynomials such as Hermite, Laguerre, Chebyshev and Legendre polynomials are most widely used in approximation theory and numerical analysis. These polynomials are used frequently to approximate solution of various kinds of differential and integral equations. In this section, a very short overview of Legendre, Chebyshev, Bernstein, Hermite and Laguerre polynomials is presented. After that, a new set of polynomials as a product of Laguerre and Hermite polynomials is introduced.

2.1 Legendre Polynomials

The Legendre polynomials $P_n(u)$ are set of orthogonal polynomials over the domain $[-1,1]$ and are solutions of the Legendre differential equations. Explicit formula for $P_n(u)$is

$$P_n(u) = \sum_{r=0}^{n} \binom{n}{r} \binom{-n-1}{r} \left(\frac{1-u}{2}\right)^r, \quad n = 0,1,2, \dots. \tag{1}$$

And, the recurrence relation for Legendre polynomials are follows:

$$P_0(u) = 1, \qquad P_1(u) = u$$
$$(n+1)P_{n+1}(u) = (2n+1)uP_n(u) - nP_{n-1}(u), \qquad n = 1,2,3, \dots$$

2.2 Chebyshev Polynomials

Chebyshev polynomials of first kind $T_n(u)$are set of orthogonal polynomials over the domain $[-1,1]$ and are solutions of the Chebyshev differential equations. Explicit formula for $T_n(u)$ is

$$T_n(u) = u^n \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2r} (1-u^{-2})^r, \quad n = 0,1,2, \dots. \tag{2}$$

And, the recurrence relation for Chebyshev polynomials of first kind are follows:

$$T_0(u) = 1, \qquad T_1(u) = u$$

$$T_{n+1}(u) = 2uT_n(u) - T_{n-1}(u), \qquad n = 1,2,3, \dots$$

2.3 Bernstein Polynomials

The n^{th} degree Bernstein polynomials defined on the interval $[a, b]$ are

$$B_{r,n}(u) = \binom{n}{r} \frac{(u-a)^r (b-u)^{n-r}}{(b-a)^n}; \quad a \le u \le b; \quad r = 0,1,2, \dots, n \tag{3}$$

where

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

There are $(n+1)$ Bernstein polynomials of i^{th} degree with following properties:

i) $B_{r,n}(u) = 0, \; if \;\; r < 0 \; or \; r > n$

ii) $B_{r,n}(a) = B_{r,n}(b) = 0, \; 1 \le r \le n-1$

2.4 Hermite Polynomials

The Hermite polynomials $H_n(u)$ are set of orthogonal polynomials over the domain $(-\infty, \infty)$ with weighting function e^{-u^2}[14]. The explicit formula for Hermite polynomials are

$$H_n(u) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r \frac{n!}{r!\,(n-2r)!} (2u)^{n-2r} \tag{4}$$

And, the recurrence relations for Hermite polynomials are as follows:

$$H_0(u) = 1,\ H_1(u) = 2u$$

$$H_{n+1}(u) = 2uH_n(u) - 2nH_{n-1}(u); \quad n = 1,2,3, \ldots.$$

2.5 Laguerre Polynomials

The Laguerre polynomials $L_n(u)$ are set of orthogonal polynomials over the domain $(0, \infty)$ with weighting function e^{-u}. The explicit formula for Laguerre polynomials are

$$L_n(u) = \sum_{r=0}^{n} (-1)^r \frac{n!}{r!^2\,(n-r)!} u^r \tag{5}$$

and, the recurrence relations for Laguerre polynomials are as follows:

$$L_0(u) = 1,\ L_1(u) = 1 - u$$

$$(n+1)L_{n+1}(u) = (2n+1-u)L_n(u) - nL_{n-1}(u); \quad n = 1,2,3, \ldots.$$

2.6 Product of Laguerre and Hermite Polynomials

Here we will consider the following product of Laguerre and Hermite polynomials:

$$LH_n(u) = L_n(u)H_n(u)$$

And degree of the polynomial $LH_n(u)$ is $2n$. Also, first five LH polynomials are presented underneath:

$$LH_0(u) = 1$$

$$LH_1(u) = 2u - 2u^2$$

$$LH_2(u) = -2 + 4u + 3u^2 - 8u^3 + 2u^4$$

$$LH_3(u) = -12u + 36u^2 - 10u^3 - 22u^4 + 12u^5 - \frac{4u^6}{3}$$

$$LH_4(u) = 12 - 48u - 12u^2 + 184u^3 - \frac{255u^4}{2} - 32u^5 + 46u^6 - \frac{32u^7}{3} + \frac{2u^8}{3}.$$

3. Matrix Formulation of FIE

Integral equations have enormous applications in mathematical physics. Many IVP and BVP can be transformed into different integral equations. Fredholm integral equations arise in various physical problems [15] like elasticity, fluid mechanics, electromagnetic theory, signal processing and so on.

