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SEMI DERIVATIONS OF PRIME GAMMA RINGS

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ABSTRACT

Let M be a prime Γ-ring satisfying a certain assumption (*). An additive mapping $f : M \to M$ is a semi-derivation if $f(x\alpha y) = f(x)\alpha g(y) + x\alpha f(y) = f(x)\alpha y + g(x)\alpha f(y)$ and $f(g(x)) = g(f(x))$ for all x, y∈M and α∈Γ, where $g : M \to M$ is an associated function. In this paper, we generalize some properties of prime rings with semi-derivations to the prime Γ-rings with semi-derivations.

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1. Introduction

J. C. Chang [6] worked on semi-derivations of prime rings. He obtained some results of derivations of prime rings into semi-derivations. H. E. Bell and W. S. Martindale III [1] investigated the commutativity property of a prime ring by means of semi-derivations. C. L. Chuang [7] studied on the structure of semi-derivations in prime rings. He obtained some remarkable results in connection with the semi-derivations. J. Bergen and P. Grzesczuk [3] obtained the commutativity properties of semiprime rings with the help of skew (semi)-derivations. A. Firat [8] generalized some results of prime rings with derivations to the prime rings with semi-derivations.

In this paper, we generalize some results of prime rings with semi-derivations to the prime Γ-rings with semi-derivations.

2. Preliminaries

Let *M* and Γ be additive abelian groups. *M* is called a Γ-ring if for all *x*, *y*, $z \in M$, $\alpha, \beta \in \Gamma$ the following conditions are satisfied:

(i) $x\beta y \in M$,

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)y = x\alpha y + x\beta y$, $x\alpha(y + z) = x\alpha y + x\alpha z$,

(iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$.

Let M be a Γ-ring with center C(M). For any x, y∈M, the notation $[x, y]_\alpha$ and $(x, y)_\alpha$ will denote $x\alpha y - y\alpha x$ and $x\alpha y + y\alpha x$ respectively. We know that $[x\beta y, z]_\alpha = x\beta[y,z]_\alpha + [x,z]_\alpha\beta y + x[\beta,\alpha]_z y$ and $[x,y\beta z]_\alpha = y\beta[x,z]_\alpha + [x,y]_\alpha\beta z + y[\beta,\alpha]_x z$, for all x,y,z∈M and for all $\alpha,\beta\in\Gamma$. We shall take an assumption (*) $x\alpha y\beta z = x\beta y\alpha z$ for all $x,y,z\in M$, $\alpha,\beta\in\Gamma$. Using the assumption (*) the identities $[x\beta y, z]_\alpha = x\beta[y,z]_\alpha + [x,z]_\alpha\beta y$ and $[x,y\beta z]_\alpha=y\beta[x,z]_\alpha+[x,y]_\alpha\beta z$, for all $x,y,z\in M$ and for all $\alpha,\beta\in\Gamma$ are used extensively in our results. So we make extensive use of the basic commutator identities: $(x\beta y, z)_\alpha = (x, z)_\alpha\beta y + x\beta[y, z]_\alpha = [x, z]_\alpha\beta y + x\beta(y, z)_\alpha$. A Γ-ring M is to be n-torsion free if $nx = 0$, $x\in M$ implies $x = 0$. Recall that a Γ-ring M is prime if $x\Gamma M\Gamma y = 0$ implies that $x = 0$ or $y = 0$.

A mapping D from M to M is said to be commuting on M if $[D(x), x]_\alpha = 0$ holds for all $x\in M$, $\alpha\in\Gamma$, and is said to be centralizing on M if $[D(x), x]_\alpha\in C(M)$ holds for all $x\in M$, $\alpha\in\Gamma$. An additive mapping D from M to M is called a derivation if $D(x\alpha y) = D(x)\alpha y + x\alpha D(y)$ holds for all $x, y\in M$, $\alpha\in\Gamma$.

