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THE LOTKA-VOLTERRA MODEL: AN APPROACH BY THE CAS

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ABSTRACT

In this paper, we investigate and compare the trajectories of a well-known prey-predator model named Lotka - Volterra Model including the effects of the trajectories of this model by changing its different parameters. The Computer Algebra System (CAS) MATHEMATICA5.0 is used to get the graphical consequences.

1. Introduction

One of the first models to incorporate interactions between predators and prey was proposed in 1925 by the American biophysicist Alfred Lotka and the Italian mathematician Vito Volterra. Unlike the Malthusian and Logistic models the Lotka-Volterra model is based on differential equations very deeply. The Lotka-Volterra model describes interactions between two species in an ecosystem, a predator and a prey. Since we are considering two species, the model will involve two equations, one which describes how the prey population changes and the second which describes how the predator population changes .Since this model is studied as a basic model in modeling of two species population, so studying of this model occupies great importance in modeling science. In the Lotka-Volterra-model it is assumed that the death rate of the prey depends on the number of predators. The larger the predator population, the more prey animals will fall a victim. On the other hand, the predators are better off if many prey animals are available. The Lotka-Volterra-model is also a feedback model, i.e. the prey population has a positive effect on the size of the predator population, whereas the latter has a negative (inhibiting) effect on the size of the prey population.

2. The Lotka-Volterra Model

We consider the Lotka-Volterra Predator-Prey Model [5] as follows:

$$\left.\begin{aligned} &\frac{dP_1}{dt} = \alpha P_1 - \beta P_1 P_2 \\ &\frac{dP_2}{dt} = -\gamma P_2 + \eta P_1 P_2 \\ &P_1(0) = P_{1_0} > 0, P_2(0) = P_{2_0} > 0 \end{aligned}\right\} \qquad (2.1)$$

Where $P_1(t)$ and $P_2(t)$ denotes the number of prey and predator at time t respectively and the parameters α, β, γ and η are positive constants.

3. Preliminaries

As the system (2.1) consists a system of nonlinear differential equations that cannot be separated from each other and that cannot be solved in closed form. In this regards we will discuss the qualitative and quantitative behavior of the system.

3.1 Theorem: Each trajectory for the Lotka - Volterra model (2.1) through each point $\left(P_{1_0}, P_{2_0}\right)$ on the positive quadrant, in P_1P_2 - plane is a closed oval curve.

Proof: From (2.1) we get, $\dfrac{dP_2}{dP_1} = \dfrac{P_2(\eta P_1 - \gamma)}{P_1(\alpha - \beta P_2)}$

$$\Rightarrow \frac{dP_2(\alpha - \beta P_2)}{P_2} = \frac{dP_1(\eta P_1 - \gamma)}{P_1}$$

$$\Rightarrow \left(\frac{\alpha}{P_2} - \beta\right) dP_2 = \left(\eta - \frac{\gamma}{P_1}\right) dP_1 \qquad (3.1.1)$$

Integrating (3.1.1), we obtain

$$\alpha \ln \frac{P_2}{P_{2_0}} - \beta\left(P_2 - P_{2_0}\right) = \eta\left(P_1 - P_{1_0}\right) - \gamma \ln \frac{P_1}{P_{1_0}}$$

$$\Rightarrow \ln\left(\frac{P_2}{P_{2_0}}\right)^{\alpha} + \ln\left(\frac{P_1}{P_{1_0}}\right)^{\gamma} = \beta\left(P_2 - P_{2_0}\right) + \eta\left(P_1 - P_{1_0}\right)$$

$$\Rightarrow \ln\left(\frac{P_2}{P_{2_0}}\right)^{\alpha} \cdot \left(\frac{P_1}{P_{1_0}}\right)^{\gamma} = \ln\left\{e^{\beta\left(P_2 - P_{2_0}\right) + \eta\left(P_1 - P_{1_0}\right)}\right\}$$

$$\Rightarrow \left(\frac{P_2}{P_{2_0}}\right)^{\alpha} \cdot \left(\frac{P_1}{P_{1_0}}\right)^{\gamma} = e^{\beta\left(P_2 - P_{2_0}\right) + \eta\left(P_1 - P_{1_0}\right)}$$

