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An Algorithmic Procedure for Finding Nash Equilibrium

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ABSTRACT

This paper proposes a heuristic algorithm for the computation of Nash equilibrium of a bi-matrix game, which extends the idea of a single payoff matrix of two-person zero-sum game problems. As for auxiliary but making the comparison, we also introduce here the well-known definition of Nash equilibrium and a mathematical construction via a set-valued map for finding the Nash equilibrium and illustrates them. An important feature of our algorithm is that it finds a perfect equilibrium when at the start of all actions are played. Furthermore, we can find all Nash equilibria of repeated use of this algorithm. It is found from our illustrative examples and extensive experiment on the current phenomenon that some games have a single Nash equilibrium, some possess no Nash equilibrium, and others had many Nash equilibria. These suggest that our proposed algorithm is capable of solving all types of problems. Finally, we explore the economic behaviour of game theory and its social implications to draw a conclusion stating the privilege of our algorithm.

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1. Introduction

With the advancement in mathematics of modern science, game theory, however, did not confined to a particular sector rather than is used at present for predicting the outcome of a strategic interaction in the social sciences. These situations are organized in a matrix by a set of players, a set of actions (or pure strategies) available to each player and a payoff (or utility) function for each player. Our aim in this paper is to find the best strategy or strategies called Nash equilibrium or equilibria in a bi-matrix game for a mixed strategies problem.

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The Nash equilibrium is the fundamental concept in the theory of non-cooperative games. It is an action profile (equilibrium strategy) with the property that no single player can obtain a payoff by deviating from his strategy. In every case, the key point is how to find the best outcome among the strategies. In this paper we (mostly) examine the Nash equilibrium, but we introduce a new algorithm as well by checking the payoff along row or column considering one player is fixed. This is done repeatedly until we find the Nash equilibrium. This concept can be understood by looking at some illustrative examples.

Rest of the paper is organized as follows. First, we briefly review the relevant oncepts of this paper and set up standard terminology and important definitions in Section 2. Section 3 is devoted to find an easy and new convenient approach for finding Nash equilibrium. We illustrate the algorithm by analytically and compare with a number of methods through some well-known problems of finite games in this field in Section 4. Finally, we have explored the findings and practical implications in Section 5 and drawn a conclusion stating the privilege of our method of computation in Section 6.

Background

John von Neumann, one of the greatest mathematicians of the 20th century who was closely associated with the creation of the theory of games called two person zero sum game. Moreover, Neumann and Morgenstern [19] work culminated in their fundamental book "Theory of Games and Economic Behavior" on game theory in 1947. This book also contains a theory of $n-$person zero sum game of a type which we would call cooperative. Recently Das and Hasan [5], Das et. al. [6], Das and Dhar [7], Das [8], Das and Chakroborty [9], Das [10], and Saha et. al. [25] have studied on the two-person zero-sum game problems with the necessary computations. In 1984, Bernhein [1] discovered the rationalizable strategic behaviour of Nash equilibrium. In addition, there are so many standard books likely Davis [4], Myerson[17], Osborne[22], Osborne and Rubinstein [23], Nisan et. al. [21] e.t.c. for the theory of games and economic behavior. Beside this many research articles, likely in 1991, Elzen and Talman [11], Elzen et. al. [12] proposed how to find an equilibrium by solving a related stationary point problem. Latter, in 1996 Moreno and Wooders [16] proved that any correlated strategy whose support is contained in the set of actions that survive the iterated elimination of strictly dominated strategies and weakly Pareto dominates every other correlated strategy whose support is contained in that set, is a coalition-proof equilibrium. Chiappori et. al. [2] tested the mixed-strategy equilibria when players are heterogeneous in 2002. In the same year, Herings et. al. [15] developed a computational procedure of the Nash equilibrium selected by the tracing procedure in n-person games. In 2009, Shoham and Brown [26] introduced multiagent systems: algorithmic, game theoric, and logical foundation. In the same year, Carmona et. al. [3] had shown the existence of pure strategy Nash equilibria of large games. Later in 2015 Pharaon [24] solved game theory problems by using set-Valued Maps. But, so far several authors proposed different types methods for finding nash equilibria for bi-matrix game problems. Since 1950's many scholars have been done on game theory and is continuing today to improve the existing game theory methods and to develop new techniques or models. However, no one did not discuss the whole problem in systematically together with algorithmically. This suggests, in very general terms, the kind of approach that is required for games lacking for the manual procedure. Therefore, in this research the focus is to develop a new and easy algorithm for finding the Nash equilibriums by extending the idea of single payoff matrix of two person zero-sum game problems. Moreover, the basics of game theory models, methods, real life applications, and an algorithmic procedure are illustrated in this works.