Let M be a Γ-ring. An additive mapping $d: M \to M$, is called a semi-derivation associated with a function $g: M \to M$, if, for all $x, y\in M$, $\alpha\in\Gamma$,

(i) $d(x\alpha y) = d(x)\alpha g(y) + x\alpha d(y) = d(x)\alpha y + g(x)\alpha d(y)$,

(ii) $d(g(x)) = g(d(x))$.

If $g = I$, i.e., an identity mapping of M, then all semi-derivations associated with g are merely ordinary derivations. If g is any endomorphism of M, then other examples of semi-derivations are of the form $d(x) = x - g(x)$.

Example 2.1

Let M_1 be a $\Gamma_1$ring and M_2 be a Γ_2-ring. Consider $M = M_1 \times M_2$ and $\Gamma = \Gamma_1 \times \Gamma_2$.

Define addition and multiplication on M and Γ by

$(m_1, m_2) + (m_3, m_4) = (m_1 + m_3, m_2 + m_4)$,

$(\alpha_1, \alpha_2) + (\alpha_3, \alpha_4) = (\alpha_1 + \alpha_3, \alpha_2 + \alpha_4)$,

$(m_1, m_2)(\alpha_1, \alpha_2)(m_3, m_4) = (m_1\alpha_1m_3, m_2\alpha_2m_4)$,

for every $(m_1, m_2), (m_3, m_4)\in M$ and $(\alpha_1, \alpha_2), (\alpha_3, \alpha_4)\in\Gamma$.

Under these addition and multiplication M is a Γ-ring. Let $\delta: M_1 \to M_1$ be an additive map and $\tau: M_2 \to M_2$ be a left and right $M_2{}^\Gamma$-module which is not a derivation. Define $d: M \to M$ such that $d((m_1, m_2)) = (0, \tau(m_2))$ and $g: M \to M$ such that $g((m_1, m_2)) = (\delta(m_1), 0)$, $m_1\in M_1$, $m_2\in M_2$. Then it is clear that d is a semi-derivation of M (with associated map g) which is not a derivation.

3. Semi Derivations of Prime Γ-rings

We obtain our results.

Lemma 3.1

Let M be a prime Γ-ring satisfying the assumption (*) and let $m\in M$. If

$[[m, x]_\alpha, x]_\alpha = 0$ for all $x\in M$, $\alpha\in\Gamma$, then $x\in C(M)$.

Proof

A linearization of $[[m, x]_\alpha, x]_\alpha = 0$ for all $x\in M$, $\alpha\in\Gamma$, gives

$$[[m, x]_\alpha, y]_\alpha + [[m, y]_\alpha, x]_\alpha = 0 \text{ for all } x, y\in M, \alpha\in\Gamma. \quad (1)$$

Replacing y by $y\beta x$ in (1) and using $[[m, x]_\alpha, x]_\alpha = 0$ for all $x\in M$, $\alpha\in\Gamma$, we obtain

$$0 = [[m, x]_\alpha, y\beta x]_\alpha + [[m, y\beta x]_\alpha, x]_\alpha = [[m, x]_\alpha, y]_\alpha\beta x + [[m, y]_\alpha\beta x + y\beta[m, x]_\alpha$$

$$= [[m, x]_\alpha, y]_\alpha\beta x + [[m, y]\alpha, x]_\alpha\beta x + [y, x]_\alpha\beta[m,x]_\alpha, \text{ for all } x\in M, \alpha\in\Gamma,$$

Applying (1), we then get $[y, x]_\alpha\beta[m, x]_\alpha = 0$, for all $x\in M$, $\alpha\in\Gamma$. Taking $y\beta z$ for y in this relation and using $[y\beta z, x]_\alpha = [y, x]_\alpha\beta z + y\beta[z, x]_\alpha$, we see that

$[y, x]_\alpha\beta z\beta[m, x]_\alpha = 0$, for all $x, y, z\in M$, $\alpha\in\Gamma$. In particular, $[m, x]_\alpha\beta z\beta[m,x]_\alpha = 0$, for all $x\in M$, $\alpha\in\Gamma$. Since M is prime, $[m, x]_\alpha = 0$. This implies $x\in C(M)$.