$$\Rightarrow \frac{P_1^{\gamma}}{P_{1_0}^{\gamma}} \cdot \frac{P_2^{\alpha}}{P_{2_0}^{\alpha}} = \frac{e^{\beta P_2}}{e^{\beta P_{2_0}}} \cdot \frac{e^{\eta P_{1_0}}}{e^{\eta P_{2_0}}}$$

$$\Rightarrow \frac{P_1^{\gamma}}{e^{\eta P_1}} \cdot \frac{e^{\eta P_{1_0}}}{P_{1_0}^{\gamma}} = \frac{e^{\beta P_2}}{P_2^{\alpha}} \cdot \frac{P_{2_0}^{\alpha}}{e^{\beta P_{2_0}}}$$

$$e^{-\eta P_1} P_1^{\gamma} = \lambda e^{\beta P_2} P_2^{-\alpha} \qquad (3.1.3)$$

where, λ is a constant and $\lambda = \frac{P_{1_0}^{\ \gamma}}{e^{\eta P_{1_0}}} \cdot \frac{P_{2_0}^{\ \alpha}}{e^{\beta P_{2_0}}}$

These curves in the P_1P_2-Plane are called the trajectories for our model.

To discuss the nature of these curves, we first find their points of intersection with lines parallel to the axis of coordinates. If we consider the point of intersection with the line P_2=k, we get,

$$e^{-\eta P_1} P_1^{\gamma} = \lambda e^{\beta k} k^{P_1} = \frac{1}{\mu}_{(say)}$$

$$\text{or} \quad F(P_1) \equiv e^{\eta P_1} - \mu P_1^{\gamma} = 0 \tag{3.1.5}$$

Since F(0) >0, F(∞)>0, (3.1.4) either does not give any positive real root or it gives an even number of positive values of P_1. Further, the roots of (3.1.4) are determined by the abscissae of the points of intersection of the curves

$$P_2 = e^{\eta P_1} \text{ and } P_2 = \mu P_1^{\gamma} \tag{3.1.6}$$

It is easily seen that the two curves intersect in two distinct real points (when $\mu = \mu_2$) or in two coincident real points (when $\mu = \mu^*$) or do not intersect in any point $\mu = \mu_1$, (Figure 1)

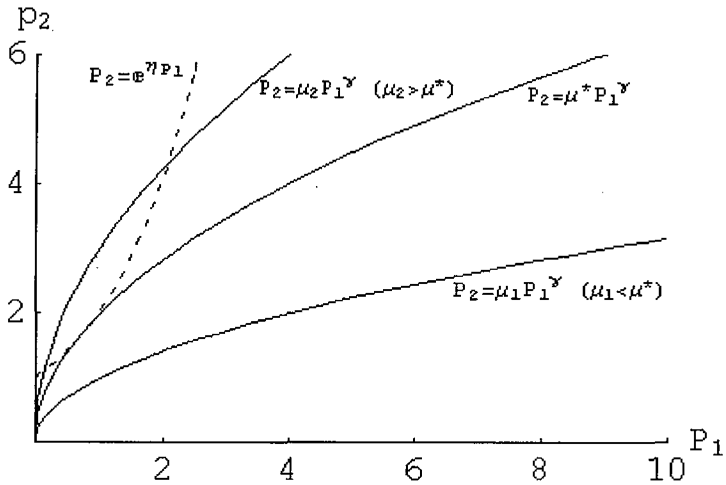

Figure1: Determining the point of intersection of curves.

The condition that the curves touch each other is obtained by eliminating P_1 between the equations

$$\mu P_1^{\gamma} = e^{\eta P_1} \quad \text{and} \quad \mu \gamma P_1^{\gamma - 1} = \eta e^{\eta P_1} \tag{3.1.7}$$

With the critical value of μ and the value P_1 as

$$\mu^* = \frac{e^{\gamma}}{\left(\frac{\gamma}{\eta}\right)^{\gamma}}, P_1^* = \frac{\gamma}{\eta} \tag{3.1.8}$$

If μ > μ*, (3.1.4) has two real roots and if μ < μ*, it has no real roots. Thus every straight line parallel to the P_1 -axis cuts each trajectory in two real (coincident or distinct) points or does not cut in any point. Similarly, every straight line parallel to the P_2-axis cuts each trajectory in two real points or does not cut in any points. This suggests that each trajectory is a closed oval curve of the shape shown in Figure 2.