2. Formal Definitions and Terminology

In this section, we set up standard terminology and important definitions. Moreover, Nash equilibrium through mathematical construction via set valued map also introduce here. However, the contents here are the materials appeared in any standard textbook of game theory. Besides, it is necessary to repeat those materials again in the paper to define the basic concepts of this paper.

2.1 Nash Equilibrium with ordinal preferences

Briefly, a Nash equilibrium is an action profile a^* with the property that no player i can do better by choosing an action different from a_i^*, given that every other player j adheres to a_j^*. In mathematically, let a be an action profile, in which the action of each player i is a_i. Let a_i' be any action of player i (either equal to a_i, or different from it). Then (a_i', a_{-i}) denotes the action profile in which every player j except i chooses her action a_j as specified by a, whereas player i chooses a_i'. (The $-i$ subscript on a stands for "except i".) That is, (a_i', a_{-i}) is the action profile in which all the players other than i adhere to a while i "deviates" to a_i'. (If $a_i' = a_i$ then of course $(a_i', a_{-i}) = (a_i, a_{-i}) = a$). If there are three players, for example, then (a_2', a_{-2}) is the action profile in which players 1 and 3 adhere to a (player 1 chooses a_1 player 3 chooses a_3) and player 2 deviates to a_2'. If we now assume that the action profile a^* in a strategic game with ordinal preferences is a Nash equilibrium if, for every player i and every action a_i of player i, a^* is at least as good according to player i's preferences as the action profile (a_i, a_{-i}^*) in which player i chooses a_i while every other player j chooses a_j^*. Equivalently, for every player i,

$$u_i(a^*) \geq u_i(a_i, a_{-i}^*) \text{ for every action } a_i \text{ of player } i, \tag{2.1}$$

where u_i is a payoff function that represents player i's preferences.

2.2 Nash Equilibrium with Set valued map

We start here by considering two players: say player A and player B. The game requires A to pick a strategy $x \in D$ and B to pick a strategy $y \in S$. The pair $(x, y) \in D \times S$ is called a strategy pair or a bistrategy. A natural mechanism for the selection of strategies by the two players is coming up with *decision rules*.

Definition 1

A decision rule for A is a set-valued map $C_D : S \to D$ which associates each strategy $y \in S$ played by B with the strategies $x \in C_D(y)$ which may be played by A.

Similarly, a decision rule for B is a set-valued map $C_S : D \to S$ which associates each strategy $x \in D$ played by A with the strategies $y \in C_s(x)$ which may be played by B. Once player A and player B come up with their decision rules C_D and C_S respectively, we become interested in pairs of strategies (x, y) that are in static equilibrium, in the sense that: $\bar{x} \in C_D(\bar{y})$ and $\bar{y} \in C_S(\bar{x})$. This leads to the following definition:

Definition 2

A pair of strategies $(\bar{x}, \bar{y})$ which is in static equilibrium is called a consistent pair of strategies or a consistent bistrategy. The set of consistent bistrategies may be empty or very large. However, Game theorist are interested for the non-empty and small set. The problem of finding consistent bistrategies is a fixed point problem. We use C to denote the set-valued map from $D \times S$ into itself: $\forall (x, y) \in D \times S$, $C(x, y) := C_D(y) \times C_S(x)$. Here, we are looking for pairs $(\bar{x}, \bar{y})$ that satisfy the condition: $(\bar{x}, \bar{y}) \in C(\bar{x}, \bar{y})$.

Definition 3 (Normal Strategic Form)

A two-person game in normal (strategic) form is defined by a mapping $f : D \times S \to \Re^2$ is called a biloss mapping.
If player A knows that B is playing strategy $y \in S$, then he may be tempted to choose a strategy $x \in D$ that minimizes his loss. From this idea, we create the canonical decision rule $C_D : S \to D$ in the following way:

$$\overline{C}_D(y) = \left\{ \overline{x} \in D \mid f_D(\overline{x}, y) = \inf_{x \in E} f_D(x, y) \right\}$$

If B knows that A is playing strategy $x \in D$, then he may be tempted to choose a strategy $y \in S$ that minimizes his loss. From this idea, we create the canonical decision rule $C_S : D \to S$ in the following way:

$$\overline{C}_S(x) = \left\{ \overline{y} \in S \mid f_S(x, \overline{y}) = \inf_{y \in S} f_S(x, y) \right\}$$

The above canonical decision rule of two-person game in normal (strategic) form is helped to introduce the following definition of Nash equilibrium.

