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COMPUTATION OF HOMOLOGY AND COHOMOLOGY GROUPS OF A METACYCLIC GROUP AND A FACTOR GROUP OF THE HEISENBERG GROUP

Subrata Majumdar and Nasima Akhter

Department of Mathematics
University of Rajshahi, Rajshahi-6205, Bangladesh
Email: nasima.math.ru@gmail.com

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ABSTRACT

Free resolutions for a metacyclic group *M* and a factor group *G* of the Heisenberg group from their presentations constructed by solving system of linear equations over the integral group ring and determined the homology and cohomology groups. The method is straightforward, and the free resolutions are explicitly expressed in terms of the free generators. The resolutions yield the homology and the cohomology groups immediately.

Keywords: Group presentation, Metacyclic group, Heisenberg group, Free resolution, Huebschmann perturbation method, Homology, Cohomology.

1. Introduction

We have computed the homology and the cohomology groups of a metacyclic group M and a factor group G of the Heisenberg group by constructing free resolutions.The Heisenberg group $H_3$is the multiplicative group of the matrices $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ where x, y, z are real numbers ([22], p.467). Huebschmann[9] has determined the cohomology rings of G using the sophisticated and complicated machinery of perturbation theory developed by him. He has also dealt with the mod$p$ cohomology of *M* [11]. To construct our resolutions we do not need to use any sophisticated machinery such as Künneth relations, spectral sequence and perturbation theory of homological algebra and Identity Theorem of combinatorial group theory. The "rewriting systems" of Groves [5] and Carbone [2] also have not been used. We have obtained our resolutions by extending Lyndon's partial resolution ([6], [12], [13]) through step by step solutions of systems of linear equations over the integral group rings. The details of the construction are given in [19]. Let $G = F/R$ where F is the free group on $\{x_1, \cdots, x_m\}$ and R the normal closure of $\{r_1, \cdots, r_n\}$. Let $\pi : \mathbb{Z}F \to \mathbb{Z}G$ be the ring homomorphism which extends the canonical homomorphism from F onto G, and let $\pi(x_i) = h_i$. Then Lyndon's partial free $\mathbb{Z}G$-resolution of $\mathbb{Z}$ is the following:

where

Y_0 is a free right ℤG-module on $\{\alpha_1, \cdots, \alpha_m\}$,

Y_1 is a free right ℤG-module on $\{\beta_1, \cdots, \beta_n\}$,

and ε, d_0, d_1 are $\mathbb{Z}G$-homomorphisms given by

$\varepsilon(g) = 1$, for all $g \in G$

$d_0(\alpha_i) = h_i - 1,\ i = 1, \cdots, m,$

$d_1(\beta_j) = \sum_{i=1}^{m} a_i \pi\left(\frac{\partial r_j}{\partial x_i}\right), j = 1, \cdots n$,Here $\frac{\partial r_j}{\partial x_i}$ is the Fox derivative [4] of r_j with respect to x_i and is defined by

$$r - 1 = \sum_{i=1}^{n} (x_i - 1)\frac{\partial r_j}{\partial x_i},\ j = 1, \cdots n,$$

For the extension of this partial resolution, the first set of equations is obtained by equating the right hand side expression for the map d_1 in (P). The solutions of this set of equations is the kernel of d_1. From these solutions we can read off the generators of the next term Y_2 and the next homomorphism d_2 in the extension of (P) such that

Im d_2 = Ker d_1.

Similarly, we can get the next term Y_3 and the next homomorphism d_3 in the extension from the set of equations obtained from d_2 as in the case of d_1. An extension of (P) to a full resolution is obtained by continuing this process.

As the terms of this resolution are free modules on explicitly given generators and the homomorphisms are explicitly defined on these generators, the resolutions immediately yield the homology and the cohomology groups with arbitrary coefficient modules. These have been determined and the integral homology and the cohomology groups have been explicitly calculated. One of the objectives in choosing these two groups is to exhibit the relative simplicity of our method in comparison to that of Huebschmann. While the latter method is applicable to a wide class of groups including nilpotent groups of class 2, our method too has been applied to different classes of groups ([1], [14], [15], [16], [17], [18], [20], [21]) and is applicable to many others.

2. Determination of Homology and Cohomology of a factor group of the Heisenberg group

The Group G considered here is given by

generators : $h_1, h_2, h_3;$

relations :$[h_1, h_2] = h_3, [h_2, h_3] = 1 = [h_1, h_3], h_3^l = 1$, where $l \in \mathbb{N}$

Then G can be viewed as a central extension of $\mathbb{Z}_l$ by $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}_l$ is a cyclic group of order l. Also we can see that G is a factor group of Heisenberg group H with generators : h_1, h_2, h_3; relations : $[h_1, h_2] = h_3, [h_2, h_3] = 1 = [h_1, h_3]$, $h_3^l = 1$.

Thus $G = F/R$, where F is the free group generated by x_1, x_2, x_3 and R is the normal subgroup of F generated by r_1, r_2, r_3, r_4 where

$$r_1 = x_3^{-1} x_2^{-1} x_1^{-1} x_2 x_1,\ r_2 = x_2^{-1} x_3^{-1} x_2 x_3,\ r_3 = x_1^{-1} x_3^{-1} x_1 x_3,\ r_4 = x_3^1.$$

Then the fox derivatives are:

$$\frac{\partial r_1}{\partial x_1} = -x_1^{-1} x_2 x_1 + 1, \quad \frac{\partial r_1}{\partial x_2} = -x_2^{-1} x_1^{-1} x_2 x_1 + x_1, \quad \frac{\partial r_1}{\partial x_3} = -r_1,$$

$$\frac{\partial r_2}{\partial x_1} = 0, \quad \frac{\partial r_2}{\partial x_2} = -r_2 + x_3, \quad \frac{\partial r_2}{\partial x_3} = -x_3^{-1} x_2 x_3 + 1,$$

$$\frac{\partial r_3}{\partial x_1} = -r_3 + x_3, \quad \frac{\partial r_3}{\partial x_2} = 0, \quad \frac{\partial r_3}{\partial x_3} = -x_3^{-1} x_1 x_3 + 1,$$

$$\frac{\partial r_4}{\partial x_1} = 0, \quad \frac{\partial r_4}{\partial x_2} = 0, \quad \frac{\partial r_4}{\partial x_3} = -x_3^{l-1} + \ldots\ldots + 1,$$