Galerkin method is one of the popular method in approximation theory which is used to find the approximate solution of ODE, Integral equations, and so on. Details of the method are given in Lewis and Ward [16]. In both Galerkin and Collocation methods, solution is approximated by a finite sum of a set of known functions called as basis functions. To approximate solution, once choice of a set of basis functions are made, next step is the determination of unknown parameters in trial solution also known as expansion coefficients. To determine expansion coefficients Galerkin and Collocation methods use two different approaches. Now details of matrix formulation from linear FIE to determine expansion coefficients by both Galerkin and Collocation method are presented in this section.

General form of linear FIE of 2nd kind is presented below:

$$\alpha(x)\phi(x) + \lambda \int_a^b k(t,x)\phi(t)dt = f(x)\,, \quad a \le x \le b \tag{6}$$

where $\alpha(x)$, $f(x)$and $k(t,x)$ are known functions, λ is a known parameter, and $\phi(x)$ is the unknown solution of Eq. (6), needed to be resolved.

In order to approximate the solution of Eq. (6), let the trial solution be $\tilde{\phi}(x)$, where

$$\tilde{\phi}(x) = \sum_{i=0}^{n} c_i P_i(x) \tag{7}$$

Here $P_i(x)$ are some known polynomials called as basis functions and $c_i$are the unknown parameters also known as expansion coefficients, to be determined.

Substituting the trial solution into Eq. (6), we have

$$\alpha(x)\sum_{i=0}^{n} c_i P_i(x) + \lambda \int_a^b k(t,x)\sum_{i=0}^{n} c_i P_i(t)\,dt = f(x)$$

$$\Rightarrow \sum_{i=0}^{n} c_i \left[\alpha(x)P_i(x) + \lambda \int_a^b k(t,x)P_i(t)dt\right] = f(x) \tag{8}$$

In order to obtain the Galerkin equation to determine expansion coefficients, we first multiply Eq. (8) by $P_j(x)$ and then integrate with respect to x from a to b. Thus Eq. (8) reduces to

$$\sum_{i=0}^{n} c_i \left[\int_a^b \left[\alpha(x)P_i(x) + \lambda \int_a^b k(t,x)P_i(t)dt\right] P_j(x)dx\right] = \int_a^b f(x)P_j(x)\mathrm{dx}\,, \quad j = 0,1,..,n$$

This reduces to the following system of $(n+1)$ linear equations in $(n+1)$ unknowns c_i

$$\sum_{i=0}^{n} c_i M_{i,j} = F_j\,, \quad j = 0,1,2,\ldots,n \tag{9}$$

where

$$M_{i,j} = \int_a^b \left[\alpha(x)P_i(x) + \lambda \int_a^b k(t,x)P_i(t)dt\right] P_j(x)dx\,; \qquad i,j = 0,1,2,\ldots,n$$

$$F_j = \int_a^b f(x)P_j(x)\mathrm{dx} \ , \quad j = 0,1,2,\dots,n$$

The unknown parameters of trial solution in Eq. (7) can now be determined easily by solving the system of linear equations in Eq. (9). In order to determine the unknown parameters, c_i , let us considered the basis functions as the LH polynomials. Once we have the values of c_i , we will substitute these values in Eq. (7) and will get the approximate polynomial solution of FIE.

In Collocation method, for each unknown parameter c_j in the trial solution, we chose a point x_j in the domain. These points x_j are called collocation points; they may be located anywhere in the domain and on the boundary, not necessarily in any particular pattern. Then Collocation method force the trial solution to be exact at each collocation points. Hence, at each x_j, we then force to satisfy Eq. (8), that is

$$\sum_{i=0}^{n} c_i \left[\alpha(x_j)P_i(x_j) + \lambda \int_a^b k(t,x_j)P_i(t)dt\right] = f(x_j) \ ; \quad j = 0,1,2,\dots,n \tag{10}$$

A trial solution with $(n+1)$ unknown parameters will produce the following system of $(n+1)$ linear equations in $(n+1)$ unknowns c_i from Eq. (10).