Definition 4 (Nash equilibrium for bilosses)

A consistent pair of strategies $(\overline{x}, \overline{y})$ based on the canonical decision rules is called a non-cooperative equilibrium (or a Nash equilibrium) of the game. In other words $(\overline{x}, \overline{y})$ is a non-cooperative equilibrium if and only if $f_D(\overline{x}, \overline{y}) = \inf_{x \in D} f_D(x, \overline{y})$ and $f_S(\overline{x}, \overline{y}) = \inf_{y \in S} f_S(\overline{x}, y)$.

A convenient way to find non-cooperative equilibria is to introduce the following functions:

$$f_D^b(y) = \inf_{x \in D} f_D(x, y) \text{and} f_S^b(y) = \inf_{x \in S} f_S(x, y) \tag{2.2}$$

and so $(\overline{x}, \overline{y})$ is a non-cooperative equilibrium if and only if

$$f_D^b(\overline{y}) = f_D(\overline{x}, \overline{y}) \text{and} f_S^b(\overline{x}) = f_S(\overline{x}, \overline{y}) \tag{2.3}$$

Definition 5 (Nash equilibrium for not bilosses)

A consistent pair of strategies $(\overline{x}, \overline{y})$ based on the canonical decision rules is called a non-cooperative equilibrium (or a Nash equilibrium) of the game. In other words $(\overline{x}, \overline{y})$ is a non-cooperative equilibrium if and only if $f_D(\overline{x}, \overline{y}) = \sup_{x \hat{I} D} f_D(x, \overline{y})$ and $f_S(\overline{x}, \overline{y}) = \sup_{y \hat{I} S} f_S(\overline{x}, y)$.

A convenient way to find non-cooperative equilibria is to introduce the following functions:

$$f_D^b(y) = \sup_{x \in D} f_D(x, y) \text{ and } f_S^b(y) = \sup_{x \in S} f_S(x, y)$$

and so $(\overline{x}, \overline{y})$ is a non-cooperative equilibrium if and only if $f_D^b(\overline{y}) = f_D(\overline{x}, \overline{y})$ and $f_S^b(\overline{x}) = f_S(\overline{x}, \overline{y})$.

Definition 6 (Conservative Strategies)

There is a behaviour where B's only goal is to annoy A, and A is aware of this. Hence, it would be wise for A to evaluate the loss associated with a strategy $x \in D$ using the function f_D^{Ξ} given by: $f_D^{\Xi}(x) = \sup_{y \in S} f_D(x, y)$.

This is called the worst-loss function. In this case, A's behaviour consists of finding $x^{\Xi} \in D$ which minimizes his worst loss, namely:

$$f_D^{\Xi}(x^{\Xi}) = \inf_{x\in D} f_D^{\Xi}(x) = \inf_{x\in D} \sup_{y\in S} f_E(x, y) \tag{2.4}$$

This strategy is conservative and $v_D^{\Xi} := \inf_{x\in D} f_D^{\Xi}(x)$ is value is called A's conservative value.

Similarly, there is another behaviour where A 's only goal is to annoy B and B is aware of this. Therefore, it would be wise for B to evaluate the loss associated with a strategy $y \in S$ using the function f_S^{Ξ} given by: $f_S^{\Xi}(y) = \sup_{x\in D} f_S(x, y)$. This is called the worst-loss function.

In this case, B's behavior consists of finding $y^{\Xi} \in S$ which minimizes his worst loss, namely:

$$f_S^{\Xi}(y^{\Xi}) = \inf_{y\in S} f_S^{\Xi}(y) = \inf_{y\in S} \sup_{x\in D} f_S(x, y) \tag{2.5}$$

This strategy is conservative and $v_S^{\Xi} := \inf_{y\hat{I}S} f_S^{\Xi}(y)$ it's value is called B's conservative value. The vector $v^{\Xi} = (v_D^{\Xi}, v_S^{\Xi})$ is called the conservative vector of the game.

3. A new approach of Nash equilibrium

The main objective of this research is to find an easy and new convenient approach for finding Nash equilibrium in comparison with other techniques prescribed above which are also useful in this field.