Let $\pi : \mathbb{Z}F \rightarrow \mathbb{Z}G$ be the ring homomorphism induced by the canonical homomorphism of F onto G with R as the kernel. Let $\pi(x_i) = h_i$, $i = 1, 2, 3$. Then,

$$\pi\left(\frac{\partial r_1}{\partial x_1}\right) = -h_1^{-1} h_2 h_1 + 1, \quad \pi\left(\frac{\partial r_1}{\partial x_2}\right) = h_1 - h_3, \quad \pi\left(\frac{\partial r_1}{\partial x_3}\right) = -1,$$

$$\pi\left(\frac{\partial r_2}{\partial x_1}\right) = 0, \quad \pi\left(\frac{\partial r_2}{\partial x_2}\right) = h_3 - 1, \quad \pi\left(\frac{\partial r_2}{\partial x_3}\right) = 1 - h_2,$$

$$\pi\left(\frac{\partial r_3}{\partial x_1}\right) = h_3 - 1, \quad \pi\left(\frac{\partial r_3}{\partial x_2}\right) = 0, \quad \pi\left(\frac{\partial r_3}{\partial x_3}\right) = 1 - h_1,$$

$$\pi\left(\frac{\partial r_4}{\partial x_1}\right) = 0, \quad \pi\left(\frac{\partial r_4}{\partial x_2}\right) = 0, \quad \pi\left(\frac{\partial r_4}{\partial x_3}\right) = h_3^{l-1} + \ldots\ldots + 1,$$

Theorem 1.1

The following is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$:

$$\ldots\ldots\ldots\ldots\ldots\ldots.\ Y_1 \xrightarrow{d_2} Y_1 \xrightarrow{d_3} Y_1 \xrightarrow{d_2} Y_1 \xrightarrow{d_2} Y_1 \xrightarrow{d_0} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0$$

where

Y_0 is a right $\mathbb{Z}G$-module free on $\alpha_1, \alpha_2, \alpha_3$;

Y_1 is a right $\mathbb{Z}G$-module free on $\beta_1, \beta_2, \beta_3, \beta_4$;

and $\varepsilon, d_0, d_1, d_2, d_3$ are the $\mathbb{Z}G$-homomorphisms and given by

$\varepsilon(g) = 1, \text{ for all } g \in G$

$d_0(\alpha_i) = h_i - 1, i = 1, 2, 3,$

$d_1(\beta_j) = \sum_{i=1}^{3} \alpha_i \pi\left(\frac{\partial r_j}{\partial x_i}\right), j = 1, 2, 3, 4$

$d_2(\beta_1) = \beta_1(h_3 - 1) - \beta_2(h_1 - h_2) + \beta_3(1 - h_2 h_3)$

$d_2(\beta_2) = \beta_2(h_3^{l-1} + \ldots\ldots + 1) + \beta_4(h_2 - 1),$

$d_2(\beta_3) = \beta_3(h_3^{l-1} + \ldots\ldots + 1) + \beta_4(h_1 - 1),$

$d_2(\beta_4) = \beta_4(h_3 - 1),$

$d_3(\beta_1) = \beta_1(h_3^{l-1} + \ldots\ldots + 1) + \beta_2(h_1 - h_3) + \beta_3(1 - h_2 h_3) - \beta_4,$

$d_3(\beta_2) = \beta_2(h_3 - 1) + \beta_4(1 - h_2),$

$d_3(\beta_3) = \beta_3(h_3 - 1) + \beta_4(1 - h_1),$

$d_3(\beta_4) = \beta_4(h_3^{l-1} + \ldots\ldots + 1).$

Proof

By (P) it is sufficient to verify the exactness of the sequence only at the first, second and third Y_1's from the right.

<u>Exactness at Y_1</u> (first from the right)

$$d_1 d_2(\beta_1) = d_1[\beta_1(h_3 - 1) - \beta_2(h_1 - h_3) - \beta_3(1 - h_2 h_3)]$$
$$= (\alpha_1(1 - h_2 h_3) + \alpha_2(h_1 - h_3) - \alpha_3)(h_3 - 1) - (\alpha_2(h_3 - 1) + \alpha_3(1 - h_2))(h_1 - h_3)$$
$$-(\alpha_1(h_3 - 1) + \alpha_3(1 - h_1))(1 - h_2 h_3))$$
$$= 0,$$

Similarly, one can easily show that, $d_1 d_2(\beta_2) = 0 = d_1 d_2(\beta_3) = d_1 d_2(\beta_4)$.

Ker $d_1 \subset$ Im d_2.

Conversely, let $d_1(\beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 + \beta_4\gamma_4) = 0$, for some $\gamma_i \in \mathbb{Z}G, i = 1, 2, 3, 4$.

Then $[\alpha_1(1 - h_2 h_3) + \alpha_2(h_1 - h_3) - \alpha_3]\gamma_1 + [\alpha_2(h_3 - 1) + \alpha_3(1 - h_2)]\gamma_2$
$+[\alpha_1(h_3 - 1) + \alpha_3(1 - h_1)]\gamma_3 + \alpha_3(h_3^{l-1} + \ldots\ldots + 1)\gamma_4 = 0.$

Since, Y_0 is free on $\alpha_1, \alpha_2, \alpha_3$, this implies that

$(1 - h_2 h_3)\gamma_1 + (h_3 - 1)\gamma_3 = 0$ … (1)

$(h_1 - h_3)\gamma_1 + (h_3 - 1)\gamma_2 = 0$ … (2)

$-\gamma_1 + (1 - h_2)\gamma_2 + (1 - h_1)\gamma_3 + (h_3^{l-1} + \ldots\ldots + 1)\gamma_4 = 0$ … (3)

Multiplying (1) by $h_3^{l-1}+\ldots\ldots+1$, we have

$(h_3^{l-1}+\ldots\ldots+1)(1-h_2)h_3\gamma_1=0$. Since the order of h_2 is not finite,

$(h_3^{l-1}+\ldots\ldots+1)\gamma_1=0$. Hence by Proposition 1 of [14]

$\gamma_1=(h_3-1)\,\gamma'_1$ *for some* $\gamma'_1\in\mathbb{Z}G$.

