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AROUND A CENTRAL ELEMENT OF A NEARLATTICE

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ABSTRACT

A nearlattice S is a meet semilattice together with the property that any two elements possessing a common upper bound have a supremum. It is well known that if $n \in S$ is a neutral and upper element then its isotope $S_n = (S; \cap)$ is again a nearlattice, where $x \cap y = (x \wedge y) \vee (x \wedge n) \vee (y \wedge n)$ for all $x, y \in S$. In this paper we have discussed the central elements in a nearlattice and also in a lattice. We included several characterizations of these elements. We showed that for a central element $n \in S$, $P_n(S) \cong (n]^d \times [n)$, where $P_n(S)$ is the set of principal n-ideals of S. Then we proved that for a central element $n \in S$, an element $t \in S$ is central if and only if it is central in S_n. We also proved that for a lattice L, L_n is again a lattice if and only if n is central. Finally we showed that B is a Boolean algebra if and only if B_n is a Boolean algebra with same complement when n is central. Moreover, $B \cong B_n$.

Keywords: Central element, Nearlattice, Isotope, Boolean algebra.

1. Introduction

By a nearlattice S, we will always mean a (lower) semilattice which has the property that any two elements possessing a common upper bound, have a supremum. Nearlattice will form a lattice if it has a largest element. A nearlattice S is distributive if and only if for all $x, y, z \in S$, $t \wedge ((x \wedge y) \vee (x \wedge z) = (t \wedge x \wedge y) \vee (t \wedge x \wedge z)$. Let S be a nearlattice and $S \in S$. Then S is called a standard element if for all $x, y, t \in S$, $t \wedge [(x \wedge y) \vee (x \wedge s)] = (t \wedge x \wedge y) \vee (t \wedge x \wedge s)$. In a nearlattice, an element s is neutral if for any $t, x, y \in S$, s is standard and $s \wedge [(t \wedge x) \vee (t \wedge y)] = (s \wedge t \wedge x) \vee (s \wedge t \wedge y)$. An element s of a nearlattice S is called a medial element if $\mathrm{m}(x, s, y) = (x \wedge y) \vee (x \wedge s) \vee (y \wedge s)$ exists for all $x, y \in S$. An element s of a nearlattice S is called sesquimedial if for all $x, y, z \in S$, $J_s(x, y, z)$ exists in S where $J_s(x, y, z) = [(x \wedge s) \vee (y \wedge s)] \wedge [(y \wedge s) \vee (z \wedge s)] \vee (x \wedge y) \vee (y \wedge z)$. Every sesquimedial element is medial. An element n of a nearlattice S is called an upper element if $x \vee n$ exists for all $x \in S$. Every upper element is of course sesquimedial. An element is called a central element of S if it is neutral, upper and complemented in each interval containing it. We know by [2] that for a neutral element $n \in S$ if n is sesquimedial then $S_n = (S; \cap)$ is again a nearlattice where $x \cap y - (x \wedge y) \vee (x \wedge n) \vee (y \wedge n)$. For a fixed element n of a nearlattice

S, a convex subnearlattice containing n is called an n-ideal. An n-ideal generated by a finite number of elements $a_1, ..., a_m$ is called a finitely generated n-ideal denoted by $(a_1, ..., a_m)_n$. Set of all finitely generated n-ideals of S is denoted by $F_n(S)$. An n-ideal generated by a single element is called a principal n-ideal. The set of all principal n-ideals of S is denoted by $P_n(S)$. If S is a lattice then $(a_1, ..., a_m)n = [a_1 \wedge ... \wedge a_m \wedge n, a1 \vee ... \vee a_m \vee n]$. Thus $(a)_n = [a \wedge n, a \vee n]$. For detailed literature on n-ideal of lattices and nearlatices we refer the reader to consult [3], [5], [7], [8]. In this paper we have given several characterizations of central elements of a nearlattice. We proved that for a central element $n \in S$, $P_n(S) \cong (n]^d \times [n)$. Then we proved that for a central element $n \in S$, an element $t \in S$ is central if and only if it is central in S_n. We also showed that for a lattice L, L_n is again a lattice if and only if n is central. Finally we extended a result of Goetz's result on isotopes of Boolean algebras.