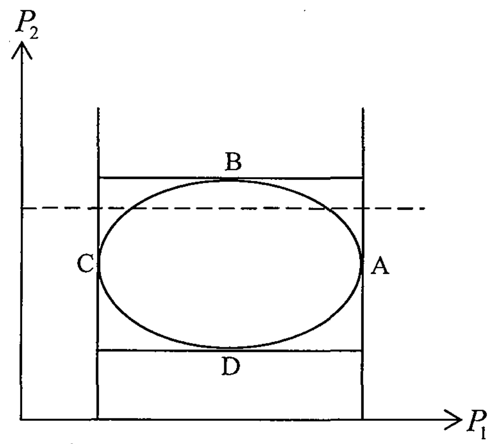

Figure 2: Typical trajectories.

It also appears from (3.1.8) that for every trajectory, points B and D, where the lines parallel to the P_1-axis touch the trajectory, lie on the line $P_1 = \frac{\gamma}{\eta}$. Similarly, the points A and C, where the lines parallel to the P_2 -axis touch the trajectory, lie on the line $P_2 = \frac{\alpha}{\beta}$.

There are four degenerate trajectories which are of special interest. If the starting point is the equilibrium point, $P_{1_E} = \frac{\gamma}{\eta}, P_{2_E} = \frac{\alpha}{\beta}$ i.e. the point at which $\frac{dP_1}{dt}, \frac{dP_2}{dt}$ vanish, then (P_1, P_2) always stays at this equilibrium point and we get a point trajectory. Similarly, if the initial position is $(0,0)$, then (P_1, P_2) always stays at $(0,0)$ and we get another point trajectory. If the trajectory starts from a point $(P_{1_0},0)$ on the P_1-axis, then P_2 always remains zero and the trajectory is the part of the P_1-axis for which $P_1 \geq P_{1_0}$. Similarly if the trajectory starts from a point $(0,P_{2_0})$ on the P_2-axis, then P_1 always remains zero and the trajectory is the part of the P_2-axes for which $P_2 \leq P_{2_0}$. These degenerate trajectories are shown in the following Figure-3.

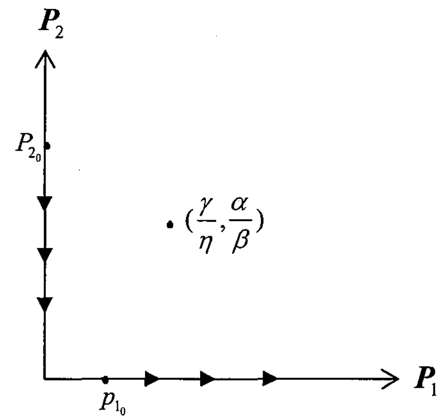

Figure 3: Special trajectories.

It is easily seen that two different trajectories cannot cross each other at a point; if, they do, $\frac{dP_2}{dP_1}$ will have two values at one point. In particular, no non point trajectory can pass through the equilibrium points $\left(\frac{\gamma}{\eta},\frac{\alpha}{\beta}\right)$ and $(0,0)$.Further, the trajectory starting at (P_{1_0},P_{2_0}), where $P_{1_0}>0, P_{2_0}>0$, cannot intersect the P_1 - , P_2-axis so that, if we start with positive values of P_1, P_2, we continue to get positive values of P_1, P_2.

We can also get some idea of the shapes of the trajectories by considering the signs of $\frac{dP_1}{dt}$ and $\frac{dP_2}{dt}$ in the four regions in which the first quadrant is divided by the lines $P_1=\frac{\gamma}{\eta}$ and $P_2=\frac{\alpha}{\beta}$. From $\frac{dP_1}{dt}=\beta P_1\left(\frac{\alpha}{\beta}-P_2\right), \frac{dP_2}{dt}=\eta P_2\left(-\frac{\gamma}{\eta}+P_1\right)$, we get the signs of P_1' and P_2' in regions I, II, III and IV as shown in Figure 4 .