Put the value of γ_1 in (2), we get

$(h_3-1)[(h_1-h_2)\,\gamma'_1+\gamma_2]=0$. Therefore,

$(h_1-h_3)\,\gamma'_1+\gamma_2=(h_3^{l-1}+\ldots\ldots+1)\,\gamma'_2$, for some $\gamma'_2\in\mathbb{Z}G$, by Proposition 2 of [14].

$\therefore\ \gamma_2=(h_3^{l-1}+\ldots\ldots+1)\,\gamma'_2-(h_1-h_3)\,\gamma'_1.$

Substituting for γ_1 in (1) and applying Proposition 2 of [14] we have,

$\gamma_3=(h_3^{l-1}+\ldots\ldots+1)\,\gamma'_3-(1-h_2h_3)\,\gamma'_1.$

Now putting the value of γ_1, γ_2 and γ_3 in (3), we get,

$(h_3^{l-1}+\ldots\ldots+1)[(1-h_2)\,\gamma'_2+(1-h_1)\,\gamma'_3+\gamma_4]=0.$

Thus, $\gamma_4=(h_3-1)\,\gamma'_4+(h_2-1)\,\gamma'_2+(h_1-1)\,\gamma'_3$, *for some* $\gamma'_4\in\mathbb{Z}G$.

Hence $\beta_1\gamma_1+\beta_2\gamma_2+\beta_3\gamma_3+\beta_4\gamma_4$

$=\{\beta_1(h_3-1)-\beta_2(h_1-h_3)-\beta_3(1-h_2h_3)\}\,\gamma'_1+\{\beta_2(h_3^{l-1}+\ldots\ldots+1)+\beta_4(h_2-1)\}\,\gamma'_2$

$+\{\beta_3(h_3^{l-1}+\ldots\ldots+1)+\beta_4(h_1-1)\}\,\gamma'_3+\beta_4(h_3-1)\,\gamma'_4$

$=d_2(\beta_1\gamma'_1+\beta_2\gamma'_2+\beta_3\gamma'_3+\beta_4\gamma'_4)\subset \mathrm{Im}\,d_2.$

$\therefore$ Ker $d_1=$ Im d_2.

Exactness at Y_1 (second from the right)

The kernel of d_2 contains the image of d_3 is obvious; so we only prove its converse.

Let $\beta_1\gamma_1+\beta_2\gamma_2+\beta_3\gamma_3+\beta_4\gamma_4\in Ker\ d_2$,

Then $d_2(\beta_1\gamma_1+\beta_2\gamma_2+\beta_3\gamma_3+\beta_4\gamma_4)=0.$

i.e., $[\beta_1(h_3-1)-\beta_2(h_1-h_3)-\beta_3(1-h_2h_3)]\gamma_1+\{\beta_2(h_3^{l-1}+\ldots\ldots+1)+\beta_4(h_2-1)\}\gamma_2$

$+\{\beta_3(h_3^{l-1}+\ldots\ldots+1)+\beta_4(h_1-1)\}\gamma_3+\beta_4(h_3-1)\gamma_4=0.$

Since $\beta_1, \beta_2, \beta_3, \beta_4$, are linearly independent

$(h_3-1)\gamma_1=0$

$(h_3^{l-1}+\ldots\ldots+1)\gamma_2-(h_1-h_3)\gamma_1=0$

$(h_3^{l-1}+\ldots\ldots+1)\gamma_3-(1-h_2h_3)\gamma_1=0$

$(h_3-1)\gamma_4+(h_1-1)\gamma_3+(h_2-1)\gamma_2=0$. Thus, from the first equation, we have

$\gamma_1=(h_3^{l-1}+\ldots\ldots+1)\,\gamma'_1$, for some $\gamma'_1\in\mathbb{Z}G$, by Proposition 2 of [14].

Putting the value of γ_1 in the second and third equations and applying Proposition 1 of [14], we get

$$\gamma_2 = (h_3 - 1)\,\gamma'_2 + (h_1 - h_3)\,\gamma'_1$$

$\gamma_3 = (h_3 - 1)\,\gamma'_3 + (1 - h_2h_3)\,\gamma'_1$, for some γ'_2 , $\gamma'_3 \in \mathbb{Z}G$ respectively.

Substituting for γ_2 and γ_3 in the last equation and an application of Proposition 2 of [14], gives

$\gamma_4 = (h_3^{l-1} + \ldots\ldots + 1)\gamma'_4 - \gamma'_1 + (1 - h_2)\,\gamma'_2 + (1 - h_1)\,\gamma'_1$, for some $\gamma'_4 \in \mathbb{Z}G$.

Therefore, $\beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 + \beta_4\gamma_4 = d_3(\beta_1\,\gamma'_1 + \beta_2\gamma'_2 + \beta_3\gamma'_3 + \beta_4\gamma'_4) \in$ Im d_3.

$\therefore$ Ker d_2 = Im d_3.

<u>Exactness at Y_1</u>(third from the right)

By the definitions of d_2 and d_3

Ker $d_3 \subset$ Im d_2.

So let $\beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 + \beta_4\gamma_4 \subset$ Ker d_3, then $d_3(\beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 + \beta_4\gamma_4) = 0$,

i.e., $\beta_1(h_3^{l-1} + \ldots\ldots + 1)\gamma_1 + \beta_2(h_1 - h_3)\gamma_1 + \beta_3(1 - h_2h_3)\gamma_1 - \beta_4\gamma_1 + \beta_2(h_3 - 1)\gamma_2 + \beta_4(1 - h_2)\gamma_2$

$$+ \beta_3(h_3 - 1)\gamma_3 + \beta_4(1 - h_1)\gamma_3 + \beta_4(h_3^{l-1} + \ldots\ldots + 1)\gamma_4 = 0.$$

Since Y_1 is free on $\beta_1, \beta_2, \beta_3, \beta_4$, we have equations

$$(h_3^{l-1} + \ldots\ldots + 1)\gamma_1 = 0,$$

$$(h_1 - h_3)\gamma_1 + (h_3 - 1)\gamma_2 = 0,$$

$$(1 - h_2h_3)\gamma_1 + (h_3 - 1)\gamma_3 = 0,$$

$$-\gamma_1 + (1 - h_2)\gamma_2 + (1 - h_1)\gamma_3 + (h_3^{l-1} + \ldots\ldots + 1)\gamma_4 = 0.$$

