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ON THE HYDROMAGNETIC STABILITY OF FLUID ROTATING BETWEEN TWO CONCENTRIC CYLINDERS

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ABSTRACT

A theoretical study of the hydromagnetic instability due to slow hydromagnetic waves has been carried out after simplifying the complexity of governing equations. A series of stability conditions have been derived for wide range of magnetic field profiles. It has been shown that regardless of the magnetic field profile, any unstable disturbances propagate against the basic rotation.

1. Introduction

Acheson [1] considered the Magnetohydrodynamic (MHD) stability of a unifrom rotating fluid in the annular region between two concentric infinitely long cylinders. Following his analysis a detailed MHD stability of this system has been examined theoretically in this paper. Michael [2] earlier had shown that for sufficiently rapid rotation the system is stable if the magnetic field is azimuthal and the disturbances are axisymmetric. Couette [3] initiated the investigation of flow in an annulus region between two rotating cylinders. Acheson [4] examined the hydromagnetic stability of a radially stratified fluid rotating between two co-axial cylinders, with particular emphasis on the case when the angular velocity exceeds both buoyant and Alfven′ frequencies. Sung [5] concerns the stability of the Von-Kar′man′ swirling flow between two coaxial disks in the presence of an axial and toroidal magnetic field. Howard and Gupta [6] investigated the stability of inviscid flows between two concentric cylinders which have an axial velocity component depending only on r in addition to a swirl velocity component in the direction of increasing azimuthal angle θ. Bhattacharyya et al. [7] studied the hydro-magnetic stability of a non dissipative flow of an incompressible conducting fluid for non-axisymmetric disturbances. Zhang and Busse [8] investigated the instability of an electrically conducting fluid of magnetic diffusivity and viscosity in a rapidly rotating spherical container in the presence of a toroidal magnetic field. The transformation of initially turbulent flow of electrically incompressible viscous fluid under the influence of an imposed homogeneous magnetic field is investigated using direct numerical simulation by Zikanov and Thess [9]. Vorobev et al. [10] investigated fluid flow of low magnetic Reynolds number using direct numerical simulations large eddy forced flow in a periodic box. A series of simulation is performed with different strengths of the magnetic field, varying Reynolds number. Instability and transition to turbulence in a temporally evolving free shear layer of an electrically conducting fluid affected by an imposed parallel magnetic field is investigated numerically by Vorobev and Zikanov [11]. Kenny [12] investigated the flow experience a degree of breaking, mostly in the vicinity of the neutral point, owing to the effect of Lorentz forces acting upon the liquid-metal. Thess and Zikanov [13] investigated of the robustness of two-dimensional inviscid magnetohydrodynamic flows at low magnetic Reynolds numbers with respect to three dimensional perturbations. In the presence of a vertical magnetic field, convection may occur in vigorous cells separated by regions of strong magnetic field. This occurrence was studied by Dawes [14].

Investigation of the MHD instability in this paper due to slow hydromagnetic waves in the rotating fluid is encountered with complicated mathematics. However after simplifying the governing equations, we have a series of stability conditions. Simplifications have led us to investigate a very wide range of magnetic field profiles. It has been shown that regardless of the magnetic field profile, any unstable disturbances must have (Angular velocity)/(phase velocity in the azimuthal direction) with negative sign and must therefore propagate against the basic rotation. Sufficient stability conditions for azimuthal magnetic field have also been deduced.

2. Mathematical Formulation

To investigate the hydromagnetic stability of an inviscid, perfectly conducting incompressible and homogeneous fluid rotating with angular velocity Ω, it is convenient to choose a set of uniformly rotating cylindrical polar co-ordinates (r, θ, z) relative to which the fluid is at rest. The imposed magnetic field $H_0 = (0, A/r^n, B/r^n)$ varies in both magnitude and direction with distance from the rotational axis and the fluid is bounded by two infinitely long cylinders $r = r_1$ and $r = r_2$. The appropriate MHD equations relative to the rotating co-ordinate system are.

