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FIRST-ORDER OPTIMALITY CONDITIONS IN MULTIOBJECTIVE OPTIMIZATION PROBLEMS: DIFFERENTIABLE CASE

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ABSTRACT

T. Maeda gave some constraint qualifications to get positive Lagrange multipliers associated with the vector-valued objective function and under these conditions, he derived Karush-Kuhn-Tucker (KKT) type necessary conditions for inequality constraints. In this paper, we have defined these Maeda-type constraint qualifications under different sets and have derived KKT type necessary conditions for both equality and inequality constraints.

1. Introduction

Investigation on optimality conditions has been one of the most attracting topics in the theory of multiobjective optimization problems. Many authors have derived the necessary conditions for an efficient solution under the same constraint qualification as that used in scalar-valued objective function [1, 2, 3]. As some of the multipliers may be equal to zero, the components of the vector valued objective functions corresponding to zero multipliers have no role in the necessary conditions for efficiency. To remove this shortcoming getting positive Lagrange multipliers, T. Maeda [4] first gave some constraint qualifications, which ensures the existence of positive Lagrange multipliers. For getting positive Lagrange multipliers, much work has been done [5, 6, 7], starting from the T. Maeda's paper [4].

In this paper, we have used these Maeda-type constraint qualifications under more general sets that are more easily determinable than Maeda's sets. Consequently we have been able to derive KKT type necessary conditions in a new way for both equality and inequality constraints. Our result has been illustrated with a suitable example.

2. Preliminaries

In this section, we introduce some notations and definitions, which are used throughout the paper.

For $\mathbf{x}, \mathbf{y} \in E_n$, we use the following conventions.

$\mathbf{x} \geqq \mathbf{y}$, iff $x_i \geqq y_i$, i=1,…,n

$\mathbf{x} \geq \mathbf{y}$, iff $\mathbf{x} \geqq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$,

$\mathbf{x} > \mathbf{y}$, iff $x_i > y_i$ i=1,…,n

At first, we consider the following multiobjective optimization problem P :

$\min \mathbf{f}(\mathbf{x})$, subject to the conditions that the minimizing point (or vector) $\bar{\mathbf{x}}$ should lie in the set X:

$$\bar{x} \in X = \{x \in E_n | g(x) \leqq 0, h(x) = 0\}$$

Let, $f : E_n \to E_l$, $g : E_n \to E_m$ and $h : E_n \to E_p$ are continuously differentiable vector-valued functions defined by $f(x) \equiv (f_1(x), f_2(x), ..., f_l(x))$, $g(x) \equiv (g_1(x), g_2(x), ..., g_m(x))$ and $h(x) \equiv (h_1(x), h_2(x), ..., h_p(x))$ where $f_i : E_n \to E_1$ for i=1,..,l, $g_j : E_n \to E_1$ for j=1,…,m and $h_k : E_n \to E_1$ for k = 1,…,p. Assume that $I(\bar{x}) = \{j : g_j(\bar{x}) = 0\}$ for j=1,…,m.

Due to the conflicting nature of the objectives, an optimal solution that simultaneously minimizes all the objectives is usually not obtainable. Thus, for Problem P, the solution is defined in terms of an efficient solution [8].

Definition 2.1. A point $\bar{x} \in X$ is called an *efficient* solution to Problem P if there is no $x \in X$ such that $f(x) \leq f(\bar{x})$.

Now, we shall define the nonempty sets M^i and M by

$$M^i \equiv \{x \in E_n | x \in X,\ f_i(x) \leqq f_i(\bar{x})\},\ i = 1,2,...,l$$

and $M \equiv \{x \in E_n | x \in X,\ f(x) \leqq f(\bar{x})\} = \bigcap_{i=1}^{l} M^i$ = Set of efficient solution.