Theorem 3.2

Let M be a non-commutative 2-torsion free prime Γ-ring satisfying the condition (*) and d is a semi-derivation of M with $g: M \to M$ is an onto endomorphism. If the mapping $x \to [a\beta d(x), x]_\alpha$ for all $\alpha,\beta\in\Gamma$, is commuting on M, then a = 0 or d = 0.

Proof

Firstly, we assume that *a* be a nonzero element of M. Then we know that the mapping $x \to [a\beta d(x), x]_\alpha$ is commuting on M. Thus we have $[[a\beta d(x), x]_\alpha, x]_\alpha = 0$. By lemma 3.1, we have

$$[a\beta d(x), x]_\alpha = 0, \text{ for all } x\in M, \alpha,\beta\in\Gamma. \quad (2)$$

By linearizing (2), we have

$$[a\beta d(x), y]_\alpha + [a\beta d(y), x]_\alpha = 0, \text{ for all } x, y\in M, \alpha,\beta\in\Gamma. \quad (3)$$

From this relation it follows that

$$a\beta[d(x), y]_\alpha + [a, y]_\alpha\beta d(x) + a\beta[d(y), x]_\alpha + [a, x]_\alpha\beta d(y) = 0, \text{ for all } x, y\in M, \alpha,\beta\in\Gamma. \quad (4)$$

Replacing y by $y\delta x$ in (3) and using (2), we get

$$[a\beta d(x), y\delta x]_\alpha + [a\beta d(y\delta x), x]_\alpha = 0, \text{ for all } x, y\in M, \alpha,\beta\in\Gamma.$$

We get

$$y\delta[a\beta d(x), x]_\alpha + [a\beta d(x), y]_\alpha\delta x + [a\beta(d(y)\delta x + g(y)\delta d(x)), x]_\alpha$$

$$= [a\beta d(x), y]_\alpha\delta x + [a\beta d(y)\delta x, x]_\alpha + [a\beta g(y)\delta d(x), x]_\alpha$$

$$= a\beta[d(x), y]_\alpha\delta x + [a, y]_\alpha\beta d(x)\delta x + a\beta d(y)\delta[x, x]_\alpha + [a\beta d(y), x]_\alpha\delta x + a\beta g(y)\delta[d(x), x]_\alpha + [a\beta g(y), x]_\alpha\delta d(x)$$

$$= a\beta[d(x), y]_\alpha\delta x + [a, y]_\alpha\beta d(x)\delta x + a\beta[d(y), x]_\alpha\delta x + [a, x]_\alpha\beta d(y)\delta x + a\beta g(y)\delta[d(x), x]_\alpha + a\beta[g(y), x]_\alpha\delta d(x) + [a, x]_\alpha\beta g(y)\delta d(x) = 0 \text{ for all } x,y\in M, \alpha,\beta\in\Gamma. \quad (5)$$

Right multiplication of (3) by δx gives

$a\beta[d(x), y]_\alpha\delta x + [a, y]_\alpha\beta d(x)\delta x + a\beta[d(y), x]_\alpha\delta x + [a, x]_\alpha\beta d(y)\delta x = 0$, for all $x, y\in M$, $\alpha,\beta\in\Gamma$. (6)

Subtracting (6) from (5), we obtain

$a\beta g(y)\delta[d(x), x]_\alpha + a\beta[g(y), x]_\alpha\delta d(x) + [a, x]_\alpha\beta g(y)\delta d(x) = 0$ for all $x, y\in M$, $\alpha, \beta, \delta\in\Gamma$. (7)

Taking $a\lambda g(y)$ instead of $g(y)$ in (7), we have

$$a\beta a\lambda g(y)\delta[d(x), x]_\alpha + a\beta[a\lambda g(y), x]_\alpha\delta d(x) + [a, x]_\alpha\beta a\lambda g(y)\delta d(x)$$
$$= a\beta a\lambda g(y)\delta[d(x), x]_\alpha + a\beta a\lambda[g(y), x]_\alpha\delta d(x) + a\beta[a, x]_\alpha\lambda g(y)\delta d(x)$$
$$+ [a, x]_\alpha\beta a\lambda g(y)\delta d(x)$$
$$= a\beta a\lambda g(y)\delta[d(x), x]_\alpha + a\beta a\lambda[g(y), x]_\alpha\delta d(x) + a\beta[a, x]_\alpha\lambda g(y)\delta d(x)$$
$$+ [a, x]_\alpha\beta a\lambda g(y)\delta d(x)$$
$$=0 \text{ for all } x, y\in M, \alpha,\beta,\lambda,\delta\in\Gamma. \quad (8)$$