From the first equation, by Proposition 1 of [14], we have

$$\gamma_1 = (h_3 - 1)\gamma'_1, \textit{ for some } \gamma'_1 \in \mathbb{Z}G.$$

Substituting in the second equation and applying Proposition 1 of [14], we get

$$\gamma_2 = (h_3^{l-1} + \ldots\ldots + 1)\,\gamma'_2 - (h_1 - h_3)\,\gamma'_1, \textit{ for some } \gamma'_2 \in \mathbb{Z}G.$$

Similarly, the third equation gives

$$\gamma_3 = (h_3^{l-1} + \ldots\ldots + 1)\,\gamma'_3 - (1 - h_2h_3)\,\gamma'_1, \textit{ for some } \gamma'_3 \in \mathbb{Z}G.$$

Substituting for γ_1, γ_2 and γ_3 in the last equation and applying Proposition 1 of [14], we get

$$\gamma_4 = (h_3 - 1)\,\gamma'_4 - (1 - h_2)\,\gamma'_2 - (1 - h_1)\,\gamma'_3, \textit{ for some } \gamma'_4 \in \mathbb{Z}G.$$

Thus, $\beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 + \beta_4\gamma_4 = d_2(\beta_1\,\gamma'_1 + \beta_2\gamma'_2 + \beta_3\gamma'_3 + \beta_4\gamma'_4)$.

$\therefore$ Ker d_3 = Im d_2.

By the nature of the sequence we do not need to consider any more terms, and the proof of the theorem is complete.

Homology Groups of *G*

Let A be a left $\mathbb{Z}G$-*module*. The homology groups $H_n(G, A)$ are given by the homology of the complex:

$$\ldots\ldots\ldots\ldots\ldots\ A^4 \xrightarrow{\bar{d}_2} A^4 \xrightarrow{\bar{d}_3} A^4 \xrightarrow{\bar{d}_2} A^4 \xrightarrow{\bar{d}_2} A^4 \xrightarrow{\bar{d}_0} A \to 0$$

Where A^n stands for the direct sum of n isomorphic copies of A and the homomorphisms $\bar{d}_0, \bar{d}_1, \bar{d}_2, \bar{d}_3$ are induced by d_0, d_1, d_2, d_3 respectively and are given by

$$\bar{d}_0\ (a_1, a_2, a_3) = (h_1 - 1)a_1 + (h_2 - 1)a_2 + (h_3 - 1)a_3,$$

$$\bar{d}_1\ (a_1, a_2, a_3, a_4) = ((1 - h_2h_3)a_1 + (h_3 - 1)a_3,\ (h_1 - h_3)a_1 + (h_3 - 1)a_2,\ -a_1 + (1 - h_2)a_2 + (1 - h_1)a_3 + (h_3^{l-1} + \ldots\ldots + 1)a_4),$$

$$\bar{d}_2\ (a_1, a_2, a_3, a_4) = ((h_3 - 1)a_1,\ (h_3^{l-1} + \ldots\ldots + 1)a_2 - (h_1 - h_3)a_1,\ (h_3^{l-1} + \ldots\ldots + 1)a_3 - (1 - h_2h_3)a_1,\ (h_3 - 1)a_4 + (h_1 - 1)a_3 + (1 - h_2)a_2),$$

$$\bar{d}_3\ (a_1, a_2, a_3, a_4) = (\,(h_3^{l-1} + \ldots\ldots + 1)a_1,\ (h_3 - 1)a_2 + (h_1 - h_3)a_1,\ (h_3 - 1)a_3 + (1 - h_2h_3)a_3,\ (h_3^{l-1} + \ldots\ldots + 1)a_4 + (1 - h_1)a_3 + (1 - h_2)a_2 - a_1)$$

for all $a_1, a_2, a_3, a_4 \in A$

When A is a trivial $\mathbb{Z}G$-module $\mathbb{Z}$, the above homomorphisms of the complex reduce to the following:

$$\bar{d}_0\ (a_1, a_2, a_3) = 0,$$

$$\bar{d}_1\ (a_1, a_2, a_3, a_4) = (0, 0, -a_1 + la_4),$$

$$\bar{d}_2\ (a_1, a_2, a_3, a_4) = (0, la_2, la_3, 0),$$

$$\bar{d}_3\ (a_1, a_2, a_3, a_4) = (la_1, 0, 0, la_4 - a_1).$$

Therefore, in this case, we have

$H_0(G, \mathbb{Z}) \cong \mathbb{Z}$

$$H_1(G, \mathbb{Z}) = \frac{Ker\bar{d}_0}{lm\bar{d}_2} = \frac{\{(a_2, a_2, a_3) \mid a_2, a_2, a_3, \in Z\}}{\{(0, 0, -a_2 + a_4) \mid a_2, a_4 \in Z\}} \cong (x, y, z \mid z = 0,\ lz = 0) \cong \mathbb{Z}^2$$

$$H_2(G, \mathbb{Z}) = \frac{Ker\bar{d}_2}{lm\bar{d}_2} = \frac{\{(a_1, a_2, a_3, a_4) \mid a_2 = la_4, a_z, a_3, a_4 \in Z\}}{\{(0, la_2 + la_4, 0) \mid a_2, a_3 \in Z\}} = \frac{(x) \oplus (y) \oplus (z)}{(ly) + (lz)}$$

$\cong (x, y, z \mid ly = 0, lz = 0) \cong \mathbb{Z}\oplus\mathbb{Z}_l\oplus\mathbb{Z}_l$

$$H_3(G, \mathbb{Z})\ \frac{Ker\bar{d}_2}{lm\bar{d}_3} = \frac{\{(a_1, a_2, a_3, a_4) \mid a_2 = 0 = a_3; a_3, a_2, a_2, a_4 \in Z\}}{\{(la_2, 0, 0, la_4 - a_2) \mid a_2, a_4 \in Z\}}$$

$= \dfrac{(x)\oplus(y)}{(lx - y) + (ly)}$, *writing x = (1, 0) and y = (0, 1).*

$= (x, y \mid y = lx, ly = 0) = (x \mid l^2 x = 0) \cong \mathbb{Z}_{l^2}$

$$= H_4(G, \mathbb{Z}) = \frac{Ker\bar{d}_2}{lm\bar{d}_3} = \frac{\{(a_2, a_2, a_3, a_4) \mid la_2 = 0, a_2 = la_4; a_2, a_2, a_3, a_4 \in Z\}}{\{(0, la_2, la_3, 0) \mid a_2, a_3 \in Z\}} = \frac{(x)\oplus(y)}{(lx) + (ly)}$$

$\cong (x, y \mid lx = 0, ly = 0) = (x \mid l^2 x = 0) \cong \mathbb{Z}_l \oplus \mathbb{Z}_l$

Thus $H_{2n-1}(G, \mathbb{Z}) \cong \mathbb{Z}_{l^2}$, $n \geq 2$.