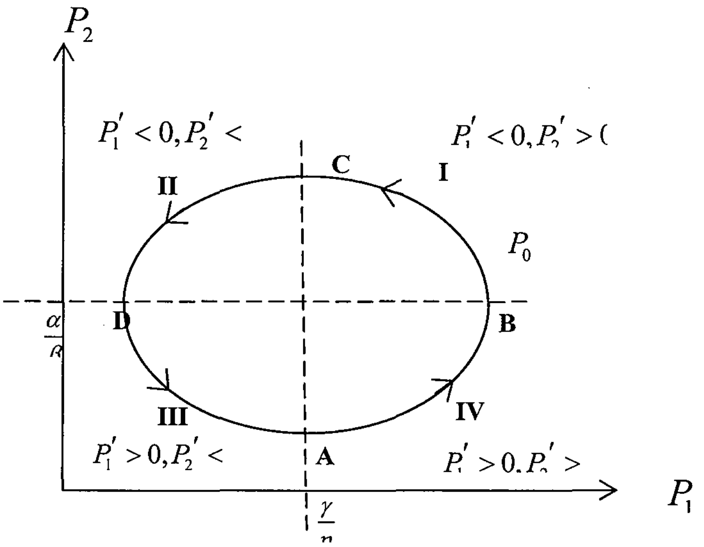

Figure 4 : Phase diagram of a predator-prey system. In the right lower part the predator and the prey population increase ($P'_1 > 0$, $P'_2 > 0$), in the right upper part the predator population increases and the prey population decreases ($P'_1 > 0$, $P'_2 > 0$), in the left upper part the predator- and prey population decreases ($P'_1 > 0$, $P'_2 > 0$), in the left lower part the predator population decreases and the prey population increases ($P'_1 > 0, P'_2 > 0$).

Let the initial point $P_0 = (P_{1_0}, P_{2_0})$ be in region **I**. Here P_1 decreases and P_2 increases so that the point moves in the counter-clockwise direction till it reaches C where $P_2' = 0$ and the tangent is parallel to P_1 -axis. In region **II**, both P_1 and P_2 decreases and the point continues to move in the counter clockwise direction till it reaches D where $P_1' = 0$. In region **III**, P_1 increases and P_2 decreases till A. In region **IV** both P_1 and P_2 increases till B.

All the trajectories are thus described in the counter clockwise sense and they appear to be cramped near the axes since they can only approach, but cannot cross them.

3.2 Theorem: The equilibrium positions for small oscillations, the trajectories of Lotka-Volterra Predator-Prey Model are stable.

3.2.1 Analytical Proof

There are two equilibrium positions, namely, $(0,0)$ and $\left(\frac{\gamma}{\eta}, \frac{\alpha}{\beta}\right)$. The first position is easily seen to be unstable. To discuss the stability of the second position, we substitute

$$P_1(t)=\frac{\gamma}{\eta}+u_1(t),P_2(t)=\frac{\alpha}{\beta}+u_2(t)$$

In (2.1) we get,

$$\left.\begin{aligned}\frac{dP_1}{dt}&=\beta\left(\frac{\gamma}{\eta}+u_1\right)\left(\frac{\alpha}{\beta}-\frac{\alpha}{\beta}-u_2\right)\Rightarrow\frac{dP_1}{dt}=-\frac{\beta\gamma}{\eta}u_2-\beta u_1u_2\\\frac{dP_2}{dt}&=\eta\left(\frac{\alpha}{\beta}-u_2\right)\left(-\frac{\gamma}{\eta}+\frac{\gamma}{\eta}+u_1\right)\Rightarrow\frac{dP_2}{dt}=-\frac{\alpha\eta}{\beta}u_1-\eta u_2u_1\end{aligned}\right\}\tag{3.2.2}$$

So by linearize (3.2.2) we get,

$$\frac{du_1}{dt}=-\frac{\gamma}{\eta}\beta u_2,\frac{du_2}{dt}=\frac{\alpha}{\beta}\eta u_1\tag{3.2.3}$$

The secular equation determined by (3.2.3) is

$$\lambda^2+\alpha\gamma=0\tag{3.2.4}$$

So that the real parts of both the roots are zero. Solving (3.2.3), we obtain

$$u_1(t)=A_1Cos\left(\sqrt{\alpha\gamma}t+k_1\right),u_2(t)=A_2Cos\left(\sqrt{\alpha\gamma}t+k_2\right)\tag{3.2.5}$$