Algorithm for finding Nash equilibrium

Step 1: Check whether the game is bilosses or not.
Step 2: If the game problem is bilosses, then go to the following step. If not go to step 4.
Step 3: Select randomly a pair from the table, claiming that this pair is a Nash equilibrium. Then we have to check either our claim is true or not. For checking this follow the following sub-steps:

- **Sub-Step (i):** Suppose Player 2 is fixed. It means that the concerning strategy's (the strategy of the selected pair for Player 2) payoffs are fixed i.e. the column containing Player 2's payoffs is fixed.
- **Sub-Step (ii):** Checking for Player 1: To obtain the best strategy for Player 1, Player 1's selected strategy's payoff must be less than any other strategy's pay offs against the Player 2.
- **Sub-Step (iii):** If Sub-Step (ii) is true then Player 1's strategy's payoff is more convenient than any other strategy's payoffs and then go to the following Sub-Step. If not true, then the selected pair is not Nash equilibrium.
- **Sub-Step (iv):** Suppose Player 1 is fixed. It means that the concerning strategy's (the strategy of the selected pair for Player 1) payoffs are fixed i.e. the row containing that Player 1's payoff is fixed.
- **Sub-Step (v):** Checking for Player 2: To obtain the best strategy for Player 2, Player 2's selected strategy's payoff must be less than any other corresponding strategy's payoffs against the Player 1.
- **Sub-Step (vi):** If Sub-Step (v) is true then Player 2's strategy's payoff is more convenient than any other strategy's payoffs of that row and the selected pair is Nash equilibrium. If not true, then the selected pair is not Nash equilibrium.

Step 4: Select randomly a pair from the table, claiming that this pair is a Nash equilibrium. Then we have to check either our claim is true or not. For checking this follow the following Sub-Steps.

- **Sub-Step (i):** Suppose Player 2 is fixed. It means that the concerning strategy's (the strategy of the selected pair for Player 2) payoffs are fixed i.e. the column containing Player 2's payoffs is fixed.
- **Sub-Step (ii):** Checking for Player 1: To obtain the best strategy for Player 1, Player 1's selected strategy's payoff must be greater than any other strategy's payoffs against the Player 2.
- **Sub-Step (iii):** If Sub-Step (ii) is true then Player 1's strategy's payoff is more convenient than any other strategy's payoffs, and then go to the following Sub-Step. If not, then selected pair is not Nash equilibrium.

Sub-Step (iv): Suppose Player 1 is fixed. It means that the concerning strategy's (the strategy of the selected pair for Player 1) payoffs are fixed i.e. the row containing that Player 1's payoff is fixed.
Sub-Step (v): Checking for Player 2: To obtain the best strategy for Player 2, Player 2's selected strategy's payoff must be greater than any other corresponding strategy's payoffs against the Player 1.
Sub-Step (vi): If Sub-Step (v) is true then Player 2's strategy's payoff is more convenient than any other strategy's payoffs of that row and the selected pair is Nash equilibrium. If not true, then the selected pair is not Nash equilibrium.

Stop

We explain the above methodology more clearly in the following section, which shows that some games have a single Nash equilibrium, some possess no Nash equilibrium, and others have many Nash equilibria.

4. Illustrative Examples

In this section, we are intended to illustrate the concepts defined in the paper and display experimentally these methods through some well-known problems of finite games in this field. A number of phenomena, which occur in these games, are illustrated by our algorithmic procedure. Moreover, the paper would like to compare the proposed algorithm with two existing methods and the explanation of two existing methods would be clear to the reader. By our developed algorithm, it is easy to search for the Nash equilibrium of bi-matrix games because the motivation of the proposed algorithm to search for the Nash equilibrium for bi-matrix games is clear. The following examples have taken from Osborne [22], Myerson [17], Pharaon [24] and Osborne et. al. [23].

4.1 Finding Nash Equilibrium Games Using Definition

We find the Nash equilibrium using the well-known definition of Nash equilibrium in equation (2.1).

Example 1: Prisoner's Dilemma

Two suspects in a major crime are held in separate cells. There is enough evidence to convict each of them of a minor offense, but not enough evidence to convict either of them of the major crime unless one of them acts as an informer against the other (finks). If they both stay quiet, each will be convicted of the minor offense and spend one year in prison. If one and only one of them finks, he will be freed and used as a witness against the other, who will spend four years in prison. If they both fink, each will spend three years in prison. This situation can be modelled as the following Nash game:

Table 4.1.1: Prisoner's Dilemma

		Player 2 : B	
		Quiet	Fink
Player 1: A	Quiet	$(2,2)$	$(0,3)$
	Fink	$(3,0)$	$(1,1)$

By using definition in equation (2.1) of the four possible pairs of actions in the *Prisoner's Dilemma* (Table 4.1.1), we see that (*Fink, Fink*) is the unique Nash equilibrium. No other action profile is a Nash equilibrium.