Comparison between Maeda's sets and Present sets:

Maeda's sets:

$$Q^i \equiv \{x \in E_n | x \in X,\ f_k(x) \leqq f_k(\bar{x}),\ k = 1,2,...,l \ \text{and}\ k \neq i\}$$

Present sets:

$$M^i \equiv \{x \in E_n | x \in X,\ f_i(x) \leqq f_i(\bar{x})\},\ i = 1,2,...,l$$

Relation between two types of sets are

$$Q^i = \bigcap_{\substack{k=1 \\ k \neq i}}^{l} M^k,\ i = 1,...,l$$

Definition 2.2. The linearizing cone to M at $\bar{x} \in M$ is the set defined by

$$\Omega(M;\bar{x}) \equiv \left\{ d \in E_n \,\middle|\, \begin{array}{l} \nabla f_i(\bar{x})^T d \leqq 0, \mathrm{i}=1,2,...,l, \nabla g_j(\bar{x})^T d \leqq 0, \quad \mathrm{j} \in \mathrm{I}(\bar{x}) \\ \text{and } \nabla h_k(\bar{x}) d = 0, k = 1,2,..., p \end{array} \right\}.$$

Here $\Omega(M;\bar{x})$ is a nonempty closed convex cone.

Definition 2.3. Let X be a subset of E_n. The tangent cone to X at $\bar{x} \in cl\ X$ is the set defined by $T(X;\bar{x}) \equiv \left\{ \begin{array}{l} d \in E_n \middle| d = \lim\limits_{n\to\infty} t_n (x_n - \bar{x})\ such\ that\ x_n \in X, with\ x_n \to \bar{x} \\ and\ t_n > 0,\ for\ all\ n = 1,2,... \end{array} \right\}$,

where $cl\ X$ denotes the closure of X and $T(X;\bar{x})$ is a nonempty closed cone and enjoys some important properties[9, 10]; let's just recall that it is isotone, i.e. $T(X_1;\bar{x}) \subseteq T(X_2;\bar{x})$ whenever $X_1 \subseteq X_2$. It is convex if the original set is convex.

3. Generalized constraint qualification

The following lemma 3.1 shows that the relationship between the tangent cone $T(M^i;\bar{x})$ and linearzing cone $\Omega(M;\bar{x})$.

Lemma 3.1. We assume that $\bar{x}$ is a feasible solution to problem P then we have

$$\bigcap_{i=1}^{l} cl\, convT(M^i;\bar{x}) \subseteq \Omega(M;\bar{x})$$

The proof is similar as Maeda did in [4].

Remark: 3.1 In general, the converse inclusion in lemma 3.1 does not hold. So for obtain the necessary conditions that a feasible solution to Problem P be an efficient solution, it is reasonable to assume that

$$\Omega(M;\bar{x}) \subseteq \bigcap_{i=1}^{l} cl\, convT(M^i;\bar{x}) \tag{3.1}$$

The condition (3.1) is considered as a Generalized Guignard Constraint Qualification (GGCQ) [4].

Theorem 3.1. Let $\bar{x} \in X$ be any feasible solution to problem P and f_i, $i = 1,2,...,l$, g_j, $j \in I(\bar{x})$ and h_k, $k = 1,2,..., p$ are continuously differentiable at $\bar{x}$. Assume that the GGCQ holds at $\bar{x}$. If $\bar{x} \in X$ is an efficient solution to Problem P, then the system

$$\left. \begin{array}{ll} \nabla f_i(\bar{x})^T d \leq 0 & i = 1,2,...,l \\ \nabla g_j(\bar{x})^T d \leqq 0 & j \in I(\bar{x}) \\ \nabla h_k(\bar{x})^T d = 0 & k = 1,2,..., p \end{array} \right\} \tag{3.2}$$

has no solution $d \in E_n$.

Proof: Assume that, (3.2) has solution $d \in E_n$. Then we can write $d \in \Omega(M;\bar{x})$.

By assumption we can write $d \in cl\, convT(M^i;\bar{x}),\ i = 1,2,...,l$. Without loss of generality, we may assume that $d \in cl\, convT(M^1;\bar{x})$. Therefore, there exists a sequence $\{d_m\} \subseteq convT(M^1;\bar{x})$ such that $\lim_{n\to\infty} d_m = d$.