Therefore, periodic oscillations occur with period $\frac{2\pi}{\sqrt{\alpha\gamma}}$. In fact, from (3.2.3),

$$\frac{d^2u_1}{dt^2}=-\alpha\gamma u_1,\frac{d^2u_2}{dt^2}=-\alpha\gamma u_2\tag{3.2.6}$$

and there is elliptic harmonic motion in the u_1u_2-plane. Also from (3.2.3),

$$\frac{du_2}{du_1}=-\frac{\alpha\eta^2}{\gamma\beta^2}\cdot\frac{u_1}{u_2}\tag{3.2.7}$$

Integrating (3.2.7), we get

$$\frac{u_1^2}{\gamma\beta^2}+\frac{u_2^2}{\alpha\eta^2}=\text{Constant}\tag{3.2.8}$$

So that for small oscillations the trajectories are ellipses. The equilibrium position $\left(\frac{\gamma}{\eta},\frac{\alpha}{\beta}\right)$ is therefore a center, or the equilibrium is neutral, and (P_1,P_2) performs conservative oscillations about $\left(\frac{\gamma}{\eta},\frac{\alpha}{\beta}\right)$.

The existence of the conservative oscillations about the equilibrium point has been established only for local or neighborhood stability. These oscillations exist also for global stability, i.e.. they occur even for large deviations from the equilibrium positions, if we can establish the existence of a Lypunov function $G(P_1, P_2)$ which is always positive and whose derivative with respect to t is always less than or equal to zero. Such a function is,

$$G(P_1,P_2) = P_1^{\gamma} P_2^{\alpha} e^{-\beta P_2} e^{-\eta P_1} \tag{3.2.9}$$

Since it is always positive and its derivative is zero because from (3.1.4) G is constant. This result confirms that all non-degenerate trajectories are closed oval curves with a non-zero equilibrium point inside them.

3.2.2 Proof (By CAS

Here we replace the parameters $\alpha, \beta, \gamma,$ and η of our model (2.1) by the letters **a, b, c** and **d** and also P_1, P_2 by x, y for simplicity of writing *MATHEMATICA* codes and take the values:

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a= 1.5; b= 0.03; c= 0.5; d= 0.01;
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as a standard one.

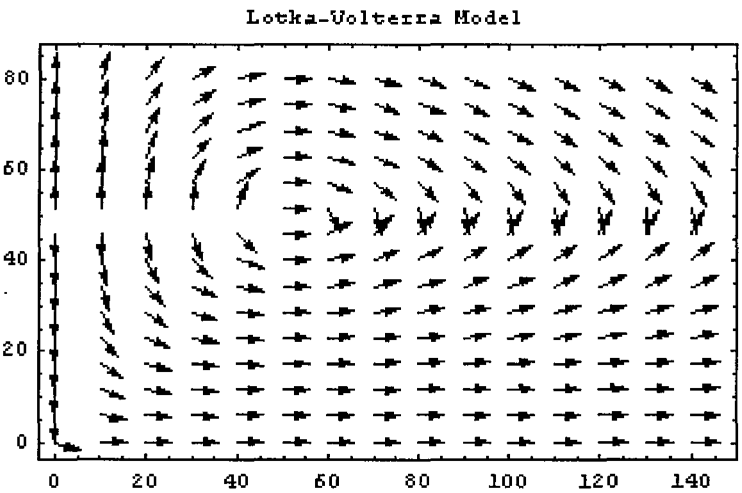


Figure 5: Direction field of $\frac{dP_2}{dP_1}$

Direction field allow us to get the possible shape of the solution curves. Hence the solution curves of $\frac{dP_2}{dP_1}$ is the trajectories of the given systems. So ultimately we get the shapes of the family of trajectories. Likely the trajectories would be oval shaped.

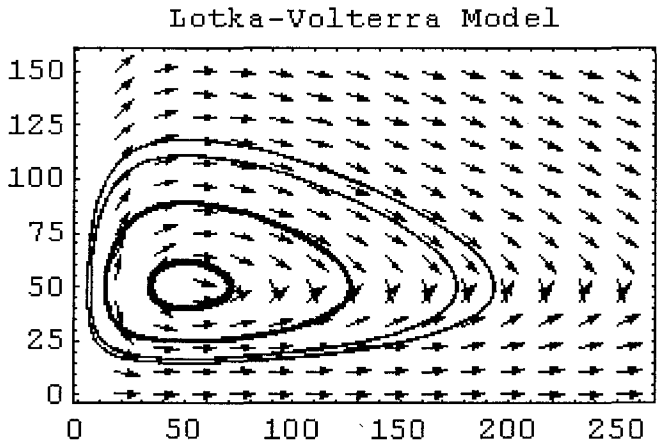


Figure 6: For different initial conditions different trajectories are shown in bold block curves.