Solution

The complete procedure can be found in any standard book like Martin. In summary, the only Nash equilibrium of the Prisoner's Dilemma both players choose Fink.

Example 2: Battle of the Friends

Two people wish to go out together. Two concerts are available: one of music by Bach, and one of music by Stravinsky. One person prefers Bach and the other prefers Stravinsky. If they go to different concerts, each of them is equally unhappy listening to the music of either composer. We can model this situation as the two-player strategic game in Table 4.1.2, in which the person who prefers Bach chooses a row and the person who prefers Stravinsky chooses a column. This situation can be modelled as the following Nash game:

Table 4.1.2: Battle of the Friends

		Player 2: B	
		Bach	Stravinsky
Player 1: A	Bach	$(2,1)$	$(0,0)$
	Stravinsky	$(0,0)$	$(1,2)$

Solution

To examine each pair of actions, we find two Nash equilibria: (Bach, Bach) and (Stravinsky, Stravinsky) i.e. both of these outcomes are compatible with a steady state and both outcomes are stable social norms.

Example 3: Prisoner's Dilemma for Bilosses

Suppose that player A and player B are accomplices to a crime which leads to their imprisonment. Each has to choose between the strategies of confession ("I" for A and "1" for B) or accusation ("II" for A and "2" for B). The strategy sets are therefore $D=\{I,II\}$ and $S=\{1,2\}$. If neither confesses, moderate sentences are given (b years in prison). If A confesses and B accuses him, B is freed and A is sentenced to $c>b$ years in prison. If B confesses and A accuses her, A is freed and B is sentenced to $c>b$ years in prison. If both confess, they will each serve a years in prison where, $a<b<c$. This situation may be modelled as the bilosses of the different bistrategies in the following table:

Table 4.1.3: Prisoner's dilemma for Bilosses

		Player 2: B	
		1	2
Player 1: A	I	(a,a)	$(c,0)$
	II	$(0,c)$	(b,b)

Solution

Using the equation (2.1) of the definition of Nash equilibrium, we can easily see that (II, 2) is the unique Nash equilibrium. No other action profile is a Nash equilibrium.

Example 4: General Battle of Friends for Bilosses

The strategies of player A and B consist of watching a political debate or going to the mall. B prefers going to the mall while A prefers watching a political debate, but they both prefer to be together. A's strategies are I and II for watching a political debate and going to the mall, respectively. As for B, his strategies are 1 and 2, in the same order as well. Following table summarizes the bilosses based on the bistrategies $(0<a<b)$:

Table 4.1.4: Battle of friends for Bilosses

		Player 2: B	
		1	2
Player 1 : A	I	$(0,a)$	(b,b)
	II	(b,b)	$(a,0)$

Solution

Using the equation (2.1) of the definition of Nash equilibrium, we can easily see that (I,1) and (II, 2) are the Nash equilibrium. We now introduce the mathematical constructions of Nash game in the following Section.

4.2 Nash Games Using Set Valued Map

We illustrate the Nash equilibrium using set valued map stated above for biloss and for not bilosses.

Example 5: Bilosses Prisoner's dilemma

Suppose that player A and player B are accomplices to a crime which leads to their imprisonment. Each has to choose between the strategies of confession (”I” for A and ”1” for B) or accusation(”II” for A and”2” for B). The strategy sets are therefore $D=\{I,II\}$ and $S=\{1,2\}$. If neither confesses, moderate sentences are given (b years in prison). If A confesses and B accuses him, B is freed and A is sentenced to $c>b$ years in prison. If B confesses and A accuses him, A is freed and B is sentenced to $c>b$ years in prison. If both confess, they will each serve a years in prison where $a<b<c$. We summarize the bistrategies in the following table:

Table 4.2.1: Bilosses Prisoner's dilemma

		Player 2: B	
		1	2
Player 1: A	I	(a,a)	$(c,0)$
	II	$(0,c)$	(b,b)

Solution

Using the equations (2.4) and (2.5) we have, $f_D^{\Xi}(I)=c, f_D^{\Xi}(II)=b, f_S^{\Xi}(1)=c, f_S^{\Xi}(2)=b$. Now, the A's conservative value is $v_D^{\Xi}=b$, and the B's conservative value is $v_D^{\Xi}=b$. Also, $f_D^b(1)=\inf\{a,0\}=0$, $f_D^b(2)=\inf\{c,b\}=b$ since $a<b<c$, and $f_S^b(I)=\inf\{a,0\}=0, f_S^b(II)=\inf\{c,b\}=b$. Examining the four possible pairs of actions in the Prisoner's Dilemma (produced in Table 4.2.1), we see that (II, 2) is the unique Nash equilibrium. No other action profile is a Nash equilibrium.