$$\sum_{i=0}^{n} c_i G_{i,j} = H_j \ ; \quad j = 0,1,2,\dots,n \tag{11}$$

where,

$$G_{i,j} = \alpha(x_j)P_i(x_j) + \lambda \int_a^b k(t,x_j)\, P_i(t)\, dt \ ; \quad i,j = 0,1,2,\dots,n$$

$$H_j = f(x_j) \ , \quad j = 0,1,2,\dots,n$$

The unknown parameters of trial solution in Eq. (7) can now be determined easily by solving the system of linear equations in Eq. (11). After that substitution of the parameters into trial solution will give us the approximate solution of Eq. (6).

4. Results and Discussion

In this research, Galerkin and Collocation methods have been applied in order to find the numerical solution of linear FIE of 2nd kind, and investigation is carried out to see the performance of Galerkin and Collocation methods using different types of polynomials and different sets of collocation points. None of these methods need exact solution to approximate the solution of FIE. Exact solution is used only to validate our code and to investigate the performance in terms of absolute errors. In order to generate numerical results and graphs we have used in-house code written in Wolfram Mathematica 9 using 64 bit Windows operating system. Moreover, investigation of numerically approximate solution of linear FIE of 2nd kind is divided into following three sections:

Firstly, we will investigate the errors in approximations using product of Hermite and Laguerre polynomials $LH_n(x)$by both Galerkin and Collocation methods for different values of n where n is the degree of a polynomial. Secondly, we will compare approximate solutions obtained by Galerkin method using five different well known polynomials namely Bernstein, Legendre, Chebyshev first kind, Hermite and Laguerre with $n = 5$. Finally, we will consider five different sets of collocation points each for one of the five polynomials such as Bernstein, Chebyshev first kind, Legendre, Laguerre and Hermite polynomials.

4.1 Numerical Example 1:

Consider the linear FIE of 2nd kind given by [17]

$$\phi(x) + \frac{1}{3}\int_0^1 e^{2x-\frac{5}{3}t}\phi(t)dt = e^{2x+\frac{1}{3}}, \quad 0 \le x \le 1$$

with exact solution $\phi(x) = e^{2x}$.

Now, in first part of our numerical investigation, absolute errors in Galerkin and Collocation methods are presented in the Table 1 for LH polynomials, $LH_n(x)$with $n = 1, 2, 3, 4, 5$ respectively.

Table 1: Numerical Results for LH Polynomial

x	Exact	Galerkin sol	Collocation sol	Galerkin err	Collocation
0.	1.000000000	3.813395029	$0. \times 10^{-10}$	2.813395029	1.
0.1	1.221402758	3.332758525	4.856526518	2.11136	3.63512
0.2	1.491824698	2.958930133	8.6338249216	1.46711	7.142
0.3	1.8221188004	2.691909853	11.3318952096	0.869791	9.50978
0.4	2.225540928	2.531697685	12.9507373824	0.306157	10.7252
0.5	2.718281828	2.478293629	13.490351440	0.239988	10.7721
0.6	3.320116923	2.531697685	12.9507373824	0.788419	9.63062
0.7	4.055199967	2.691909853	11.3318952096	1.36329	7.2767
0.8	4.953032424	2.958930133	8.6338249216	1.9941	3.68079
0.9	6.049647464	3.332758525	4.856526518	2.71689	1.19312
1.	7.389056099	3.813395029	$0. \times 10^{-10}$	3.57566	7.38906