Since each $d_m \in convT(M^1;\bar{x}),\ m = 1,2,...$ so we can write

$$d_m = \sum_{k=1}^{K_m} \lambda_{mk} d_{mk}\ ,\ \sum_{k=1}^{K_m} \lambda_{mk} = 1 \text{ and } \lambda_{mk} \geq 0 \text{ for } k = 1,2,...,K_m$$

where K_m is a positive integer and $d_{m1}, d_{m2}, ..., d_{mK_m} \in T(M^1;\bar{x})$.

By definition of $T(M^1;\bar{x})$, there exist sequences $\{x_{mk}^n\} \subseteq M^1$ with $\{t_{mk}^n\} > 0$ for all n, such that, $\lim_{n\to\infty} x_{mk}^n = \bar{x}$, $\lim_{n\to\infty} t_{mk}^n (x_{mk}^n - \bar{x}) = d_{mk}$, for any m, k.

If $d_{mk}^n = t_{mk}^n (x_{mk}^n - \bar{x})$ then for any n, we have

$$f_1(x_{mk}^n) - f_1(\bar{x}) = \nabla f_1(\bar{x})^T (x_{mk}^n - \bar{x}) + \mathrm{o}(\|x_{mk}^n - \bar{x}\|) \leqq 0 \tag{3.3}$$

where $\dfrac{\mathrm{o}(\|x_{mk}^n - \bar{x}\|)}{\|x_{mk}^n - \bar{x}\|} \to 0$ as $x_{mk}^n \to \bar{x}$

Since $t_{mk}^n > 0$ and taking the limit as $n \to \infty$, the above inequality implies

$$\nabla f_1(\bar{x})^T d_{mk} \leqq 0 \Rightarrow \nabla f_1(\bar{x})^T d \leqq 0 .$$

Hence, $d \in T(M^i;\bar{x}) \Rightarrow \nabla f_i(\bar{x})^T d \leqq 0$, $i = 1,2,...,l$.

Let $F_i = \{d : \nabla f_i(\bar{x})^T d \leqq 0\},\ \ i = 1,2,...,l$. So we can write that $T(M^i;\bar{x}) \subseteq F_i,\ i = 1,2,...,l$. Since T is closed cone and *i* is arbitrary, we have

$$\bigcap_{i=1}^{l} T(M^i;\bar{x}) \subseteq \bigcap_{i=1}^{l} F_i = F = \{d : \nabla f_i(\bar{x})^T d \leqq 0, i = 1,...,l.\} \text{ (let)} \tag{3.4}$$

Also, $\bigcap_{i=1}^{l} T(M^i;\bar{x}) \subseteq T(X;\bar{x})$ so that we can write

$$d \in \bigcap_{i=1}^{l} T(M^i;\bar{x}) \Rightarrow d \in T(X;\bar{x}).$$

That is $d = \lim_{p\to\infty} t_p (x_p - \bar{x})$, where $t_p > 0$, $x_p \in X$ for each p, and $\lim_{p\to\infty} x_p = \bar{x}$.

Since $\bar{x}$ is an efficient solution to Problem P, so there is no point $x_p \in X$, where $f(x_p) \leq f(\bar{x})$.

i.e. $\nabla f(\bar{x})^T (x_p - \bar{x}) + o(\|x_p - \bar{x}\|) \leq 0$

$$\Rightarrow \nabla f(\bar{x})^T (x_p - \bar{x}) t_p + o(\|x_p - \bar{x}\|) t_p \leq 0$$

Since $t_p > 0$ and taking the limit as $p \to \infty$, the above inequality implies $\nabla f(\bar{x})^T d \leq 0$.(3.5)

It means that if $\bar{x}$ is an efficient solution then we do not get (3.5).