1. Isotopes L_n when n is a Central Element

We start this paper with the following characterization of a central element of a nearlattice.

Theorem 1.1. *Let S be a nearlattice and $n \in S$. Then the following conditions are equivalent :*

(i) n is central ;
(ii) n is standard, upper and complemented in each interval containing it.

Proof. (i) $\Rightarrow$ (ii) is trivial from the definition .

(ii) $\Rightarrow$ (i). Suppose n is standard and complemented in each interval containing it. It is enough to prove that $n \wedge ((t \wedge x) \vee (t \wedge y) = (n \wedge t \wedge x) \vee (n \wedge t \wedge y)$

Since $(n \wedge t \wedge x) \vee (n \wedge t \wedge y) \leq n \leq (t \wedge x) \vee (t \wedge y) \vee n$, there exists $r \in S$ such that $n \wedge r = (n \wedge t \wedge x) \vee (n \wedge t \wedge y)$ and $n \vee r = (t \wedge x) \vee (t \wedge y) \vee n$

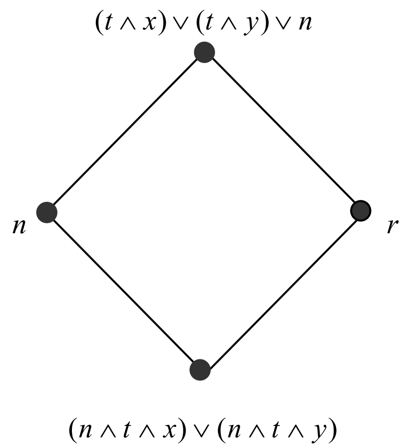

Figure 1

Now $t \wedge x = (t \wedge x) \wedge [(t \wedge x) \vee (t \wedge y) \vee n]$

$= (t \wedge x) \wedge (n \vee r)$

$= (t \wedge x \wedge n) \vee (t \wedge x \wedge r)$ (as n is standard).

Similarly $t \wedge y = (t \wedge y \wedge n) \vee (t \wedge y \wedge r)$

So $(t \wedge x) \vee (t \wedge y) = (t \wedge x \wedge n) \vee (t \wedge x \wedge r) \vee (t \wedge y \wedge n) \vee (t \wedge y \wedge r)$

$= (t \wedge x \wedge r) \vee (t \wedge y \wedge r) \vee (n \wedge r) \leq r$

Therefore $n \wedge ((t \wedge x) \vee (t \wedge y)) \leq n \wedge r$

$= (n \wedge t \wedge x) \vee (n \wedge t \wedge y)$

But the reverse inequality is trivial.

Hence $n \wedge ((t \wedge x) \vee (t \wedge y)) = (n \wedge t \wedge x) \vee (n \wedge t \wedge y)$

Therefore *n* is neutral, and so *n* is central □

We know from [1] that for a neutral element *n* of a lattice L, L_n is a medial nearlattice. Now we have the following result.

Theorem 1.2. *Suppose L is a lattice and* $n \in L$ *is standard. Then the isotope* L_n *is a lattice if and only if n is central in L.*

Proof. Since *n* is standard, so by [2, Theorem 2.1]

$(L; \subseteq) \cong (P_n(L); \subseteq)$

Thus L_n is a lattice if and only if $(P_n(L); \subseteq)$ is a lattice, But by [7],

$P_n(L)$ is a lattice if and only if *n* is complemented in each interval containing it. Therefore L_n *is a lattice* if and only if *n* is central in *L* □

Corollary 1.3. *For a central element* $n \in L$ *of a bounded lattice,* L_n *is also a bounded lattice with n as the smallest and n′ as the largest element.*

Moreover, $x \cap y = m(x, n, y)$ *and* $x \cup y = m(x, n', y)$ □

Theorem 1.4. *Let L be a bounded lattice and* $n \in L$ *be central. If* n' *is the complement of n then* n' *is also central.*

Proof. Let $a \leq n' \leq b$

Consider $(a \vee n) \wedge b$

Now $n' \wedge [(a \vee n) \wedge b] = [n' \wedge (a \vee n)] \wedge b$

$= [(a \wedge n') \vee (n \wedge n')] \wedge b$ (as n is standard)

$= [(a \wedge n') \vee 0] \wedge b$

$= a \wedge n' \wedge b = a$

$n' \vee [(a \vee n) \wedge b] = n' \vee (a \wedge b) \vee (b \wedge n)$ (as n is standard)

$= n' \vee a \vee (b \wedge n)$

$= n' \vee (b \wedge n)$

$= (b \wedge n') \vee (b \wedge n)$

$= b \wedge (n \vee n')$ (as n is standard)

$= b \wedge 1 = b$.