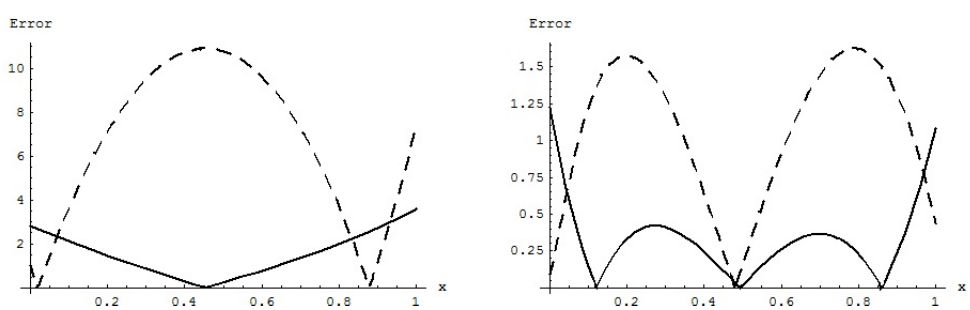
x	Exact	Galerkin sol	Collocation sol	Galerkin err	Collocation
0.	1.000000000	2.221173044	1.058711633	1.2211730	0.0587116
0.1	1.221402758	1.374363011	0.003629344	0.15296	1.21777
0.2	1.491824698	1.159832991	-0.0822512520	0.331992	1.57408
0.3	1.8221188004	1.410772916	0.525091789	0.411346	1.29703
0.4	2.225540928	1.979997433	1.582148418	0.245543	0.643393
0.5	2.718281828	2.739945901	2.8778765934	0.0216641	0.159595
0.6	3.320116923	3.5826823952	4.233702280	0.262565	0.913585
0.7	4.055199967	4.419895703	5.503519448	0.364696	1.44832
0.8	4.953032424	5.1828993255	6.5736900752	0.229867	1.62066
0.9	6.049647464	5.8226314786	7.363044145	0.227016	1.3134
1.	7.389056099	6.3096550915	7.8228796484	1.0794	0.433824

x	Exact	Galerkin sol	Collocation sol	Galerkin err	Collocation
0.	1.000000000	1.326095505	1.010321191	0.326096	0.0103212
0.1	1.221402758	1.148281150	0.950211412	0.0731216	0.271191
0.2	1.491824698	1.383855426	1.288675643	0.107969	0.203149
0.3	1.8221188004	1.805591212	1.7893177519	0.0165276	0.032801
0.4	2.225540928	2.2916357836	2.326642422	0.0660949	0.101101
0.5	2.718281828	2.802165457	2.8613894646	0.0838836	0.143108
0.6	3.320116923	3.357907868	3.417841393	0.0377909	0.0977245
0.7	4.055199967	4.020531877	4.063104243	0.0346681	0.00790428
0.8	4.953032424	4.874905104	4.888361651	0.0781273	0.0646708
0.9	6.049647464	6.013219082	5.992102191	0.0364284	0.0575453
1.	7.389056099	7.520982055	7.465319961	0.131926	0.0762639

x	Exact	Galerkin sol	Collocation sol	Galerkin err	Collocation
0.	1.000000000	1.044458789	1.001063421	0.0445	0.00106342
0.1	1.221402758	1.203846736	1.176618754	0.017556	0.044784
0.2	1.491824698	1.4884460677	1.477167751	0.00337863	0.0146569
0.3	1.8221188004	1.834716208	1.834948020	0.0125974	0.0128292
0.4	2.225540928	2.235601375	2.241500071	0.0100604	0.0159591
0.5	2.718281828	2.7146375087	2.721172506	0.00364432	0.00289068
0.6	3.320116923	3.307181371	3.311758619	0.0129356	0.0083583
0.7	4.055199967	4.047666667	4.051177669	0.0075333	0.0040223
0.8	4.953032424	4.9618583739	4.969179969	0.00882595	0.0161475
0.9	6.049647464	6.063142874	6.083120763	0.0134954	0.0334733
1.	7.389056099	7.3519578301	7.396913775	0.0370983	0.00785768

x	Exact	Galerkin sol	Collocation sol	Galerkin err	Collocation err
0.	1.000000000	1.002183614	1.000064756	$0. \times 10^{-3}$	0.000064756
0.1	1.221402758	1.2208041060	1.2189225218	0.000598652	0.00248024
0.2	1.491824698	1.492411202	1.491921302	0.000586504	0.0000966047
0.3	1.8221188004	1.822278107	1.822747682	0.000159307	0.000628882
0.4	2.225540928	2.225078427	2.225685046	0.000462502	0.000144117
0.5	2.718281828	2.718065843	2.718297578	0.000215986	0.0000157491
0.6	3.320116923	3.320439240	3.320331920	0.000322317	0.000214998
0.7	4.055199967	4.055452720	4.055450993	0.000252753	0.000251026
0.8	4.953032424	4.9527900799	4.953353163	0.000242344	0.000320739
0.9	6.049647464	6.049556877	6.050645134	0.0000905871	0.00099767
1.	7.389056099	7.388956443	7.389534585	0.0000996555	0.000478486

The absolute error graphs by both methods for $n = 1, 2, 3, 4, 5$ are given in Fig. 1 arranged row wise.