$$\frac{\delta \vec{V}}{\delta t} + (\vec{V}.\nabla)\vec{V} + 2\vec{\Omega}\times\vec{V} = -\frac{1}{\rho}\nabla P + \frac{\mu}{\rho}(\nabla\times\vec{H})\times\vec{H}, \tag{1}$$

where $\vec{V}$ is the velocity vector, t, p, ρ, μ, and $\vec{H}$ represent time, pressure, density, magnetic permeability, and magnetic intensity of the fluid, respectively. The basic state $\vec{V} = 0$, and $\vec{H} = \vec{H}_0$ are an exact solution of the equations

$$\nabla.\vec{V} = 0, \qquad 2(a)$$

$$\nabla.\vec{H} = 0, \qquad 2(b)$$

$$\frac{\delta \vec{H}}{\delta t} = \nabla\times(\vec{H}\times\vec{H}) . \qquad 2(c)$$

We perturb this basic state by small amounts $\vec{v}$ and $\vec{h}$ respectively, linearize the equations in the usual way and seek solutions in which all perturbation quantities ψ may be written

$$\psi = \hat{\psi}(r)\, e^{i(m\theta + kz - \omega t)} \quad (3)$$

where ω is in general complex number, m is an integer and k is any real number. We thus find

$$\hat{h}_r = \frac{\hat{v}_r}{\omega r^{n+1}} S(r) \quad 4(a)$$

$$\hat{h}_\theta = \frac{A\hat{v}^{/}{}_r}{i\omega r^n} - \frac{An\hat{v}_r}{i\omega r^{n+1}} - \frac{k}{\omega r^n}(B\hat{v}_\theta - A\hat{v}_z) \quad 4(b)$$

$$\hat{h}_z = \frac{B}{i\omega r} - \left(\frac{\hat{v}_r}{r^{n+1}}\right)^{/} - \frac{m}{\omega r^{n+1}}(A\hat{v}_z - B\hat{v}_\theta) \quad 4(c)$$

$$\hat{v}_z = -\frac{(r\hat{v}_r)^{/}}{rik} - \frac{m\hat{v}_\theta}{kr} \quad 4(d)$$

Eliminating $\hat{v}_z$ from equations (4) we get

$$\left(1 + \frac{m^2}{r^2k^2}\right) ri\hat{v}_\theta = -\frac{m}{k^2 r}(r\hat{v}_r)' - \frac{\hat{v}_r}{E(r)}\left(\frac{2\Omega r}{\omega} + \frac{2kM_\theta}{\omega^2}\left(M_z + \frac{mM_\theta}{rk}\right)\right), \quad (5)$$

where

$$E(r) = \left(M_z + \frac{mM_\theta}{rk}\right)^2 \frac{k^2}{\omega^2} - 1 = \frac{\mu}{\omega^2 \rho\, r^{2n+2}}(S(r))^2 - 1 \quad (6)$$

S(r) = mA + r k B (7)

$$M_\theta(r) = \frac{A\sqrt{\mu}}{r^n\sqrt{\rho}}, \qquad M_z(r) = \frac{B\sqrt{\mu}}{r^n\sqrt{\rho}} \quad (8)$$

therefore equation (5) becomes

$$\left(1 + \frac{m^2}{r^2k^2}\right) ri\hat{v}_\theta = -\frac{m}{k^2 r}(r\hat{v}_r)' - \frac{\hat{v}_r}{E(r)}\left(\frac{2\Omega r}{\omega} + \frac{2A\mu}{\omega^2 \rho r^{2n+1}} S(r)\right). \quad (9)$$