$H_{2n}(G, \mathbb{Z}) \cong \mathbb{Z}_l \oplus \mathbb{Z}_l$, $n \geq 2$.

Cohomology Groups of *G*

Let A be a right $\mathbb{Z}G$-module, then the cohomology groups $H^n(G, A)$ are given by the homology of the complex:

$$\ldots\ldots\ldots\ldots\ldots \leftarrow A^4 \overset{d_2^*}{\leftarrow} A^4 \overset{d_3^*}{\leftarrow} A^4 \overset{d_2^*}{\leftarrow} A^4 \overset{d_1^*}{\leftarrow} A^4 \overset{d_0^*}{\leftarrow} A \leftarrow 0$$

where A^n is the direct sum of n isomorphic copies of A and the homomorphisms $d_0^*, d_1^*, d_2^*, d_3^*$ are given by d_0, d_1, d_2, d_3 respectively and given by

$d_0^*(a) = (a(h_1 - 1), a(h_2 - 1), a(h_3 - 1)),$

$d_1^*(a_1, a_2, a_3) = (a_1(1 - h_2h_3) + a_2(h_1 - h_3) - a_3, a_2(h_3 - 1) + a_3(1 - h_2),$
$a_1(h_3 - 1) + a_3(1 - h_1), a_3(h_3^{l-1} + \ldots\ldots + 1)),$

$d_2^*(a_1, a_2, a_3, a_4) = (a_1(h_3 - 1) - a_2(h_1 - h_3) - a_3(1 - h_2h_3), a_2(h_3^{l-1} + \ldots\ldots + 1)$
$+ a_4(h_2 - 1), a_3(h_3^{l-1} + \ldots\ldots + 1) + a_4(h_1 - 1), a_4(h_3 - 1)),$

$d_3^*(a_1, a_2, a_3, a_4) = (a_1(h_3^{l-1} + \ldots\ldots + 1) + a_2(h_1 - h_3) + a_3(1 - h_1h_2) - a_4, a_2(h_3 - 1))$
$+ a_4(1 - h_2), a_3(h_3 - 1) + a_4(1 - h_1), a_4(h_3^{l-1} + \ldots\ldots + 1)).$

When A is a trivial $\mathbb{Z}G$-module $\mathbb{Z}$, the above homomorphisms of the complex reduce to the following:

$d_0^*(a) = (0, 0, 0),$

$d_1^*(a_1, a_2, a_3) = (-a_3, 0, 0, la_3),$

$d_2^*(a_1, a_2, a_3, a_4) = (0, la_2, la_3, 0),$

$d_3^*(a_1, a_2, a_3, a_4) = (la_1 - a_4, 0, 0, la_4).$

Therefore, in this case, we have

$H^0(G, \mathbb{Z}) \cong \mathbb{Z}$

$$H^1(G, \mathbb{Z}) = \frac{Kerd_1^*}{lmd_0^*} = \frac{\{(a_2,a_2,0)|a_2,a_2 \in Z\}}{\{0,0,0\}} \cong \mathbb{Z} \oplus \mathbb{Z}$$

$$H^2(G, \mathbb{Z}) = \frac{Kerd_2^*}{lmd_1^*} = \frac{\{(a_2,0,0a_4)|a_2,a_4 \in Z\}}{\{-a_3,0,0,la_3)|a_3 \in Z\}} \cong \frac{(x)\oplus(y)}{(-x+ly)} \cong (x,y|x=ly) \cong \mathbb{Z}_{l^2}$$

$$H^3(G, \mathbb{Z}) = \frac{Kerd_3^*}{lmd_2^*} = \frac{\{(a_2,a_2,a_3,a_4)|la_2=a_4, la_4=0; a_1,a_2,a_3,a_4 \in Z\}}{\{0,la_2,la_3,0)|a_2,a_3 \in Z\}} \cong \mathbb{Z}_l \oplus \mathbb{Z}_l$$

$$H^4(G, \mathbb{Z})\ \frac{Kerd_2^*}{lmd_3^*} = \frac{\{(a_2,a_2,a_3,a_4)|a_2=0=a_3; a_1,a_2,a_3,a_4 \in Z\}}{\{(la_2-a_4,0,0,la_4)|a_2,a_4 \in Z\}}$$

$$\cong \frac{(x)\oplus(y)}{(lx,-x+ly)} \cong (x, y|lx = 0, x = ly) \cong (y|l^2 = 0) \cong \mathbb{Z}_{l^2}.$$

Therefore $H^{2n-1}(G, \mathbb{Z}) \cong \mathbb{Z}_l \oplus \mathbb{Z}_l$, $n \geq 2$.

H2n $(G, \mathbb{Z}) \cong \mathbb{Z}_{l^2}$, $n \geq 1$.

3. Determination of Homology and Cohomology of the Metacyclic Group

An infinite Metacyclic group M has a presentation:

generators : h_1, h_2;

relations : $h_2^r = 1$, $h_1 h_2\ h_1^{-1} = h_2^t$; $(r, t) = 1$.

Every finite metacyclic group can be regarded as a factor group of M.