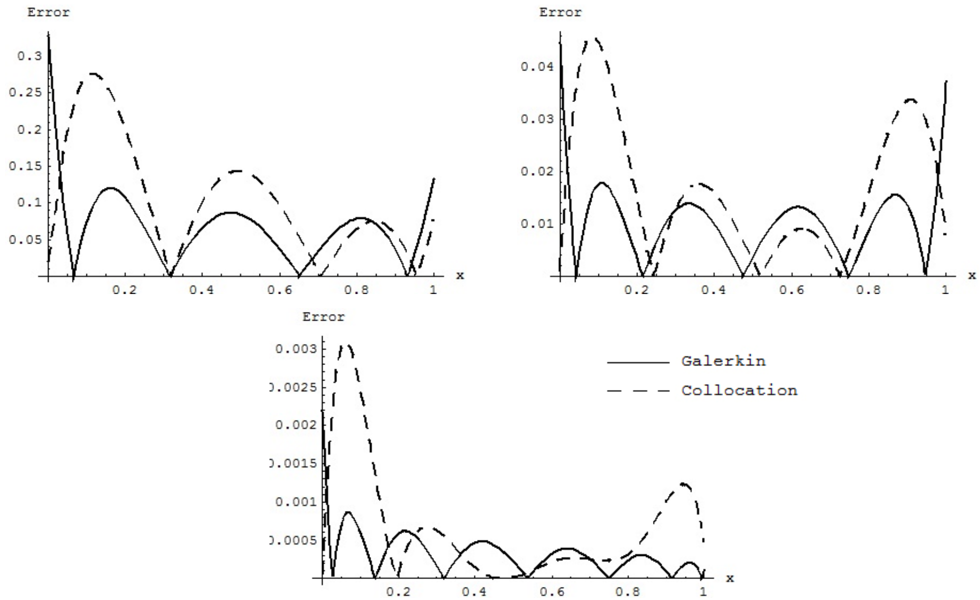


Fig. 1: Absolute error graph for $n = 1, 2, 3, 4, 5$

It is clear that as the value of n increase, absolute error decrease for both the Galerkin and Collocation methods, as shown in Fig. 1. It is observed that error in Collocation method is higher compared to Galerkin method for smaller values of n. But as the value of n increases, error in Collocation method tends to be lower than error in Galerkin method.Here in second part of numerical investigation, values of unknown parameters c_i determined by Galerkin method with five different polynomials and $n = 5$ are given in Table 2.

Table 2: Unknown parameters determined by Galerkin method

	Bernstein	Legendre	Chebyshev	Hermite	Laguerre
c_0	0.99975	1.63588	1.95493	1.95984	107.783
c_1	1.40181	3.36022	3.76496	3.70134	-493.616
c_2	1.99392	1.27451	0.95682	0.48496	940.365
c_3	2.94727	1.01952	0.66335	0.68475	-915.392
c_4	4.43567	0.00299	0.00168	0.00082	452.246
c_5	7.38876	0.09564	0.04708	0.02353	-90.3862

Substituting the values of the parameters from Table 2 with corresponding polynomials into Eq. (7), we get same approximate polynomial for the trial solution $\tilde{\phi}(x)$ which is given below:

$$\tilde{\phi}(x) = 0.999749 + 2.01029x + 1.90053x^2 + 1.71187x^3 + 0.0131013x^4 + 0.753224x^5$$

Now the values of $\tilde{\phi}(x_i)$with exact solution are given in Table 3.

Table 3: Galerkin Solution

x_i	$\tilde{\phi}(x_i)$	Exact
0.0	0.999749	1.0000000
0.1	1.2215037	1.2214028
0.2	1.4917846	1.4918247
0.3	1.8220397	1.8221188
0.4	2.2255567	2.2255409
0.5	2.7183659	2.7182818
0.6	3.3201448	3.3201169
0.7	4.0551215	4.0551999
0.8	4.9529781	4.9530324
0.9	6.0497584	6.0496475
1.0	7.3887635	7.3890561

It is observed that approximate solution remains same for different types of polynomials of same degree using Galerkin method. It suggests that variation of polynomials have no significant effect on the solution of Fredholm integral equation. Besides, we have tested other examples and noticed that the features we report here about two different methods and, although different polynomials are being consistent though for different examples, time required to get the convergent solution is different.