Different values of the constant of integration give different trajectories. Our obtained trajectories fit in with direction-field curves very closely. We also get these trajectories by appropriate Mathematica code.

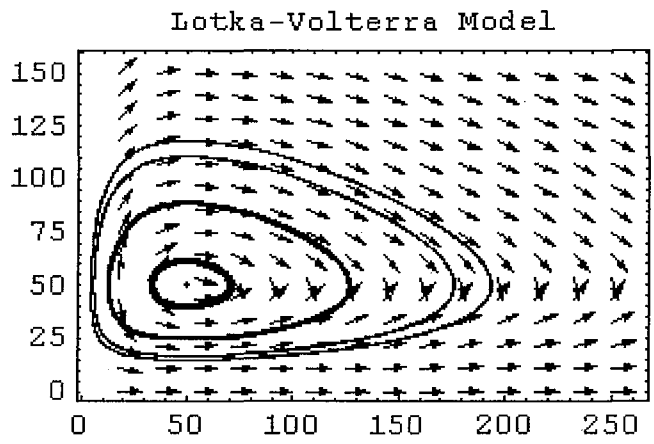


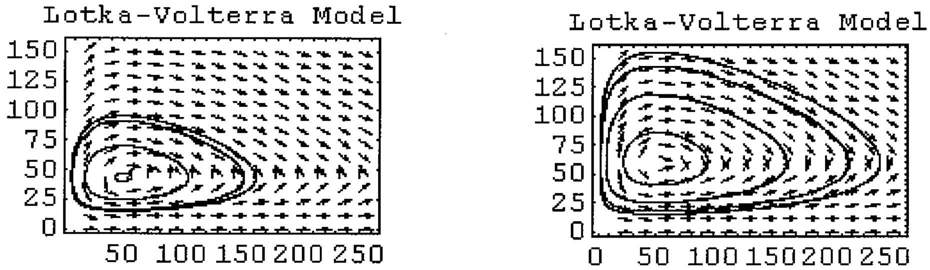
Figure 7: Our equilibrium point is perfectly set to the center of our trajectories.

The predator population growth on the prey population as a substrate, lags the prey population by 90^0. This is shown in a phase diagram in which the number of predators is plotted on the P_2 axis and the number of prey on the P_1 axis. If the parameters are adequately chosen, both populations move on a closed path around the equilibrium point.

4. Effect of Changing the Value of Parametrs and Discussion

In this part, we explore how the solutions change if one of the parameters of the model (2.1) is changed. The cases are shown by using the *Mathematica* codes.

4.1 Effect of changing the value of α

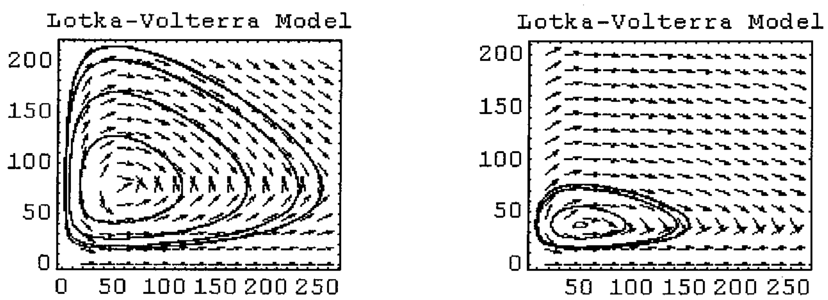


α =1.3 α =1.8

Figure 11: Effect of changing the value of α

Observation: Decreasing the value α of by 13.3% the trajectories of the system slightly reduced in both directions P_1 and P_2 i.e. the maximal growth rate of P_1 and P_2 have decreased. On the other hand, increasing the value of α by 20% the trajectories of the system expanded in both directions (P_1 and P_2 directions). i.e.. the maximal growth rate of P_1 and P_2 have increased . Thus we can conclude that the increase of α increases both population, because α implies the natural growth rate of prey and more prey means more predators.