- (II, 2) does satisfy equation (2.3). Hence, the action pair (II, 2) is a Nash equilibrium because $f_D(II,2)=b$ which is equal to $f_D^b(2)=b$ and $f_S(II,2)=b$ which is equal to $f_S^b(II)=b$.
- (I, 1) does not satisfy equation (2.3) , and hence pair (I, 1) is not Nash equilibrium because $f_D(I,1)=a$ which is not equal to $f_D^b(1)=0$ and also $f_S(I,1)=a$ which is not again equal to $f_S^b(I)=b$.
- (I, 2) does not satisfy equation (2.3). Hence, the action pair (I, 2) is not Nash equilibrium since $f_D(I,2)=c$ which is not equal to $f_D^b(2)=b$ and $f_S(I,2)=0$ which is not equal to $f_S^b(I)=0$.
- (II, 1) does not satisfy equation (2.3). Hence, the action pair (II, 1) is not Nash equilibrium because $f_D(II,1)=0$ which is equal to $f_D^b(1)=0$ but $f_S(II,1)=c$ which is not equal to $f_S^b(II)=b$.

In summary, in the only Nash equilibrium of the *Prisoner's Dilemma* both players choose 2^{nd} phase that is (II,2).

Example 6: Battle of the Friends for Bilosses

The strategies of player A and player B consist of watching a political debate or going to the mall. B prefers going to the mall while A prefers watching a political debate but they both prefer to be together. A's strategies are I and II for watching a political debate and going to the mall, respectively. As for B, his strategies are 1 and 2, in the same order as well. Here is a table that summarizes the bilosses played $0<a<b$:

Table 4.2.2: Bilosses Battle of the friends

		Player 2 : B	
		1	2
Player 1 : A	I	$(0,a)$	(b,b)
	II	(b,b)	$(a,0)$

Solution

By using the set valued map, the Nash equilibrium of the Battle of the friends both players choose either first phase of the players or the second phase of the players that is (I, 1) and (II,2).

Example 7: Prisoner's Dilemma

Two suspects in a major crime are held in separate cells. There is enough evidence to convict each of them of a minor offense, but not enough evidence to convict either of them of the major crime unless one of them acts as an informer against the other (finks). If they both stay quiet, each will be convicted of the minor offense and spend one year in prison. If one and only one of them finks, he will be freed and used as a witness against the other, who will spend four years in prison. If they both fink, each will spend three years in prison. Then we get:

Table 4.2.3: Prisoner's Dilemma

		Player 2 : B	
		Quiet (1)	Fink (2)
Player 1 : A	Quiet (I)	$(2,2)$	$(0,3)$
	Fink (II)	$(3,0)$	$(1,1)$

Solution

Examining the four possible pairs of actions we see (Fink, Fink) is the unique Nash equilibrium.

Solution

Similarly, we can use set valued map to find Nash equilibria. In summary, the only Nash equilibrium of the Prisoner's Dilemma that both players choose 2nd phase that is (II, 2).

Example 8: Battle of the Friends

Two people wish to go out together. Two concerts are available: one of music by Bach, and one of music by Stravinsky. One person prefers Bach and the other prefers Stravinsky. If they go to different concerts, each of them is equally unhappy listening to the music of either composer.

Table 4.2.4: Battle of the Friends

		Player 2 : B	
		Bach (1)	Stravinsky (2)
Player 1 : A	Bach (I)	$(2,1)$	$(0,0)$
	Stravinsky II)	$(0,0)$	$(1,2)$

We just modelled the situation as the two-player strategic game in the above table, in which the person who prefers Bach chooses a row and the person who prefers Stravinsky chooses a column.

Solution

By examining the four possible pairs of actions in the Battle of the friends (Table 4.2.4), we will have two unique Nash equilibrium. Similarly, we can use the set valued map to find Nash equilibria. In summary, the game has two Nash equilibria: (Bach, Bach) and (Stravinsky, Stravinsky) that is is (I, 1) and (II,2).