We see that, $M = F/R$, where, F is free group generated by x_1, x_2 (say) and R is the normal subgroup of F generated by r_1, r_2 where $r_1 = x_2^r$, $r_2 = x_2^{-t} x_1 x_2 x_1^{-1}$. Then the Fox derivatives are:

$$\frac{\partial r_1}{\partial x_1} = 0 \quad , \quad \frac{\partial r_1}{\partial x_2} = x_2^{r-1} + \ldots\ldots + 1,$$

$$\frac{\partial r_2}{\partial x_1} = x_2 x_1^{-1} - x_1^{-1} , \quad \frac{\partial r_2}{\partial x_2} = -\left(x_2^{-t} + x_2^{-(t-1)} + \ldots\ldots\ldots + x_2^{-1} \right) x_1 x_2 x_1^{-1} + x_1^{-1}$$

Let $\pi : \mathbb{Z}F \to \mathbb{Z}M$ be the homomorphism induced by the canonical homomorphism of F onto M with R as the kernel. Let $\pi(x_1) = h_1, \pi(x_2) = h_2$.

Theorem 3.1

The following is a free $\mathbb{Z}M$-resolution of $\mathbb{Z}$:

$$\ldots\ldots\ldots\ldots\ldots\ Y \xrightarrow{d_4} Y \xrightarrow{d_3} Y \xrightarrow{d_2} Y \xrightarrow{d_1} Y \xrightarrow{d_0} \mathbb{Z}M \xrightarrow{\varepsilon} M \to 0$$

where Y is a right $\mathbb{Z}M$-module free on α, β and ε, d_0, d_1, d_2, $d_3\ldots$ are the $\mathbb{Z}M$-homomorphisms given by

$\varepsilon(m) = 1,\quad$ *for all* $m \in M$,

$d_0(\alpha) = h_1 - 1,\quad d_0(\beta) = h_2 - 1,$

$d_1(\alpha) = \beta(h_2^{r-1} + \ldots\ldots + 1),$

$d_1(\beta) = \alpha(h_2 - 1)\ h_1^{-1} + \beta h_1^{-1} - h_2^{-t}(1 + h_2 + \cdots + h_2^{t-1}),$

$d_2(\alpha) = \alpha(h_2 - 1),$

$d_2(\beta) = \alpha(t - h_1^{-1}) + \beta(h_2^{t(r-1)} + \cdots + h_2^{t} + 1),$

$d_3(\alpha) = \alpha(h_2^{r-1} + \ldots + 1),$

$d_3(\beta) = \alpha(h_1^{-1} - t(h_2^{t-1} + \cdots + 1)) + \beta(h_2^{t} - 1),$

$d_4(\alpha) = \alpha(h_2 - 1),$

$d_4(\beta) = \alpha(t^2 - h_1^{-1}) + \beta(h_2^{t(r-1)} + \cdots + h_2^{t} + 1),$

$d_5(\alpha) = \alpha(h_2^{r-1} + \ldots\ldots + 1),$

$d_5(\beta) = \alpha(h_1^{-1} - t^2(h_2^{t-1} + \cdots + 1)) + \beta(h_2^{t} - 1),$

$d_6(\alpha) = \alpha(h_2 - 1),$

$d_6(\beta) = \alpha(t^3 - h_1^{-1}) + \beta(h_2^{t(r-1)} + \cdots + h_2^{t} + 1)$, *and so on.*

From the nature of these maps and the modules the sequence is clearly exact.

Homology Groups of *M*

Let A be a left $\mathbb{Z}M$-module, then homology groups $H_n(M, \mathbb{Z})$ are given by the homology of the complex:

$$\ldots\ldots\ldots\ldots\ldots \to A^2 \xrightarrow{\bar{d}_4} A^2 \xrightarrow{\bar{d}_3} A^2 \xrightarrow{\bar{d}_2} A^2 \xrightarrow{\bar{d}_1} A^2 \xrightarrow{\bar{d}_0} A \to 0$$

where A^2 is written for $A \times A$, and the homomorphisms $\bar{d}_0, \bar{d}_1, \bar{d}_2, \bar{d}_3, \bar{d}_4, \ldots$, are induced by d_0, d_1, d_2, d_3, d_4, $\cdots$, respectively and are given by

$\bar{d}_0(a_1, a_2) = (h_1 - 1)a_1 + (h_2 - 1)a_2,$

$\overline{d}_1(a_1, a_2) = ((h_2 - 1)\ h_1^{-1} a_2, (h_2^{r-1} + \cdots + 1)a_1 + h_1^{-1} a_2 - (h_2^{t-1} + \cdots + 1)a_2),$

$\overline{d}_2(a_1, a_2) = ((h_2 - 1)a_1 + (t - h_1^{-1})a_2, (h_2^{t(r-1)} + \cdots + h_2^{t} + 1)a_2),$

$\overline{d}_3(a_1, a_2) = ((h_2^{r-1} + \cdots + 1)a_1 + (h_1^{-1} - t(h_2^{t-1} + \cdots + 1))a_2, (h_2^{t} - 1)a_2),$

$\overline{d}_4(a_1, a_2) = ((h_2 - 1)a_1 + (t^2 - h_1^{-1})a_2, (h_2^{t(r-1)} + \cdots + h_2^{t} + 1)a_2),$

$\overline{d}_5(a_1, a_2) = ((h_2^{r-1} + \cdots + 1)a_1 + (h_1^{-1} - t^2(h_2^{t-1} + \cdots + 1))a_2, (h_2^{t} - 1)a_2),$

$\overline{d}_6(a_1, a_2) = ((h_2 - 1)a_1 + (t^3 - h_1^{-1})a_2, (h_2^{t(r-1)} + \cdots + h_2^{t} + 1)a_2),$

. . . .

When A is a trivial $\mathbb{Z}M$-module $\mathbb{Z}$, the above homomorphisms reduce to the following:

$\overline{d}_0(a_1, a_2) = 0,$

$\overline{d}_1(a_1, a_2) = (0, ra_1 + a_2 - ta_2),$

$\overline{d}_2(a_1, a_2) = ((t - 1)a_2, ra_2),$

$\overline{d}_3(a_1, a_2) = (ra_1 + a_2 - t^2 a_2, 0),$

$\overline{d}_4(a_1, a_2) = ((t^2 - 1)a_2, ra_2),$

$\overline{d}_5(a_1, a_2) = (ra_1 + a_2 - t^3 a_2, 0),$

$\overline{d}_6(a_1, a_2) = ((t^3 - 1)a_2, ra_2),$

. . . .