To replace $\hat{\psi}(r)$ by the more economical symbol $\psi(r)$ which then satisfies the following differential equation

$$Ev^{//} + \left[E^{/} + \frac{E}{r}\left(\frac{r^2 + 3m^2/k^2}{r^2 + m^2/k^2}\right)\right]v^{/} + Lv = 0 \quad (10)$$

where

$$L(r) = -\frac{2k^2\mu}{\omega^2\rho}\left[(n+1)A^2 + nB^2\right]\frac{1}{r^{2n+2}} + 4\left[\frac{\Omega}{\omega} + \frac{A\mu S(r)}{\rho\omega^2 r^{2n+2}}\right]^2 \frac{k^2}{E(r)} +$$
$$4\left[\frac{\Omega}{\omega} + \frac{AS(r)}{\mu\rho\omega^2 r^{2n+2}}\right] \frac{m}{r^2+l^2} - \frac{E(r)}{r^2+l^2}\left[r^2k^2 + l + 2m^2 + \frac{l^2}{r^2}\left(m^2-1\right)\right] \tag{11}$$

and $l = \frac{m}{k}$

Subsequent section will be concerned with the properties of this equation subject to the boundary conditions of no flow through the container walls, i.e, $v(r_1) = v(r_2) = 0$.

2.1 Azimuthal propagation of non-axisymmetric unstable modes

Equation (10) may be written

$$\left(Ev' \frac{r^3}{r^2+l^2}\right)' + \frac{r^3 Lv}{r^2+l^2} = 0 \tag{12}$$

where $E(r) = \left(M_z + \frac{mM_\theta}{kr}\right)\frac{1}{c^2} - 1, \quad c^2 = \frac{\omega^2}{k^2}$

Multiplying (12) by the complex congugate of v and integrating between the boundaries $r = r_1$ and $r = r_1$ (at which v must vanish) we find

$$\int_{r_1}^{r_2} \frac{r^3}{(r^2+l^2)}\left[L|v|^2 - E|v'|^2\right]dr = 0 \tag{13}$$

Now let $c = c_r + ic_i$, and multiply (13) by c^2. Equating the imaginary part of the left hand side to zero, we conclude that

$$2c_ic_r\int_{r_1}^{r_2} \frac{r^3}{(r^2+m^2k^{-2})}\left[|v'|^2 + E|v|^2\left(S_1(r) + \frac{2\Omega m}{(r^2+m^2k^{-2})kc_r} + \frac{4S_2(r)}{r^2|c^2E|^2}\right)\right]dr = 0 \tag{14}$$

where $S_1(r) = \frac{1}{(r^2+m^2k^{-2})}\left[r^2k^2 + l + 2m^2 + \frac{m^2}{k^2r^2}(m^2-1)\right]$ (15)

and

$$S_2(r) = \left(\Omega rc_r + \frac{A\mu S(r)}{\rho k r^{2n+1}}\right)^2 - \Omega^2r^2c_i^2 + \frac{\Omega r}{c_r}\left(\Omega rc_r + \frac{A\mu}{\rho k r^{2n+1}}S(r)\right)\left(\frac{S^2(r)\mu}{k^2\rho r^{2n+2}} - c_r^2 + c_i^2\right) \tag{16}$$

In the absence of rotation ($\Omega = 0$) the integrand in (14) is positive throughout the interval $r_1 \le r \le r_2$ (since the azimuthal wave number m may take only integral values), the integral therefore cannot vanish, (i) $c_r = 0$ but $c_i \neq 0$, in this case the motion is stable and does not propagate, (ii) $c_r \neq 0$ but $c_i = 0$, in this case the motion becomes oscillatory, (iii) $c_r = c_i = 0$, in this case there is no propagation, i.e. trivial solution.

On the other hand, non-axisymmetric disturbances may both grow in amplitude and propagate, and we now turn attention to such modes, for which $c_r c_i \neq 0$. First consider that $S_1(r)$ is always positive and that

$$S_2(r) = \frac{S(r)\mu^{1/2}}{kr^{n+1}\rho^{1/2}}\left[\frac{S(r)\mu^{1/2}}{kr^{n+1}\rho^{1/2}}\left(\frac{A^2\mu}{\rho r^{2n}}+\Omega^2 r^2\right)\right]+\frac{\Omega A\mu}{\rho r^{n+1}c_r}\left(\frac{S^2(r)\mu}{k^2\rho r^{2n+2}}+c_r^{\,2}+c_i^{\,2}\right) \tag{17}$$

unless the inequality

$$\frac{4S_2(r)}{r^2}+\frac{2\Omega m\,|\,c^2 E\,|^2}{(r^2+m^2k^{-2})kc_r}<0 \tag{18}$$

