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CHARACTERIZATION OF MODULAR JOIN-SEMILATTICES

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ABSTRACT

In this paper we have proved a classical characterization of modular join-semilattices. We have also given some characterizations of modular ideals of join-semilattices through congruences.

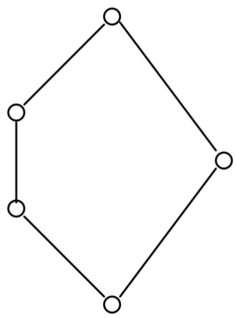
Key words: Join-semilattices, modular semilattices, distributive semilattices, quotient semilattices

1. Introduction

A classical characterization in lattices is:

- A lattice *L* is modular if and only if it has no sublattice isomorphic to the pentagonal lattice [5, 6].

For the pentagonal lattice see Figure 1.



R_5

Figure-1

Grätzer and Schmidt [4] first introduced the notion of modularity in semilattices. Rhodes [7] characterized the modular meet-semilattices like as the classical characterization for modular lattice. In section 3, we prove these results for join-semilattices. We claim that our arguments make the proof easier than Rhodes' proof.

Cornish [2] characterized the modular join-semilattices in terms of congruences. The notion of standard and distributive element (ideal) [3] has been introduced to study on lattices in general. Talukder and Noor [8, 9] introduced the notion of a modular element (ideal) in a join-semilattice. For this notion we can study the join-semilattices in general. Talukder and Noor [8, 9] proved some parallel results, of Cornish [2], for modular ideals in a join-semilattice. In section 4, we give some more results which characterize modular ideal in a join-semilattice. This paper is based on [1].

2. Preliminaries

A join-semilattice **S**: = $\langle S;\vee\rangle$ is an algebra of type $\langle 2\rangle$ that satisfies, for all $a, b, c \in S$

(i) $a \vee a = a$ ($\vee$ is idempotent)

(ii) $a \vee b = b \vee a$ ($\vee$ is commutative)

(iii) $a \vee (b \vee c) = (a \vee b) \vee c$ ($\vee$ is associative).

We will denote a join-semilattice as algebra, by **S**: = $\langle S;\vee\rangle$ or simply **S** if there is no confusion.

A join-semilattice S is said to be modular join-semilattice if for all $x, y, z \in S$ with $z \leq x$, $x \leq y \vee z$, implies $x = y_1 \vee z$ for some $y_1 \leq y$ and $y_1 \in S$.

The set $[a, b] = \{x \mid a \leq x \leq b\}$ is called the closed interval from a to b. Clearly, $[a, b]$ is a join-semilattice.

Let **S** and **T** be two join-semilattices. A map $\psi : S \to T$ is said to be a homomorphism if ψ is a join preserving map. That is, for all $a, b \in S$,

$$\psi(a \vee b) = \psi(a) \vee \psi(b) \; in \; T$$

A one-to-one homomorphism is called a monomorphism or an embedding. A onto homomorphism is called an epimorphism. If a map $\psi : A \to B$ is an epimorphism, we say that B is a homomorphic image of A. An epimorphism is called an isomorphism if it is one-to-one map.

Let **S** be a join-semilattice. A non empty set I of S is called an ideal if,

(i) $a, b \in I$ implies $a \vee b \in I$ and

(ii) $a \in S$, $b \in I$ with $a \leq b$ implies $a \in I$.

Equivalently by [7], a nonempty subset I of a join-semilattice S is called an ideal if,

$$a \vee b \in I, \text{ if and only if } a \in I \text{ and } b \in I$$

for all $a, b \in S$.

3. A classical characterization

Let P and Q be two ordered sets. A map $f : P \to Q$ is said to be order preserving if $f(a) \leq f(b)$ whenever $a \leq b$.

Lemma 3.1 *Let L and K be two join-semilattices. Every homomorphism $f : L \to K$ is an order preserving map.*

Proof: Let $a, b \in L$ with $a \leq b$. Since $f : L \to K$ is a homomorphism so, $f(a) \vee f(b) = f(a \vee b) = f(b)$. This implies $f(a) \leq f(b)$ in K. Hence f is an order preserving map.

A join-semilattice **R** is called a retract of a join-semilattice **S** if there are homomorphisms $f : S \to R$ and $g : R \to S$ such that $f \circ g = I_R$, the identity map on R. Clearly, f is an epimorphism and g is a monomorphism. If **R** is a subsemilattice of **S** and there exists an epimorphism $h : S \to R$ such that $h \uparrow_R = I_R$, then **R** is certainly a retract of **S**. In this case h is called a retraction.

The dual (that is, for meet-semilattice) of the following theorem stated in [7] without proof and the proof is given in [11]. Here we prove the result for a join-semilattice as we need in this paper.