From (3.4) and (3.5) we have,

$$d \in \bigcap_{i=1}^{l} cl\, conv T(M^i; \bar{x}) \Rightarrow \nabla f_i(\bar{x})^T d = 0, \ i = 1,2,\ldots,l$$

Therefore, (3.2) has no solution. This completes the proof.

Theorem 3.2.

Let $\bar{\mathbf{x}} \in X$ be any feasible solution to problem P and f_i, $i = 1,2,\ldots,l$, g_j, $j \in I(\bar{x})$ and h_k, $k = 1,2,\ldots,p$ are continuously differentiable at $\bar{x}$. Suppose that GGCQ holds at $\bar{x}$. If $\bar{x} \in X$ is an efficient solution to Problem P, then there exist vectors $u \in E_l$, $v \in E_m$ such that

$$\sum_{i=1}^{l} u_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} v_j \nabla g_j(\bar{x}) + \sum_{k=1}^{p} \mu_k \nabla h_k(\bar{x}) = 0 \tag{3.6}$$

$$v_j g_j(\bar{x}) = 0 \ , \text{ j=1,....m} \tag{3.7}$$

$$u > 0, \ v \geqq 0$$

Proof:

Let $\bar{x} \in X$ be an efficient solution to Problem P. Then, from Theorem 3.1 we have, the system

$$\left.\begin{array}{ll} \nabla f_i(\bar{x})^T d \leq 0 & i = 1,2,\ldots,l \\ \nabla g_j(\bar{x})^T d \leqq 0 & j \in I(\bar{x}) \\ \nabla h_k(\bar{x})^T d = 0 & k = 1,2,\ldots,p \end{array}\right\}$$

has no solution. By the Tucker's theorem [1], there exist $u > 0$, $u \in E_l$ and $v_j \geqq 0$, $j \in I(\bar{x})$, such that

$$\sum_{i=1}^{l} u_i \nabla f_i(\bar{x}) + \sum_{j \in I} v_j \nabla g_j(\bar{x}) + \sum_{k=1}^{p} \mu_k \nabla h_k(\bar{x}) = 0$$

By setting $v_j = 0$, $j \notin I(\bar{x})$, we have

$$\sum_{i=1}^{l} u_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} v_j \nabla g_j(\bar{x}) + \sum_{k=1}^{p} \mu_k \nabla h_k(\bar{x}) = 0$$

$$u > 0 \quad , \; v \geqq 0$$

Since $g_j(\bar{x}) = 0$ for $j \in I(\bar{x})$, we have

$$v_j g_j(\bar{x}) = 0 \text{ for j=1,…,m}$$

which completes the proof.

Example 3.3.

Consider the problem

min $\{x_1, x_2\}$ and $X = \{(x_1, x_2) \mid -x_1 \leqq 0, -x_2 \leqq 0, x_1 + x_2 = 2\}$

Here, $M^1 = \{x \in E_n \mid x \in X, f_1(x) \leqq f_1(\bar{x})\} = X$ and $M^2 = \{x \in E_n \mid x \in X, f_2(x) \leqq f_2(\bar{x})\} = \{\bar{x}\}$

It is easily verified that:

i) All points in X are efficient solution. We choose $\bar{x} = (2,0)$ is an efficient solution to the problem.

ii) GGCQ holds at $\bar{x} = (2,0)$.

Since $\Omega(M; \bar{x}) = \{0\}$ and $\bigcap_{i=1}^{2} clconvT(M^i; \bar{x}) = \{0\}$

We have, $u_1 \nabla f_1(\bar{x}) + u_2 \nabla f_2(\bar{x}) + v_2 \nabla g_2(\bar{x}) + \mu \nabla h(\bar{x}) = 0$

$$\Rightarrow u_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{array}{l} u_1 = -\mu \\ u_2 = v_2 - \mu \end{array}\right\}$$

If $\mu < 0$ and $v_1, v_2 \geq 0$ ($v_1 = 0$, $v_1 \notin I(\bar{x})$) then we have $u_1, u_2 > 0$ which satisfy the necessary conditions of efficiency.

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