4.3 Finding Nash Equilibrium Using Our Algorithmic Procedure

We do find the Nash equilibrium using our Nash algorithm stated above through experimentally.

Example 9: Bilosses Prisoner's dilemma

Suppose that player A and player B are accomplices to a crime which leads to their imprisonment. Each has to choose between the strategies of confession ("I" for A and "1" for B) or accusation("II" for A and "2" for B). The strategy sets are therefore $D=\{I,II\}$ and $S=\{1,2\}$. If neither confesses, moderate sentences are given (b years in prison). If A confesses and B accuses him, B is freed and A is sentenced to $c > b$ years in prison. If B confesses and A accuses him, A is freed and B is sentenced to $c > b$ years in prison. If both confess, they will each serve *a* years in prison where $a < b < c$. We summarize those in the following table:

Table 4.3.1: Bilosses Prisoner's dilemma

		Player 2 : B	
		1	2
Player 1 : A	I	(a,a)	$(c,0)$
	II	$(0,c)$	(b,b)

Solution

Check whether the game is bilosses or not. Since the game problem is bilosses then examining the four possible pairs of actions using step 3.

- Select randomly a pair (I, 1), claiming that (I, 1) pair is a Nash equilibrium. Then we have to check either our claim is true or not. For checking this follow the following:

  Suppose Player 2is fixed, so $\begin{pmatrix} a \\ c \end{pmatrix}$ fixed. Player 1is playing so checking for player 1: where $a > 0$, but the algorithm refers that, Player 1 should be $a < 0$ which implies that the claim is wrong. So we don't need to verify for player 2 . We therefore conclude that (I, 1) does not Nash equilibrium.

- Now select pair (I, 2), claiming that (I, 2) pair is a Nash equilibrium. Then we have to check either our claim is true or not. For checking this follow the following:

  Suppose Player 2is fixed, so $\begin{pmatrix} 0 \\ b \end{pmatrix}$ fixed. Player 1 is playing so checking for player 1: where $c > b$, but the algorithm refers that, Player 1 should be $c < b$ which implies that the claim is wrong. So we don't need to verify for player 2. We therefore conclude that (I, 2) does not Nash equilibrium.

- Then select a pair (II, 1), claiming that (II, 1) is a Nash equilibrium. Then we have to check either our claim is true or not. For checking this follow the following:

  Suppose Player 2is fixed, so $\begin{pmatrix} a \\ c \end{pmatrix}$ fixed. Player 1 is playing so checking for player 1: where $0 < a$, which implies that the claim is true. Now we need to verify for player 2. Suppose Player 1 is fixed, so $(0 \quad b)$ fixed for the pair (II, 1), Player 2 is playing so checking for Player 2 (B): where $c > b$, but the algorithm refers that, Player 2 should be $c < b$ which implies that the claim is wrong. We therefore conclude that (II, 1) does not Nash equilibrium.
- Finally, select pair (II, 2), claiming that (II, 2) is a Nash equilibrium. Then we have to check either our claim is true or not. For checking this follow the following:

  Suppose Player 2 is fixed, so $\begin{pmatrix} 0 \\ b \end{pmatrix}$ fixed. Player 1 is playing so checking for player 1: where $b < c$, which implies that the claim is true. Now we need to verify for Player 2. Suppose Player 1is fixed, so $(0 \quad b)$ fixed. Player 2 is playing so checking for Player 2 : where $b < c$,which implies that the claim is true. We therefore conclude that (II, 2) is the Nash equilibrium.

So, only Nash equilibrium of the Prisoner's Dilemma both players choose (II, 2).

Example 10: Bilosses Battle of the Friends

The strategies of player A and B consist of watching a political debate or going to the mall. B prefers going to the mall while A prefers watching a political debate but they both prefer to be together. A's strategies are I and II for watching a political debate and going to the mall, respectively. As for B, her strategies are 1 and 2, in the same order as well. The following summarizes the bilosses played $(0 < a < b)$:

Table 4.3.2: Bilosses Battle of the friends

		Player 2 : B	
		1	2
Player 1 : A	I	$(0, a)$	(b, b)
	II	(b, b)	$(a, 0)$

Solution

Using the algorithm in Section 4, we find the Nash equilibrium of the Battle of the friends both players choose either first phase of the players or the second phase of the players that is (I, 1) and (II,2).