Therefore, in this case, we have

$H_0(M, \mathbb{Z}) \cong \mathbb{Z},$

$$H_1(M, \mathbb{Z}) = \frac{Ker\overline{d}_0}{lm\overline{d}_1} = \frac{\{(a_1, a_2) \mid a_1, a_2 \in Z\}}{\{(0, ra_2 + (1-t)a_2) \mid a_1, a_2 \in Z\}} \cong \mathbb{Z} \oplus \mathbb{Z}_h(2) \text{ where } h^{(1)} = h.c.f.(r, 1-t)$$

$$H_2(M, \mathbb{Z}) = \frac{Ker\overline{d}_1}{lm\overline{d}_2} = \frac{\{(a_1, a_2) \mid ra_2 + (1-t)a_2 = 0; a_1, a_2 \in Z\}}{\{((t-1)a_1, ra_2) \mid a_1, a_2 \in Z\}} \cong \frac{Z_\infty(s, r)}{Z_\infty(h^{(2)}(s, r))} \cong \mathbb{Z}_h(2).$$

$$H_3(M, \mathbb{Z}) = \frac{Ker\overline{d}_2}{lm\overline{d}_3} = \frac{\{(a_1, a_2) \mid (t-1)a_1 = 0 = ra_2; a_1, a_2 \in Z\}}{\{(ra_1 + (1-t^2)a_2, 0) \mid a_1, a_2 \in Z\}} \cong \mathbb{Z}_{h^{(2)}}.$$

Where $h^{(2)} = (r, 1 - t^2)$, $r = h^{(2)}r''$, $1 - t^2 = h^{(2)}s''$.

$$H_4(M, \mathbb{Z}) = \frac{Ker\bar{d}_3}{lm\bar{d}_4} = \frac{\{(a_1,a_2)|ra_1+(1-t^2)a_2=0; a_1,a_2\in Z\}}{\{((t^2-1)a_1, ra_2)|a_1,a_2\in Z\}} \cong \frac{(s,r)}{(h^{(2)}(s,r))} \cong \mathbb{Z}_h{}^{(2)}.$$

$$H_5(M, \mathbb{Z}) = \frac{Ker\bar{d}_4}{lm\bar{d}_5} = \frac{\{(a_1,a_2)|(t^3-1)a_1=0=ra_2; a_1,a_2\in Z\}}{\{(ra_1+(1-t^3)a_2,0)|a_1,a_2\in Z\}} \cong \frac{(1,0)}{(h^{(3)}(1,0))} \cong \mathbb{Z}_h{}^{(3)},$$

where $h^{(3)}=(r, 1-t^3), r=h^{(3)}r'''$, $1-t^3 = h^{(3)}s'''$, since $1 = r''' a_1 + s''' a_2$ for some $a_1, a_2 \in \mathbb{Z}$.

From the nature of the homomorphisms $H_6(M, \mathbb{Z}) \cong \mathbb{Z}_h(3)$.

Thus, in general, $H_i(M, \mathbb{Z}) \cong \mathbb{Z}_h(n), i = 2n-1, 2n, n > 1$ *and* $h^{(n)} = (r, 1-t^n)$.

Cohomology Groups of *M*

Let A be a right $\mathbb{Z}M$-module, then the cohomology groups $H^n(M, A)$ are given by the homology of the complex:

$$\cdots\cdots\cdots\cdots\cdots \leftarrow A^2 \xleftarrow{d_6^*} A^2 \xleftarrow{d_5^*} A^2 \xleftarrow{d_4^*} A^2 \xleftarrow{d_3^*} A^2 \xleftarrow{d_2^*} A^2 \xleftarrow{d_2^*} A^2 \xleftarrow{d_0^*} A \leftarrow 0$$

where the homomorphisms $d_0^*, d_1^*, d_2^*, d_3^*, \cdots$ are induced by $d_0, d_1, d_2, d_3, \cdots$

respectively and given by

$d_0^*(a) = (a(h_1-1), a(h_2-1)),$

$d_1^*(a_1, a_2) = (a_2(h_2^{r-1}+\cdots+1), a_1(h_2-1)h_1^{-1} + a_2h_1^{-1} - a_2(h_2^{t-1}+\cdots+1)),$

$d_2^*(a_1, a_2) = (a_1(h_2-1), a_1(t-h_1^{-1}) + a_2(h_2^{t(r-1)}+\cdots+h_2^t+1))$

$d_3^*(a_1, a_2) = (a_1(h_2^{r-1}+\cdots+1), a_1(h_1^{-1} - t(h_2^{t-1}+\cdots+1)) + a_2(h_2^t-1)),$

$d_4^*(a_1, a_2) = (a_1(h_2-1), a_1(t^2-h_1^{-1}) + a_2(h_2^{t(r-1)}+\cdots+h_2^t+1)),$

$d_5^*(a_1, a_2) = (a_1(h_2^{r-1}+\cdots+1), a_1(h_1^{-1} - t^2(h_2^{t-1}+\cdots+1)) + a_2(h_2^t-1)),$

$d_6^*(a_1, a_2) = (a_1(h_2-1), a_1(t^3-h_1^{-1}) + a_2(h_2^{t(r-1)}+\cdots+h_2^t+1)),$

and so on, for all $a, a_1, a_2 \in A$.

When A is a trivial $\mathbb{Z}M$-module $\mathbb{Z}$, the above homomorphisms reduce to the following:

$d_0^*(a) = (0, 0),$

$d_1^*(a_1, a_2) = (ra_2, a_2 - ta_2),$

$d_2^*(a_1, a_2) = (0, ra_2 + a_1(t-1)),$

$d_3^*\ (a_1, a_2) = (ra_1, a_1(1-t^2)),$

$d_4^*\ (a_1, a_2) = (0, a_1(t^2-1) + ra_2),$

$d_5^*\ (a_1, a_2) = (ra_1, a_1(1-t^3)),$

$d_6^*\ (a_1, a_2) = (0, a_1(t^3-1) + ra_2).$

Therefore, in that case, we have

$H^0(M, \mathbb{Z}) \cong \mathbb{Z}$

$$H^1(M, \mathbb{Z}) \cong \frac{Kerd_1^*}{lmd_0^*} \cong \frac{\{(y_1, y_2) | y_2 = 0; y_1, y_2 \in Z\}}{\{0,0\}} \cong \mathbb{Z}$$

$$H^2(M, \mathbb{Z}) \cong \frac{Kerd_2^*}{lmd_1^*} \cong \frac{\{(y_1, y_2) | ry_2 + (t-1)y_1 = 0; y_1, y_2 \in Z\}}{\{(ry_1, (1-t)y_2) | y1, y_2 \in Z\}} \cong \frac{Z_\infty(r,s)}{Z_\infty(h^{(1)}(r', s'))} \cong \mathbb{Z}_h^{(1)}.$$

$$H_3(M, \mathbb{Z}) \cong \frac{Ker\bar{d}_2}{lm\bar{d}_3} = \frac{\{(y_2, y_2) | ry_2 = 0 = (1-t^2)y_2 = 0; y_2, y_2 \in Z\}}{\{(ry_2 + (1-t)y_1) | y_2, y_2 \in Z\}} \cong \frac{\{(0, y_2)\}}{\{(0, ((ry_2 + sy_2)h^{(1)})\}}$$