Theorem 3.2 *A retract of a modular join-semilattice is a modular join-semilattice.*

Proof: Suppose S is an modular join-semilattice and let R be a retract of S. Then there exist an epimorphism $f : S \to R$ and a monomorphism $g : R \to S$ such that $f \circ g = I_R$. Let $x, y, z \in R$ with $z \leq x$ such that $x \leq y \vee z$. Then by lemma 3.1 $g(x) \leq g(y) \vee g(z),$ as g is a homomorphism. Also $z \leq x$ implies $g(z) \leq g(x)$. Since S is modular so there exist $y_1 \leq g(y)$ such that $g(x) = y_1 \vee g(z),$ where $y_1 \in S.$ Thus $(f \circ g)(x)$ $= f(y_1) \vee (f \circ g)(z).$ This implies $x = f(y_1) \vee z$, where $y_1 \leq g(y)$ implies $f(y_1) \leq (f \circ g)(y) = y$. Therefore R is modular.

For any $a, b \in S$, the interval $[a, b]=\{x \mid a \leq x \leq b\}$ is clearly a join-semilattice. We have the following result:

Theorem 3.3 *Let S be a join-semilattice. For a, b$\in$S, the interval [a, b] is retract of S.*

Proof: Define a map $f : S \to [a,b]$ such that

$$f(x) = \begin{cases} x \vee a & if\ x \vee a \leq b \\ b & if\ x \vee a \leq b \end{cases}$$

let $y \in [a,b]$, this implies $a \leq y \leq b$. Hence $f(y) = y \vee a = y.$ Therefore, clearly f is an epimporphism. Thus [a, b] is a retract of S.

We can easily prove that if B is a retract of A and C is a retract of B, then C is a retract of A. Now we prove the following important characterization of modular join-semilattice. Rodes [7] proved the result for the case of meet-semilattice. Our case is the dual of meet-semilattice. Moreover our argument makes the proof more simpler than the proof of Rodes [7].

Theorem 3. 4 *Let S be a join-semilattice. Then the followings are equivalent:*

(a) S is modular;

(b) S is directed bellow and it does not contain a retract isomorphic to the pentagonal lattice.

Proof: $(a) \Rightarrow (b)$. Suppose S is a modular join semi lattice, then each pair of elements of S has a lower bound. Let R be a retract of S, then by theorem 3.2 R is a modular join-semilattice. Hence R can not be isomorphic to the pentagonal lattice.

$(b) \Rightarrow (a)$ Suppose S is directed below non modular join–semilattice. We shall construct a retract of S isomorphic to the pentagonal lattice. Since S is non modular, there exist $a,b,c \in S$ where $c \leq a \vee b$ with $a \leq c$ such that $c \neq y \vee a$ for all $y \leq b.$ Clearly $a \vee b = b \vee c$. Since S is directed below, there is $l \leq a,b.$ Set $L = \{l,a,b,c,a \vee b\}$. We show that L is a retract of $[l, a \vee b]$. Let $W = \{w \in [l, a \vee b] \mid w \leq b,c\}.$

Define $f : [l, a \vee b] \to L$ given by,

$$f(x) = \begin{cases} l, & if\ x \in W \\ b, & if\ x \leq b\ and\ x \not\leq c \\ c, & if\ x \not\leq b, x \leq c\ and\ x \not\leq a \vee z for\ all\ z \in W \\ a, & if\ x \not\leq b\ and\ x \leq a \vee z\ for\ some\ z \in W \\ a \vee b & if\ x \not\leq b\ and\ x \not\leq c \end{cases}$$

Clearly, f is well defined. We must have to show that f is a homomorphism. Let $x, y \in [l, a \vee b].$

Case 1: $f(x) = a.$ then $x \not\leq b$ and $x \leq a \vee z$ for some $z \in W$. Since $x \not\leq b$ we have $x \vee y \not\leq b$ for each $y \in [l, a \vee b]$.

Suppose $f(y) = a.$ then $y \not\leq b$ and $y \leq a \vee w$ for some $w \in W$. So $x \vee y \leq b$ and $x \vee y = a \vee w$ some $w \in W$, Thus $f(x \vee y) = a = f(x) \vee f(y).$

Suppose $f(y) = l$, then the proof is trival.

Suppose $f(y) = c$, then $y \not\leq b$, $y \leq c$ and $y \leq a \vee p$ for every $p \in W$. So $x \vee y \not\leq a \vee p$ for every $p \in W$. Since $x \leq a \vee z$ for some $z \in W$, we have $x \leq a \vee c = c$ so $x \vee y \leq c$, hence $f(x \vee y) = c = f(x) \vee f(y)$.