Example 11: Prisoner's Dilemma

Two suspects in a major crime are held in separate cells. There is enough evidence to convict each of them of a minor offense, but not enough evidence to convict either of them of the major crime unless one of them acts as an informer against the other (finks). If they both stay quiet, each will be convicted of the minor offense and spend one year in prison. If one and only one of them finks, he will be freed and used as a witness against the other, who will spend four years in prison. If they both fink, each will spend three years in prison. Thus, we get the following table:

Table 4.3.3: Prisoner's Dilemma

		Player 2 : B	
		Quiet (1)	Fink (2)
Player 1 : A	Quiet (I)	$(2,2)$	$(0,3)$
	Fink (II)	$(3,0)$	$(1,1)$

By examining the four possible pairs of actions in the *Prisoner's Dilemma* (reproduced in table 4.3.3), we see that (Fink, Fink) is the unique Nash equilibrium.

Solution

Similarly, we can use our algorithm easily to find Nash equilibria. In summary, in the only Nash equilibrium of the *Prisoner's Dilemma* both players choose 2nd phase that is (II, 2).

Example 12: Battle of the Friends

Two people wish to go out together. Two concerts are available: one of music by Bach, and one of music by Stravinsky. One person prefers Bach and the other prefers Stravinsky. If they go to different concerts, each of them is equally unhappy listening to the music of either composer. We can model this situation as the two-player strategic game in Figure 16, in which the person who prefers Bach chooses a row and the person who prefers Stravinsky chooses a column. This situation may be modelled as a following Nash game:

Table 4.3.4: Battle of the Friends

		Player 2: B	
		Bach(1)	Stravinsky(2)
Player 1: A	Bach (I)	$(2,1)$	$(0,0)$
	Stravinsky (II)	$(0,0)$	$(1,2)$

By examining the four possible pairs of actions in the Battle of the friends (reproduced in table 4.3.4), we see that we will have two unique Nash equilibrium.

Solution

Similarly, we can use our algorithm easily to find the Nash equilibria. In summary, we conclude that the game has two Nash equilibria: (Bach, Bach) and (Stravinsky, Stravinsky) that is (I, 1) and (II, 2).

5. Discussion

The explanation in this section of the Economic Behavior in Game Theory is expected to strengthen the contribution of this paper. Moreover, we have considered the classical and well-studied form of games called Nash equilibrium of bi-matrix games and developed a new algorithm for finding the Nash equilibriums by extending the idea of single payoff matrix of two person zero-sum game problems. After analyzing three techniques, we can easily say that our algorithmic approach is more convenient for finding Nash equilibrium in comparison with other techniques prescribed above which are also useful in this field. Moreover, illustrative examples in the above Section have shown that some games have a single Nash equilibrium, some possess no Nash equilibrium, and others have many Nash equilibria. Now, let us try to find the practical implications of the game problems. To do this, we first analyze the economic behaviour to the game problems. It is obvious in social background, everyone wanted to get one's own interest undertakes some certain ways and game theory just help to find out the best one of them. In free market economy, a vendee always tries to purchase the best goods with lower price. On the other hand, the vendor must try to sell those with higher value. It is seen that for the game problems illustrated in this paper example 7, both players always tried to minimize their imprisonment; at pair (II, 2) of this problem, we claimed that that pair is Nash equilibrium, which implies that if both players accuse with each other, immediately their punishment reduced. Similarly, it is seen that for the Example 8, at pair (I,1) and (II, 2) of that problem, we claimed that those pair are Nash equilibrium which implies that two social norms are stable. Both players choose the action associated with the outcome preferred by Player 2 (B), and both players choose the action associated with the outcome preferred by Player 1 (A). Being comparing with our game problems, the main purpose of game theory is to find in such a solution so that society remains stable and every person get their benefit minimizing their imprisonment year and maximizing to wish to go together that is minimizing their loss and maximizing their profit.

6. Conclusion

In this paper, we developed a new algorithmic procedure to find the Nash equilibrium of bi-matrix games by extending the idea of a single payoff matrix of two-person zero-sum game problems. As for auxiliary but making the comparison, we introduced here the well-known definition of Nash equilibrium and a mathematical construction via a set-valued map for finding the Nash equilibrium. An important feature of our algorithm is that it finds a perfect equilibrium when at the start of all actions are played. Furthermore, we can find all Nash equilibria of repeated use of this algorithm. Illustrative examples showed that some games have a single Nash equilibrium, some possess no Nash equilibrium, and others had many Nash equilibria. Finally, we explored the economic behaviour of game theory and drawn a conclusion stating the privilege of our algorithm.

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