is satisfied somewhere in the interval $r_1 \leq r \leq r_2$ the integrand in (14) is everywhere positive, the integrand cannot vanish, and with out initial assumption $c_r c_i \neq 0$ we are led to a contradiction. We thus conclude that modes with $c_r c_i \neq 0$ must be such that

$$\frac{4}{r^2}\left(\frac{(mA+Bkr)\mu^{1/2}}{kr^{n+1}\rho^{1/2}}\right)\left[\frac{(mA+Bkr)\mu^{1/2}}{kr^{n+1}\rho^{1/2}}\left(\frac{A^2\mu}{\rho r^{2n}}+\Omega^2 r^2\right)\right]+\frac{\Omega A\mu}{\rho^{1/2}r^{n-1}c_r}\left(\frac{(mA+Bkr)^2\mu}{k^2\rho r^{2n+2}}+c_r^{\,2}+c_i^{2}\right)$$
$$+\frac{2\Omega m}{kc_r(r^2+m^2k^{-2})}\left[\left(\frac{(mA+Bkr)}{k^2r^{2n+2}\rho}-c_r^{\,2}+c_i^2\right)^2+4c_r^{\,2}c_i^{\,2}\right]<0 \tag{19}$$

somewhere in the interval $r_1 \leq r \leq r_2$. If the magnetic field is purely azimuthal then (19) becomes

$$\frac{4}{r^2}\left[\frac{m^2M_\theta^{\,2}}{r^2k^2}\left(M_\theta+\Omega^2r^2\right)+\frac{M_\theta^{\,2}\Omega m}{kc_r}\left(\frac{m^2M_\theta^{\,2}}{k^2r^2}+c_r^{\,2}+c_i^{\,2}\right)\right]$$
$$+\frac{2\Omega m}{kc_r(r^2+m^2k^{-2})}\left[\left(\frac{m^2M_\theta^{\,2}}{k^2r^2}-c_r^{\,2}+c_i^{\,2}\right)^2+4c_r^{\,2}c_i^{\,2}\right]<0 \tag{20}$$

whence it is clear that, regardless of the details of the magnetic field profile, any unstable disturbances must have

$$\frac{\Omega m}{kc_r}=\frac{\Omega m}{\omega_r}=\frac{\Omega}{c_{\theta r}}<0 \tag{21}$$

where $c_{\theta r} = \omega_r / m$ is the phase velocity in the azimuthal direction, and must therefore propagate against the basic rotation, i.e.`westward'. Comparing (20) and (21) it is also clear that this result still holds even if the magnetic field varies in both magnitude and direction with distance from the axis of rotation provided that $|M_z| \leq |mM_\theta/rk|$ everywhere in the interval $r_1 \leq r \leq r_2$. If the axial and azimuthal dimensions of an unstable disturbance are comparable it must therefore propagate westward, provided only that the axial magnetic field is somewhat less than the azimuthal field everywhere.

Finally we note that all unstable disturbances do in fact propagate westward when the magnetic field is purely axial, then (19) becomes

$$4M_z^{\,2}\Omega^2 + \frac{2\Omega m}{kc_r(r^2+m^2k^{-2})}\left[\left(M_z^{\,2}-c_r^{\,2}+c_i^{\,2}\right)^2+4c_r^{\,2}c_i^{\,2}\right]<0 \tag{22}$$

For unstable disturbance we must have (i) $\frac{m\Omega}{c_r}<0$, (ii) $kM_z > 0$, everywhere in the interval $r_1 \le r \le r_2$ and must therefore propagate against the basic rotation i.e. `westward'.