$\cong \mathbb{Z}_h^{(1)}.$

where $h^{(1)} = (r, 1-t)$, $1 = r' y_2 + s' y_1$.

$$H_4(M, \mathbb{Z}) \cong \frac{Ker\bar{d}_4}{lm\bar{d}_3} = \frac{\{(y_1, y_2) | ry_1 + (t^2-1)y_2 = 0; y_1, y_2 \in Z\}}{\{(ry_1, (1-t^2)y_1) | y_1, y_2 \in Z\}} \cong \frac{Z_\infty(r,s)}{Z_\infty(h^{(2)}(r'', s''))} \cong \mathbb{Z}_h^{(2)}.$$

$$H_5(M, \mathbb{Z}) \cong \frac{Ker\bar{d}_5}{lm\bar{d}_4} = \frac{\{(y_1, y_2) | ry_1 = 0 = (1-t^3)y_2; y_1, y_2 \in Z\}}{\{(0, ry_1 + (t^2-1)y_1) | y_1, y_2 \in Z\}} \cong \frac{\{(0, y_2)\}}{\{(0, (r'' y_2 + s'' y^2)h^{(2)})\}} \cong \mathbb{Z}_h^{(2)}.$$

where $h^{(2)} = (r, 1 - t^2)$, $1 = r''y_2 + s''y_1$.

$$H^6(M, \mathbb{Z}) \cong \frac{Ker\bar{d}_6}{lm\bar{d}_5} = \frac{\{(y_1, y_2) | ry_1 + (t^3-1)y_2 = 0; y_1, y_2 \in Z\}}{\{(ry_1, (1-t^3)y_2) | y_1, y_2 \in Z\}} \cong \frac{Z_\infty(r,s)}{Z_\infty(h^{(3)}(r''', s'''))} \cong \mathbb{Z}_h^{(3)}.$$

Thus in general, $H^i(M, \mathbb{Z}) \cong \mathbb{Z}_h^{(n)}$, for i = 2n, 2n + 1, n ≥ 1 and $h^{(n)} = (r, 1 - t^n)$.

REFERENCES

[1] Akhter, N. and Majumdar, S.; Determination of the homology and the cohomology of NEC groups with non-orientable orbit spaces and no reflexions, Bull. Cal. Math. Soc., 95(3), 2003, 207-218.

[2] Carbone, L.J., A filtration of the chain complex of rewriting system, DIMACS Series in Discrete Math.and Theoretical Computer Science Vol.25, 1996.

[3] Cartan, H. and Eilenberg, S: Homological algebra (Oxford University Press), 1956.

[4] Fox, R.H., Free Differential Calculus I, Ann. Math. 57, 3, 1953, p. 547-560.

[5] Grove, J.R.J., Rewriting systems and homology of groups, Proc. of the third International Conference on the Theory of Groups, Canberra, (1456 LNM, Springer-Verlag), 1989.

[6] Gruenberg, K.W. Resolutions by relations, J. of London Math. Soc., 35, 1960, 481-494.

[7] Gruenberg, K.W.,Cohomological topics in group theory, Spinger-Verlag, New York, 1970.

[8] Huebschmann, J.,Cohomology theory of aspherical groups and of small can- cellation groups, J. Pure Appl. Algebra, 14, 1979,137-143.

[9] Huebschmann, J., Perturbation theory and free resolutions for nilpotent groups of class 2, J. of Algebra, 126, 2, 1989, 348-399.

[10] Huebschmann, J., Cohomology of nilpotent groups of class 2, J. of Algebra 126, 1989, 400-450.

[11] Huebschmann, J. The mod-p cohomology rings of metacyclic groups, J. of Pure Appl. Algebra 60, 1989, 53-103.

[12] Learner, A.,Cohomology of groups, Lecture notes, Queen Mary College, University of London, 1965.

[13] Lyndon, R.C.,Cohomology theory of groups with a single defining relation, Ann. of Math. 52, 3, 1950, 656-665.

[14] Majumdar, S., A free resolution for a class of groups, J. Lond. Math. Soc., 2, 2, 1970, 615-619.

[15] Majumdar, S., A free resolution for a general class of groups, Tech. Report Math.Stat. (17) 17, 1971, ISI, Calcutta.

[16] Majumdar, S., Free resolutions for certain classes of groups, Proc. Edin. Math. Soc. 26, 1983, 173-179.

[17] Majumdar, S., Homology and cohomology of a class of polycyclic groups, Internal report, IC/84/177, 1984, ICTP, Trieste, Italy.

[18] Majumdar, S.,On the homology and cohomology of certain polycyclic groups, Internal report, IC/87/305,1987, ICTP, Trieste, Italy.

[19] Majumdar, S. and Akhter, N.: On a simple method of determining the homology and the cohomology for finitely presented groups, J. Bang. Acad. Sci. Vol. 34, No. 2, 2010, 103-114.

[20] Majumdar, S. and Akhter, N., Determination of the homology and the cohomology of the trifoil knot group and the fundamental group of an interesting manifold N ,Ganit: J. Bangladesh, Math. Soc., 22, 2002, 7-14.

[21] Majumdar, S. and Akhter, N., Determination of the homology and the cohomology of a class of infinite groups associated with groups of Fibonacci type,Bull. Cal. Math. Soc., 2008, 100.

[22] Scott, P., The geometries 3- manifolds, Bull. Lond. Math. Soc. Vol. 15, No. 56, 1983, 401-487.