Left multiplication of (6) by $a\lambda$ leads to

$$a\lambda a\beta g(y)\delta[d(x), x]_\alpha + a\lambda a\beta[g(y), x]_\alpha\delta d(x) + a\lambda[a, x]_\alpha\beta g(y)\delta d(x)$$
$$=a\beta a\lambda g(y)\delta[d(x), x]_\alpha + a\beta a\lambda[g(y), x]_\alpha\delta d(x) + a\beta[a, x]_\alpha\lambda g(y)\delta d(x) = 0 \text{ for all } x, y\in M, \alpha,\beta,\delta,\lambda\in\Gamma. \quad (9)$$

Subtracting (9) from (8), we get $[a, x]_\alpha\beta a\lambda g(y)\delta d(x) = 0$ for all $x, y\in M$, $\alpha,\beta,\lambda,\delta\in\Gamma$.

Since M is prime, we obtain that for any $x \in M$ either $d(x) = 0$ or $[a, x]_\alpha = 0$.

It means that M is the union of its additive subgroups $P = \{x\in M: d(x) = 0\}$

and $Q = \{x\in M: [a, x]_\alpha\beta a = 0\}$. Since a group cannot be the union of two proper subgroups, we find that either $P = M$ or $Q = M$.

If $P = M$, then $d = 0$. If $Q = M$, then this implies that $[a, x]_\alpha\beta a = 0$, for all $x\in M$, $\alpha,\beta\in\Gamma$.

Let us take $x\delta y$ instead of x in this relation. Then we get $[a, x\delta y]_\alpha\beta a = 0$, for all $x\in M$, $\alpha,\beta\in\Gamma$.

We get $[a, x\delta y]_\alpha\beta a = x\delta[a, y]_\alpha\beta a + [a, x]_\alpha\delta y\beta a = [a, x]_\alpha\delta y\beta a = 0$, for all $x, y\in M$, $\alpha,\beta,\delta\in\Gamma$.

Since $a\in M$ is nonzero and M is prime, we obtain $a\in C(M)$. Thus by this

and (2), the relation (7) reduces to $a\beta[g(y), x]_\alpha\delta d(x) = 0$, for all $x, y\in M$, $\alpha,\beta,\delta\in\Gamma$.

Since g is onto, we see that $a\beta z\gamma[u, x]_\alpha\delta d(x) = z\beta a\gamma[u, x]_\alpha\delta d(x) = 0$, for all $x, u, z\in M$, $\alpha,\beta,\delta,\gamma\in\Gamma$. Now by primeness of M, we obtain that $[u, x]_\alpha\delta d(x) = 0$, for all $x, u\in M$, $\alpha,\beta,\delta\in\Gamma$.

Replacing u by $u\lambda w$, we get $[u, x]_\alpha \lambda w\delta d(x) = 0$, for all $x, u, w \in M$, $\alpha, \beta, \delta, \lambda \in \Gamma$. By the primeness of M, $[u, x]_\alpha = 0$ or $d(x) = 0$. Again using the fact that a group cannot be the union of two proper subgroups, it follows that $d = 0$, since M is non-commutative, i.e., $[u, x]_\alpha$. Hence we see that, in any case, $d = 0$. This completes the proof.

Theorem 3.3

Let M be a prime 2-torsion free Γ-ring satisfying the condition (*), d is a nonzero semi-derivation of M, with associated endomorphism g and $a \in M$. If $g \neq \pm I$ (I is an identity map of M), then $(d(M), a)_\alpha = 0$ if and only if $d((M, a)_\alpha) = 0$.