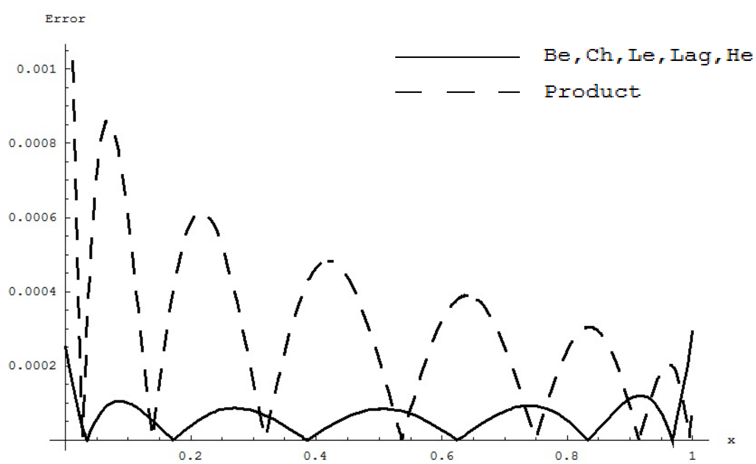


Fig. 2: Absolute error graph in Galerkin with $n = 5$

Fig. 2 shows the absolute error in the approximation in Galerkin method with $n = 5$ for two cases: one for Bernstein, Legendre, Chebyshev first kind, Hermite, Laguerre and other for product polynomials$LH_n(x)$. We can see from the graph that error for polynomial $LH_n(x)$is higher than other polynomials in most part of the domain though error is decreasing with the value of x increasing.

Finally, five different sets of collocation points xb, xc, xle, xlg and xh for five different polynomialssuch as Bernstein, Chebyshev 1st kind, Legendre, Laguerre and Hermite have used in Collocation method with $n = 5$. And then resulting approximate solutions are given in Table 4 and the absolute error graphs are given in Fig. 3.

$$xb = \left\{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}, 1\right\}$$

$$xc = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\right\}$$

$$xle = \{0, \frac{1}{3}, \frac{4}{9}, \frac{5}{9}, \frac{2}{3}, 1\}$$

$$xlg = \left\{0, \frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}, 1\right\}$$

$$xh = \{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\}$$

Table 4: Solutions using Collocation points

x	*Exact*	*Bernstein*	*Chebyshev*	*Legendre*	*Laguerre*	*Hermite*
0.0	1.0000000	0.999994739	0.999985218	0.999980844	0.999992879	0.999993504
0.1	1.2214028	1.221471553	1.221920348	1.221792104	1.221523671	1.221604762
0.2	1.4918247	1.491791947	1.491976456	1.491992503	1.491765998	1.491815006
0.3	1.8221188	1.822090895	1.821988514	1.822105987	1.822062935	1.822033149
0.4	2.2255409	2.225558520	2.225400441	2.225491226	2.225629282	2.225526471
0.5	2.7182818	2.718267537	2.718241648	2.718235041	2.718454155	2.718319911
0.6	3.3201169	3.320000692	3.320103599	3.320045839	3.320203371	3.320095354
0.7	4.0551999	4.055078198	4.055116362	4.055147040	4.055123037	4.055090919
0.8	4.9530324	4.953185177	4.952925161	4.953170513	4.952940136	4.953000248
0.9	6.0496475	6.051991010	6.049666932	6.050050001	6.049767117	6.049871790
1.0	7.3890561	7.389017224	7.388946877	7.388914554	7.389003478	7.389008097

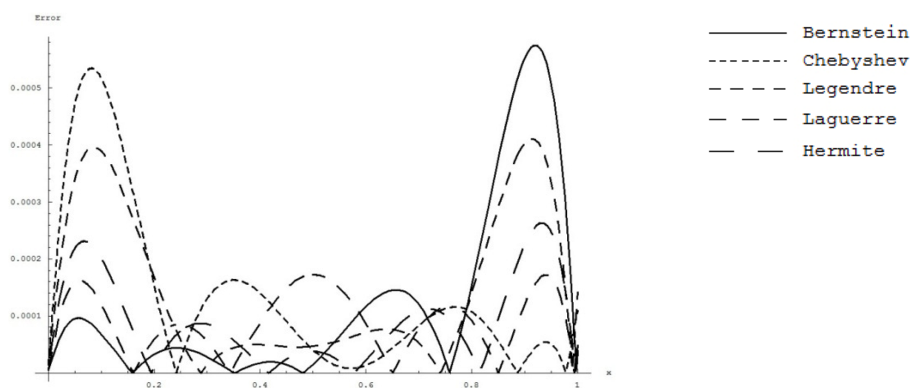


Fig. 3: Absolute error graphs in Collocation with $n = 5$

Observing the absolute error curves for the five polynomials in collocation method, it is evident that there is no way to say which one is better because there are fluctuations in the errors over the domain for all polynomials.