Suppose $f(y) \in \{b, a \vee b\}$. Then $y \leq c$ so $x \vee y \not\leq c$ and $x \vee y \leq b,$ hence $f(x \vee y) = a \vee b = f(x) \vee f(y)$.

Case 2: $f(x) = l.$ Then $x \in W$

Suppose $f(y)=l$, then $y \in W$. Hence $x \vee y \in W$, so $f(x \vee y)=l= f(x) \vee f(y)$.

Suppose $f(y)=a$. Then $x \vee y \not\leq b$ and $x \vee y = a \vee z$ for some $z \in W$.

Henve $f(x \vee y)=a= f(x) \vee f(y)$

Suppose $f(y)=b$. Then $x \vee y \leq b$ and $x \vee y \not\leq c$, hence $f(x \vee y)=b= f(x) \vee f(y)$

Suppose $f(y)=c$, then $x \vee y \not\leq b$, $x \vee y \leq c$ and $x \vee y \not\leq a \vee z$ for ever $z \in W$, hence $f(x,y)=c= f(x) \vee f(y)$

Suppose $f(y)=a \vee b$. Then $x \vee y \not\leq b$ and $x \vee y \not\leq c$ hence $f(x \vee y)=a \vee b= f(x) \vee f(y)$.

Case 3: $f(x)=a \vee b$**.** Then $x \not\leq b$ and $x \not\leq c$. Hence for any $y \in [l, a \vee b]$. We have $x \vee y \not\leq b$ and $x \vee y \not\leq c$. Therefore $f(x \vee y)=a \vee b= f(x) \vee f(y)$

Case 4: $f(x)=b$. Then $x \leq b$ and $x \not\leq c$. Since $x \not\leq c$ so $x \vee y \not\leq c$ for all $y \in [l, a \vee b]$. Suppose $f(y) \in [l,b]$, then $y \leq b$ and hence $x \vee y \leq b$. Therefore $f(x \vee y)=b= f(x) \vee f(y)$. Suppose $f(y) \in \{a,c,a \vee b\}$, then $y \leq b$ and hence $x \vee y \leq b$ therefore $f(x \vee y)=a \vee b= f(x) \vee f(y)$.

Case 5: $f(x)=c$. Then $x \not\leq b, x \leq c$ and $x \not\leq a \vee z$ for every $z \in W$. Therefore for every $y \in W$ we have $x \vee y \leq b$ and $x \vee y \leq a \vee z$ for every $z \in W$.

Suppose $f(y) \in \{l,a,c\}$. Then $y \leq c$ and hence $x \vee y \leq c$.
Therefore $f(x \vee y)=c= f(x) \vee f(y)$

Suppose $f(y) \in \{b, a \vee b\}$. then $y \not\leq c$ and hence $x \vee y \not\leq c$. Therefore $f(x \vee y)=a \vee b= f(x) \vee f(y)$.

This prove that L is an epimorphism image of $[l, a \vee b]$ and since it is obviously a subjoin-semilattice, L is a retract of $[l, a \vee b]$. Hence by theorem 3.3 L is a retract of S. This completes the proof.

4. Quotient structure

An equivalence relation Θ on a join-semilattice **S** is called a congruence relation on *S* if

$$a \equiv b(\Theta) \text{ and } c \equiv d(\Theta) \text{ implies that } a \vee c \equiv b \vee d(\Theta)$$

where $a, b, c, d \in S$.

Let **S** be a join-semilattice and *I* be an ideal of **S**. Then the congruence Θ (*I*), defined by

$$x \equiv y(\Theta(I))(x, y \in S) \text{ if and only if } x \vee i = y \vee i \text{ for some } i \in I.$$

has *I* as a congruence class. If **S** is downwards directed then $\Theta(I)$ is the smallest congruence of *S* containing *I*. We denote the quotient lattice of all the congruence classes of Θ (*I*) by $S/\Theta(I)$.

Now we have the following result.

Theorem 4.1 *Let **S** be a modular join semilattice. The every ideal J of **S** is modular and moreover S / Θ (J) is modular.*

The mapping $\varphi : S \to S/\Theta(I)$ is said to be canonical homomorphism if for all $x \in S$,

$$\varphi(x) = [x]\Theta(I)$$

The following characterizations of modular join semilattice due to [9].

Theorem 4.2 (Theorem 2.2 [10]) *Let M be an ideal of a join semilattice directed below **S**. Then M is modular if and only if* $\Theta(M)\uparrow_K = \Theta(M \cap K)\uparrow_K$ *for all* $K \in I(S)$.