4.2 Effect of changing the value of β *:*

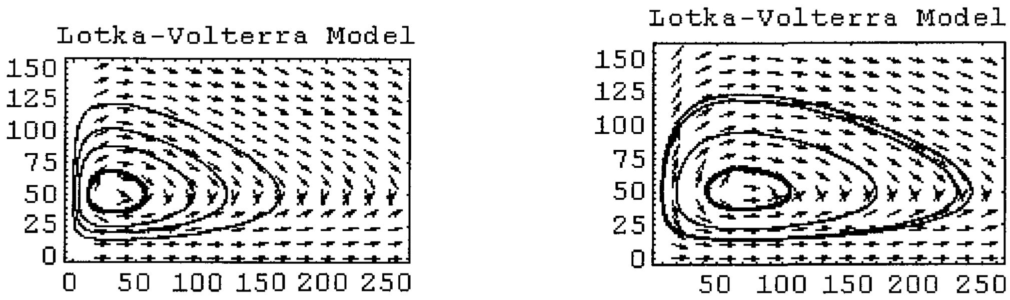


β =0.02 β =0.04

Figure12: Effect of changing the value of β

Observation: Decreasing the value of β by 33.3% the trajectories dramatically expanded in both P_1 and P_2 direction i.e. the maximal growth rate of P_1 and P_2 have increased and increasing the value of β by 33.3% the trajectories reduced suddenly in both P_1 and P_2 direction i.e. the maximal growth rate of P_1 and P_2 have decreased. Increasing the value of β reduced P_2 and P_1 to 0 level, because if predator increase their hunting capacity then the food supply is less sufficient for large number of predator and hence the prey population die out in a short time.

4.3 Effect of changing the value of γ :

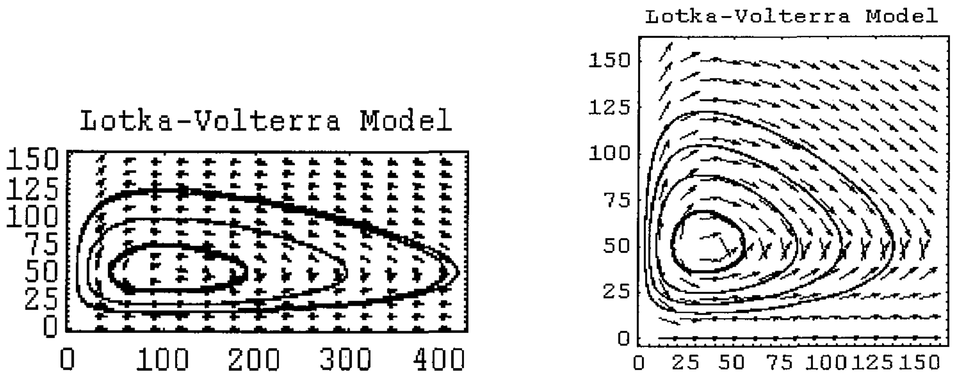


γ =0.3 γ =0.7

Figure 13: Effect of changing the value of γ

Observation: Decreasing the value of γ by 50% the trajectories mainly reduced in P_1 direction. Hence maximal growth rate of P_1 have decreased and increasing the value of γ by 50% the trajectories mainly expanded in P_1 direction. Hence maximal growth rate of P_1 have increased. But in the both cases it also happened to P_2 with small magnitude. If the parameter γ is increased the predator population P_2 is increased slowly but on the other hand prey P_1 is increased very fast.

4.4 Effect of changing the value of η :



η=0.005 η=.015

Figure 14: Effect of changing the *value of* η

Observation: Decreasing the value of η by 50% the trajectories reduced in P_1 direction. i.e.. the maximal growth rate of P_1 is reduced and increasing the value of η by 50% the trajectories reduced in P_1 direction. i.e.. the maximal growth rate of P_1 is reduced. But in the both cases it also happened to P_2 with small magnitude. Lastly when we increase the value of η the more death of the prey reduces its population and thus the population of predators also decreased.

CONCLUSION

The Lotka-Volterra model is one of the earliest predator-prey models to be based on sound mathematical principles. It forms the basis of many models used today in the analysis of population dynamics. We examine the effects of varying parameters that changed the growth rate of prey and predator population. The predator and prey populations seem to cycle endlessly without settling down quickly. While this cycling has been observed in nature, it is not overwhelmingly common. Therefore results may differ from actual situations. But the guidelines and MATHEMATICA codes that we created could be very helpful for further study to model multi –species prey predator systems.

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