Proof

Suppose $(d(M), a)_\alpha = 0$. Firstly, we will prove that $d(a) = 0$. If $a = 0$ then $d(a) = 0$. So we assume that $a \neq 0$. By our hypothesis, we have $(d(x), a)_\alpha = 0$, for all $x \in M$, $\alpha \in \Gamma$.

From this relation, we get

$$0 = (d(x\beta a), a)_\alpha = (d(x)\beta g(a) + x\beta d(a), a)_\alpha$$

$$= d(x)\beta[g(a), a]_\alpha + (d(x), a)_\alpha \beta a + x\beta(d(a), a)_\alpha + [x, a]_\alpha \beta d(a),$$

and so, $[x, a]_\alpha \beta d(a) = 0$, for all $x \in M$, $\alpha, \beta \in \Gamma$. (10)

Now, replacing x by $x\delta y$ in (10), we get

$[x\delta y, a]_\alpha \beta d(a) = 0$, for all $x \in M$, $\alpha, \beta \in \Gamma$. By calculation we get,

$x\delta[y, a]_\alpha \beta d(a) + [x, a]_\alpha \delta y\beta d(a) = [x, a]_\alpha \delta y \beta d(a) = 0$, for all $x \in M$, $\alpha, \beta, \delta \in \Gamma$ (11)

The primeness of M implies that $[x, a]_\alpha = 0$ or $d(a) = 0$ that is, $a \in C(M)$ or $d(a) = 0$.

Now suppose that $a \in C(M)$. Since $(d(a), a)_\alpha = 0$, we have $d(a)\alpha a + a\alpha d(a)$

$= 2a\alpha d(a) = 0$. Since M is 2-torsion free, $a\alpha d(a) = 0$. Since we assumed that $0 \neq a$ and M is a prime Γ-ring, we get $d(a) = 0$. Hence we have $d((x, a)_\alpha)$

$= d(x\alpha a + a\alpha x) = d(x\alpha a)+ d(a\alpha x) =d(x)\alpha a + g(x)\alpha d(a) + d(a)\alpha g(x) + a\alpha d(x)$

$= (g(x), d(a))_\alpha + (d(x), a)_\alpha = (d(x), a)_\alpha) = 0$, for all $x \in M$, $\alpha \in \Gamma$. Hence $(d(x), a)_\alpha$ $= 0$.

Conversely, for all $x \in M$,

$0 = d((a\beta x, a)_\alpha) = d(a\beta(x, a)_\alpha + [a, a]_\alpha \beta x) = d(a\beta(x, a)_\alpha) = d(a)\beta(x, a)_\alpha + g(a)\beta d((x), a)_\alpha$.

We have

$d(a)\beta(x, a)_\alpha = 0$, for all $x \in M$, $\alpha, \beta \in \Gamma$. (12)

Replacing x by $x\delta y$ in (12), we get

$$0 = d(a)\beta(x\delta y, a)_\alpha = d(a)\beta x\delta[y, a]_\alpha + d(a)\beta(x, a)_\alpha \delta y = d(a)\beta x\delta[y, a]_\alpha.$$

This implies that $d(a)\beta x\delta[x, a]_\alpha = 0$, for all $x \in M$, $\alpha, \beta, \delta \in \Gamma$.

For the primeness of M, we have either d(a) = 0 or $a\in C(M)$. If d(a) = 0,

then we have $0 = d((x), a)_\alpha = (d(x), a)_\alpha + (d(a), g(x))_\alpha = (d(x), a)_\alpha$, for all $x\in M$, $\alpha\in\Gamma$.

This yields that $(d(x), a)_\alpha = 0$. If $a\in C(M)$, then we have $0 = d((a, a)_\alpha)$

$= 2d(a)\alpha(a + g(a))$. Since M is 2-torsion free, we obtain $d(a)\alpha(a + g(a)) = 0$. Since M is prime we have d(a) = 0 or a + g(a) = 0. But since g is different from ≠ ± I, we find that d(a) = 0. Finally, $(d(x), a)_\alpha = 0$ implies the required result.

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