2.2 Sufficient conditions for stability: azimuthal magnetic field

We first note certain sufficient conditions under which unstable non-axisymmetric modes must propagate (in contrast to the non-rotating case). Supposing that they do not (i.e $c_r = 0$) in (14) becomes

$$\Omega c_i \int_{r_1}^{r_2} \frac{r^3\,|v|^2}{(r^2+m^2k^{-2})}\left[\frac{2(M_z+mM_\theta/rk)M_\theta}{r\,|(M_z+mM_\theta/rk)^2+c_i^{\,2}\,|}+\frac{m}{k(r^2+m^2k^{-2})}\right]dr=0 \tag{23}$$

we thus obtain a contradiction (for the left hand side cannot then vanish) if either (i) the field is purely aximuthal, (ii) the field is purely axial, or (iii) both components are present, but $|M_z| < |mM_\theta/rk|$ everywhere in the interval $r_1 \le r \le r_2$ and accordingly under any of these three conditions non-axisymmetric unstable disturbances in a rotating fluid must propagate.

We confine attention in the remainder of this section to the case in which the field is entirely azimuthal. When investigating non-axisymmetric unstable modes it is therefore appropriate to take both c_r and c_i as non-zero. In this case the imaginary part of (13) becomes

$$\frac{2\omega_r\omega_i}{|\omega|^2}\int_{r_1}^{r_2} \frac{r^3}{(r^2+m^2k^{-2})}\left[\frac{A^2m^2\mu}{r^{2n+2}\rho\,|\omega|^2}\,|v'|^2+S_3(r)\,|v|^2\right]dr=0\,, \tag{24}$$

where

$$S_3(r)=\frac{4k^2\Omega^2\,|\omega|^2}{|\omega^2E|^2}+\frac{2k^2A}{|\omega|^2}(n+1)r^{2n-2}-\frac{2\Omega m}{\omega_r}[\frac{2k^2A^2\mu}{|\omega^2E|^2\;r^{2n+2}\rho}.$$

$$(\frac{A^2m^2\mu}{r^{2n+2}\rho}-3\omega_r^{\;2}+\omega_i^{\;2})+\frac{1}{r^2+m^2k^{-2}}]+\frac{m^2A^2\mu}{r^{2n+2}\rho|\omega|^2}[S_1(r)-\frac{4}{r^2+m^2k^{-2}} \tag{25}$$

$$-\frac{4k^2}{m^2}(1+\frac{|\omega|^2\;r^{2n+2}\rho}{A^2m^2\mu}(\frac{A^2m^2\mu}{r^{2n+2}}-2\omega_r^{\;2}+2\omega_i^{\;2})^{-1})^{-1}]$$

and $E(r)=\dfrac{m^2A^2\mu}{\rho r^{2n+2}\omega^2}-1$. If m = 0, then E = -1 and

$$S_3(r)=\frac{k^2}{|\omega|^2}\left[4\Omega^2+\frac{2A^2(n+1)\mu}{\rho\,r^{2n+2}}\right]=\frac{k^2}{|\omega|^2}\left[4\Omega^2-r\left(\frac{M_\theta^{\;2}}{r^2}\right)^{/}\right] \tag{26}$$

As far as non-axisymmetric disturbances are concerned we know from the previous section that all unstable modes drift west. So consider here $\Omega m\omega_r < 0$. If $M_\theta m^2 > 3\,r^2\,\omega_r^2$ everywhere in the interval the second term on the right hand side of (25) will be positive. The final term exceeds.

$$\begin{aligned}&\frac{m^2A^2\mu}{r^{2n+2}\rho|\omega|^2}\left[S_1(r)-\frac{4}{r^2+m^2k^{-2}}-\frac{4k^2}{m^2}\right]\\&=\frac{m^2A^2\mu}{r^{2n+2}(r^2+k^{-2})\rho|\omega|^2}\left[k^2r^2\left(1-\frac{4}{m^2}\right)+(2m^2-7)+\frac{m^2}{r^2k^2}\left(m^2-1\right)\right]\end{aligned} \tag{27}$$

which for |m| > 1 (and, of course, integral) is patently positive. Inspection of (24) then leads to the conclusion that unstable non-axisymmetric modes, such that $M_\theta^{\;2}m^2>3r^2\omega^2$ everywhere and |m| > 1, can only occur if $M_\theta^{\;2}/r^2$ increases with radius somewhere in the interval $r_1 \le r \le r_2$.