4.2 Numerical Example 2:

Consider the linear FIE of 2[nd] kind given by

$$\phi(x) - \frac{1}{2}\int_0^1 (1 + \frac{1}{2}x^2t^2 + \frac{1}{24}x^4t^4)\phi(t)dt = x\,,\quad 0 \le x \le 1$$

Similar to the first example, in first part of numerical investigation, approximate solutions in Galerkin and Collocation methods are presented in the Table 5 and Table 6 respectively for LH polynomials, $LH_n(x\,)$with $n = 1, 2, 3, 4, 5$.

Table 5: Approximate solution using Galerkin method with LH polynomial

x	$\tilde{\phi}(x)$				
	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
0.	1.0768217268	0.7345357689	0.597420743	0.544698552	0.539678377
0.1	1.0681688403	0.662451017	0.627497758	0.637864575	0.639877857
0.2	1.061438817	0.690079480	0.7236842557	0.743214516	0.743684945
0.3	1.056631658	0.785984399	0.845436854	0.850835140	0.8493576643
0.4	1.053747363	0.922427455	0.968943166	0.9583943798	0.957144189
0.5	1.052785931	1.075368772	1.083571040	1.067114707	1.067521001
0.6	1.0537473628	1.224466917	1.1886018641	1.179013535	1.1805868454
0.7	1.056631658	1.353078900	1.290247924	1.295205773	1.296129108
0.8	1.061438817	1.448260173	1.398953827	1.415076061	1.413997756
0.9	1.068168840	1.500764628	1.526981989	1.5361406302	1.534525529
1.0	1.0768217268	1.505044602	1.686282173	1.654431152	1.6588222732

Table 6: Approximate solution using Collocation method with LH polynomial

x	$\tilde{\phi}(x)$				
	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
0.	-6.043165468	0.522585022	0.52666768	0.538536763	0.538854554
0.1	-9.3064748201	0.428750454	0.578678769	0.633631046	0.639229060
0.2	-11.8446043165	0.478132501	0.692810103	0.741003813	0.743395398
0.3	-13.657553957	0.624769379	0.827546551	0.850138962	0.84930308
0.4	-14.7453237410	0.828106572	0.958797576	0.958462307	0.957134438
0.5	-15.107913669	1.052996834	1.076174513	1.067248201	1.067437335
0.6	-14.7453237410	1.269700186	1.179565669	1.178789952	1.180421198
0.7	-13.657553957	1.4538839190	1.276009232	1.294630986	1.295944818
0.8	-11.8446043165	1.586622593	1.376864005	1.414666074	1.413838218
0.9	-9.3064748201	1.654398035	1.495277957	1.536934165	1.534300455
1.0	-6.043165468	1.6490993426	1.64395459	1.657936746	1.658201053

Now in second part of numerical investigation, values of unknown parameters c_i determined by Galerkin method with five different polynomials Bernstein, Legendre, Chebyshev, Hermite and Laguerre using $n = 5$ are given in Table 7.

Table 7: Unknown parameters determined by Galerkin method

	Bernstein	*Legendre*	*Chebyshev*	*Hermite*	*Laguerre*
c_0	0.538979	0.577959	0.597904	0.60018	1.91124
c_1	0.738979	1.	1.	0.5	-2.03584
c_2	0.950309	0.0789991	0.0596829	0.032876	1.10059
c_3	1.17297	-3.02084×10^{-11}	-3.15527×10^{-11}	9.3016×10^{-11}	-0.582676
c_4	1.40817	0.00138725	0.000758651	0.000379326	0.145679
c_5	1.65835	-2.78855×10^{-12}	-2.21595×10^{-12}	3.20319×10^{-12}	-3.96409×10^{-6}

Substituting the values of parameters from Table 7 with corresponding polynomials into Eq. (7), resulting approximate polynomials for the trial solution $\tilde{\phi}(x)$ are given in Table 8.