Theorem 4.3 (Theorem 3.4 [10]) *Let **S** be a join semilattice directed below and let J be an ideal of **S**. For an ideal I, let* $\varphi : S \to S/\Theta(I)$) *is the canonical homomorphism. Then the following conditions are equivalent:*

(i) J is modular,

(ii) For any $I \in I(S)$ *and* $x \in I \vee J$ *implies that* $x \equiv j\,\Theta\,(I)$ *for some* $j \in J$,

(iii) $\varphi(I \vee J) = \varphi(J)$,

(iv) $\varphi^{-1}\varphi(J) = I \vee J$

(v) $\varphi(J)$ *is an ideal of* $S/\Theta(I)$.

Now we prove our main results.

Theorem 4.4 *Let **S** be a join-semilattice and J be an ideal of **S**, for an ideal I of **S**, if* $\varphi : S \to S/\Theta(I)$ *is the canonical homomorphism then the following condition are equivalent:*

(i) J is modular.

(ii) For any $I \in I(S)$, $\varphi(J) = (\varphi(J)]$ *in* $S/\Theta(I)$.

(iii) For any $I, K \in I(S)$, $\varphi(J \vee K) = (\varphi(J)] \vee (\varphi(K)]$ *in* $S/\Theta(I)$.

Proof: $(i) \Rightarrow (ii)$ Suppose (i) holds. So by (v) of theorem 4.3 $\varphi(J)$ is an ideal of $S/\Theta(I)$. Since $\varphi(J)$ is an ideal it is obvious that $\varphi(J) = (\varphi(J)]$ in $S/\Theta(I)$. Thus (ii) holds.

(ii) $\Rightarrow (iii)$ Suppose (ii) holds. Hence by (iii) of theorem 4.3 we have $\varphi(J \vee K)$ $= \varphi(J) \subseteq (\varphi(J)] \vee (\varphi(K)]$. Now $\varphi(J) = \varphi(J \vee K)$. So by (ii) $(\varphi(J)] = \varphi(J \vee K)$. Again $\varphi(K) \subseteq \varphi(J \vee K) = \varphi(J)$.

So $(\varphi(K)] \subseteq \varphi(J) = \varphi(J \vee K)$. Hence $(\varphi(J)] \vee (\varphi(K)] \subseteq \varphi(J \vee K)$.

Therefore $\varphi(J \vee K) = (\varphi(J)] \vee (\varphi(K)]$ in $S/\Theta(I)$.

(iii) $\Rightarrow (ii)$ Suppose (iii) holds. If in (iii) we replace K by J we get

$\varphi(J \vee J) = (\varphi(J)] \vee (\varphi(J)]$, Hence $\varphi(J) = (\varphi(J)]$.

(ii) $\Rightarrow (i)$ Suppose (ii) holds.

Theorem 5 *Let S be a join semilattice and let J be an ideal of S. The following conditions are equivalent:*

(i) J is modular.

(ii) The canonical map $\psi : K/\Theta(J \cap K) \rightarrow J \vee K/\Theta(J)$ *for any* $K \in I(S)$ *is one-to-one.*

(iii) The canonical map $\psi : K/\Theta(J \cap K) \rightarrow J \vee K/\Theta(J)$ *for any* $K \in I(S)$ *is onto.*

(iv) The canonical map $\psi : K/\Theta(J \cap K) \rightarrow J \vee K/\Theta(J)$ *for any* $K \in I(S)$ *is an isomorphism.*

Proof. (i) $\Leftrightarrow$ (ii). Let $[x]\Theta(J) = [y]\Theta(J)$ for $x, y \in K$. By the Theorem 4.2 we have $[x]\Theta(J \cap K) = [y]\Theta(J \cap K)$. The reverse argument gives us the reverse implication.

(i) $\Leftrightarrow$ (iii). Let $[x]\Theta(J) \in J \vee K = \Theta(J)$. This implies $x \in J \vee K$. Hence by the Theorem 4.3 we have $x \equiv k\Theta(J \cap K)$ for some $k \in K$. Hence by Theorem 4.2 we have

$x \equiv k\Theta(J \cap K)$. Hence $[x]\Theta(J) = [k]\Theta(J \cap K)$ for some $k \in K$. The reverse argument gives us the reverse implication.

(i) $\Leftrightarrow$ (iv). Let $[x], [y] \in K/\Theta(J \cap K)$. Then $\psi([x] \vee [y]) = \psi([x \vee y] = [x \vee y]\Theta(J) = [x]\Theta(J) \vee [y]\Theta(J) = \psi[x] \vee \psi[y]$*:* Hence by (ii) and (iii) we have (iv) holds. The reverse argument give us the reverse implication.

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