2.3 Local stability analysis: Azimuthal magnetic field

2.3.1 Axisymmetric disturbances

If m = 0, then E= -1 and $M_z=\dfrac{B\sqrt{\mu}}{r^n\sqrt{\rho}}=0$.

Therefore $L(r)=-\dfrac{k^2}{\omega^2}\left[\dfrac{2A^2(n+1)\mu}{\rho r^{2n+2}}+4\Omega\right]+\left(k^2+\dfrac{1}{r^2}\right)$.

Equation (10) may be written as

$$v^{//}+\frac{v^{/}}{r}-\left[k^2+\frac{1}{r^2}-\frac{k^2}{\omega^2}\left(\frac{2A^2(n+1)\mu}{\rho r^{2n+2}}+4\Omega^2\right)\right]v=0 \tag{28}$$

Now consider the local solution of (28) in the neighbourhood of a particular radius $r = r_0$ so that the co-efficients may be regarded as uniform (to a first approximation) in that neighbourhood. The equation then admits solution $v = e^{lir}$ where l is a local radial wave number satisfying

$$l^2-\frac{il}{r_0}+k^2+\frac{1}{r_0}-\frac{k^2}{\omega^2}\left[\left(\frac{2A^2(n+1)\mu}{\rho r^{2n+2}}\right)_{r=r_0}+4\Omega^2\right]=0 \tag{29}$$

If $l >> 2\pi / r_0$, and we must therefore for consistency neglect the second and fourth terms in (29) compared with the first, whence

$$\omega^2=\frac{1}{l^2+k^2}\left[\frac{2A^2k^2(n+1)\mu}{\rho r^{2n+2}}+4\Omega^2k^2\right]_{r=r_0}$$

ω^2 is positive, implies that ω is real. The system will be oscillatory.

2.3.2 Non-axisymmetric disturbances

Setting $M_z = 0$ in (10) assuming $E'(r_0) \sim E(r_0)/r_0$ and $l >> 2\pi\, r_0$

$$L(r_0)=\frac{k^2}{\omega^2}\left[r_0\left(\frac{A^2\mu}{\rho r^{2n+2}}\right)^{/}_{r=r_0}\right]+4\left[\frac{\Omega}{\omega}+\frac{m}{\omega^2}\left(\frac{A^2\mu}{\rho r^{2n+2}}\right)_{r=r_0}\right]\frac{k^2}{E(r_0)}$$

$$+\frac{4m}{r_0^2+m^2k^{-2}}\left[\frac{\Omega}{\omega}+\frac{m}{\omega^2}\left(\frac{A^2\mu}{\rho r^{2n+2}}\right)_{r=r_0}\right]-\frac{E(r_0)}{r_0^2+m^2k^{-2}}\left[r_0^2k^2+l+2m^2+\frac{m^2}{k^2r_0^2}\left(m^2-1\right)\right]$$

We find

$$\frac{4k^2}{E(r_0)}\left[\frac{\Omega}{\omega}+\frac{m}{\omega^2}\left(\frac{A^2\mu}{\rho r^{2n+2}}\right)_{r=r_0}\right]^2+\frac{4m}{r_0^2+m^2k^{-2}}\left[\frac{\Omega}{\omega}+\frac{m}{\omega^2}\left(\frac{A^2\mu}{\rho r^{2n+2}}\right)_{r=r_0}\right]$$

$$-\left(l^2+k^2+\frac{m^2}{r_0^2}\right)E(r_0)+\frac{r_0k^2}{\omega^2}\left(\frac{A^2\mu}{\rho r^{2n+2}}\right)^{/}_{r=r_0}=0 \tag{30}$$