Table 8: Approximate polynomial solution in Galerkin method

Basis	$\tilde{\phi}(x)$
Bernstein	$0.538979 + 1.x + 0.113297\,x^2 - 7.63833 \times 10^{-13}x^3 + 0.00606921\,x^4 - 3.606 \times 10^{-13}x^5$
Legendre	$0.538979 + 1.x + 0.113297\,x^2 - 5.11212 \times 10^{-11}x^3 + 0.00606921\,x^4 - 2.19598 \times 10^{-11}x^5$
Chebyshev	$0.538979 + 1.x + 0.113297\,x^2 - 8.18915 \times 10^{-11}x^3 + 0.00606921\,x^4 - 3.54553 \times 10^{-11}x^5$
Hermite	$0.538979 + 1.x + 0.113297\,x^2 + 2.31619 \times 10^{-10}x^3 + 0.00606921\,x^4 + 1.02502 \times 10^{-10}x^5$
Laguerre	$0.538979 + 1.x + 0.113297\,x^2 + 7.28264 \times 10^{-8}x^3 + 0.00606913\,x^4 + 3.30341 \times 10^{-8}x^5$

Now the numerical values of $\tilde{\phi}(x_i)$ from Table 8 are given in Table 9.

Table 9: Approximations of $\tilde{\phi}(x_i)$

x_i	$\tilde{\phi}(x_i)$
0.	0.538979352
0.1	0.6401129244
0.2	0.743520924
0.3	0.849225200
0.4	0.957262169
0.5	1.0676828103
0.6	1.180552673
0.7	1.295951869
0.8	1.413975080
0.9	1.534731551
1.0	1.6583450929

Finally, approximate solutions in Collocation method using five different sets of collocation points xb, xc, xle, xlg and xh for five different polynomials such as Bernstein, Chebyshev 1st kind, Legendre, Laguerre and Hermite respectively with $n = 5$ are given in Table 10.

Table 10: Approximate solution in Collocation method

x	Bernstein	Chebyshev	Legendre	Laguerre	Hermite
0.	0.538979352	0.538979352	0.538979352	0.538979352	0.538979352
0.1	0.6401129244	0.6401129244	0.6401129244	0.6401129244	0.640112924
0.2	0.743520924	0.743520924	0.743520924	0.743520924	0.743520924
0.3	0.849225200	0.849225200	0.849225200	0.849225200	0.849225200
0.4	0.957262169	0.957262169	0.957262169	0.957262169	0.957262169
0.5	1.0676828103	1.0676828103	1.0676828103	1.0676828103	1.067682810
0.6	1.1805526727	1.1805526727	1.1805526727	1.1805526727	1.180552672
0.7	1.295951869	1.295951869	1.295951869	1.295951869	1.295951869
0.8	1.413975080	1.413975080	1.413975080	1.413975080	1.413975080
0.9	1.5347315506	1.534731551	1.534731551	1.534731551	1.534731551
1.	1.658345093	1.658345093	1.658345093	1.658345093	1.658345093

From Table 5, 6, 9 and 10 it is evident that numerical approximations by Bernstein, Chebyshev, Legendre, Laguerre, Hermite and LH polynomials in both Galerkin and Collocation methods are converging into same direction. In contrast to example 1, resulting approximate polynomials in Galerkin method using Bernstein, Chebyshev, Legendre, Laguerre and Hermite polynomials as basis are different. Although resulting numerical solutions are same for all these polynomials. Approximate solutions in Collocation methods using Bernstein, Chebyshev, Legendre, Laguerre and Hermite polynomials as basis are same where as in example 1 numerical solutions were different for different polynomials.

5. Conclusion

In this research, new polynomials, called LH-polynomials of degree $2n$ are introduced. Then, linear Fredholm integral equation of 2nd kind is solved using Galerkin and Collocation methods where LH-polynomials are considered as a basis functions. It is found that Collocation method performed better than Galerkin method. Moreover, five different well known polynomials such as Bernstein, Chebyshev 1st kind, Legendre, Laguerre and Hermite are also considered in order to solve FIE of 2nd kind using Galerkin method. And it is also found that all the approximate solutions are same. It means that different types of polynomials have insignificant effect on the solution of FIE of 2nd kind. At the end, five different collocation points set along with these polynomials are considered. It is observed that five different approximate solutions are obtained. However, no conclusion has been made about the performance of the methods in terms of absolute errors. It is seen that all the methods give quite satisfactory outcomes but time required for computer to perform the calculations varies in two methods. Collocation method takes less time than Galerkin method. Then in both methods Hermite, Legendre and Chebyshev polynomials took less time than Bernstein and Laguerre polynomials.

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