Our main interest lies in the `slow' hydromagnetic waves that propagate in a `rapidly' rotating fluid. We do not yet know the growth rate of such waves. It is helpful at this stage to replace $E=\frac{m^2A^2\mu}{\omega^2\rho r^{2n+2}}-1$ by $\frac{m^2A^2\mu}{\omega^2\rho r^{2n+2}}$. We shall have to justify this step a posteriori. Equation (30) then becomes a quadratic equation for ω with roots given by

$$\omega = \frac{m(M_\theta^2/r^2)_{r=r_0}}{2\Omega}\left[-2-\frac{m^2}{k^2(r_0^2+m^2k^{-2})} \pm \left\{(4+\frac{4m^2}{k^2(r_0^2+m^2k^{-2})}+\frac{m^4}{k^4(r_0^2+m^2k^{-2})}) - (1+\frac{m^2}{r_0^2k^2}+\frac{l^2}{k^2}+\frac{m^2}{k^4(r_0^2+m^2k^{-2})})m^2 - \frac{r_0\left(M_\theta^2/r^2\right)'_{r=r_0}}{\left(M_\theta^2/r^2\right)_{r=r_0}}\right\}^{1/2}\right] \tag{31}$$

Noting that

$$m^2\left(1+\frac{m^2}{r_0^2k^2}+\frac{l^2}{k^2}+\frac{m^2}{k^4(r_0^2+m^2k^{-2})}\right) > \left(4+\frac{4m^2}{k^2(r_0^2+m^2k^{-2})}+\frac{m^4}{k^4(r_0^2+m^2k^{-2})}\right) \tag{32}$$

if |m| > 1, we conclude that when $\left(A^2\mu/r^{2n+2}\right)'_{r=r_0} \le 0$ all modes with |m| > 1 are stable and may propagate both east and west if $\left(A^2\mu/r^{2n+2}\right)'$ is positive and sufficiently large, however, then unstable modes will result and such waves propagate westward in accord with the much more general conclusions. For unstable modes we clearly require

$$\frac{r_0\left(A^2\mu/r^{2n+2}\right)'_{r=r_0}}{\left(A^2\mu/r^{2n+2}\right)_{r=r_0}} > \frac{m^2l^2}{k^2} \tag{33}$$

and since (for a local analysis to be appropriate), we must have $l^2 \gg r_0^2$, (33) can only be satisfied if $m^2/k^2 r_0 \ll 1$ (assuming $r_0\left(A^2\mu/r^{2n+2}\rho\right)'_{r=r_0} \sim \left(A^2\mu/r^{2n+2}\rho\right)_{r=r_0}$. The character of the unstable waves is therefore displayed by a somewhat simplified version of (31)

$$\omega = \frac{\left[-2\pm(k_1-k_2)^{1/2}\right]}{D}. \tag{34}$$

where $k_1 = m^2\left(1+\frac{l^2}{k^2}\right)$, $k_2 = \frac{r_0\left(M_\theta^2/r^2\right)'_{r=r_0}}{\left(M_\theta^2/r^2\right)_{r=r_0}}$, $D = \frac{2\Omega}{m\left(\left(M_\theta^2/r^2\right)_{r=r_0}\right)}$,

k_1, k_2 and D are real. If $k_1 < k_2$, then equating real and imaginary parts from (34)

$$\omega_r = \frac{-2}{D}, \qquad \omega_i = \frac{k_1-k_2}{D}$$

(i) the motion is stable when $\omega_i > 0$, and (ii) the motion is unstable when $\omega_i < 0$.

Conclusion

A series of stability conditions have been derived for wide range of magnetic field profiles. It has been shown that regardless of the magnetic field profile, any unstable disturbances must have $\Omega/c_{\theta r} < 0$ and must therefore propagate against the basic rotation. This result still holds even if the magnetic field varies in both magnitude and direction with distance from the rotational axis provided that $|M_z| \leq |mM_\theta/rk|$ everywhere in the interval $r_1 \leq r \leq r_1$. If the axial and azimuthal dimensions of an unstable disturbances are comparable it must therefore propagate westward, provided only that the axial magnetic fields is somewhat less than the azimuthal field everywhere. All unstable disturbances propagate westward when the magnetic field is purely axial. We conclude that when $\left(A^2 \mu / r^{2n+2}\right)'_{r=r_0} \leq 0$ all modes with $|m| > 1$ are stable and propagate both east and west. When $\left(A^2 \mu / r^{2n+2}\right)'_0$ is positive and sufficiently large, however, then unstable modes will result and such waves propagate westward in accord with the very much more general conclusions.

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