Therefore $(a \vee n) \wedge b$ is the complement of n' in $[a,b]$.

Now we shall show that for all $x, y \in L$,

$x \wedge (y \vee n') = (x \wedge y) \vee (x \wedge n')$

Now $n \wedge [x \wedge (y \vee n')] = x \wedge n \wedge (y \vee n')$

$= x \wedge [(y \wedge n) \vee (n \wedge n')]$ (as n is neutral)

$= x \wedge y \wedge n$

Also $n \wedge [(x \wedge y) \vee (x \wedge n')] = (x \wedge y \wedge n) \vee (x \wedge n \wedge n')$

$= x \wedge y \wedge n$ (as n is neutral)

Again $n \vee [x \wedge (y \vee n')] = (n \vee x) \wedge (y \vee n \vee n') = n \vee x$,

and $n \vee [(x \wedge y) \vee (x \wedge n')] = (x \wedge y) \vee n \vee (x \wedge n')$

$= (x \wedge y) \vee [(n \vee x) \wedge (n \vee n')]$, (as n is distributive)

$= (x \wedge y) \vee [(n \vee x) \wedge 1]$

$= (x \wedge y) \vee (n \vee x)$

$= n \vee x$

Therefore $x \wedge (y \vee n') = (x \wedge y) \vee (x \wedge n')$, (as n is standard).

So n' is standard.

Therefore n' is central by Theorem 1.1. □

The following result is due to Kolibiar [6]

Lemma 1.5. *If an element is central in a lattice then it is also central in the dual lattice.*

Proof. Let n be central in L. Suppose $a \leq_d n \leq_d b$ in L^d.

Then $b \leq n \leq a$ in L.

So there exists $t \in [b,a]$ such that $n \wedge t = b$ and $n \vee t = a$ in L.

Then $n \vee_d t = b$ and $n \wedge_d t = a$ in L^d.

Thus n is complemented in $[a,b]$ in L^d.

Moreover for all $x, y \in L^d$, $x \wedge_d (y \vee_d n) = x \vee (y \wedge n) = x \vee (x \wedge n) \vee (y \wedge n)$

$= x \vee [n \wedge (x \vee y)]$ (as n is neutral),

$= [(x \vee y) \wedge x] \vee [(x \vee y) \wedge n]$

$= (x \vee y) \wedge (x \vee n)$ (as n is standard),

$= (x \wedge_d y) \vee_d (x \wedge_d n)$.

This implies n is standard in L^d.

Therefore by Theorem 1.1, n is neutral and hence central in L^d □

Now we give a characterization of central element in a nearlattice.

Lemma 1.6. *Suppose* $S = A \times B$ *where A is a lattice and B is a nearlattice. Then any element* $t = (t_1, t_2)$ *of* S *is central if and only if* t_1, t_2 *are central in A and B respectively.*

Proof. Suppose $t = (t_1, t_2)$ is central in S.

Let $p_1 \leq t_1 \leq q_1$ in A. Then $(p_1, t_2) \leq (t_1, t_2) \leq (q_1, t_2)$

Then there exists $(r_1, r_2) \in S$ such that $(t_1, t_2) \wedge (r_1, r_2) = (p_1, p_2)$ and

$(t_1, t_2) \vee (r_1, r_2) = (q_1, t_2)$.

This implies $r_1 \wedge t_1 = p_1$ and $r_1 \vee t_1 = q_1$.

So t_1 is complemented in each interval containing it in A.

For $x, y \in A$, $(x, t_2) \wedge [(y, t_2) \vee (t_1, t_2)]$

$= ((x, t_2) \wedge (y, t_2)) \vee (x, t_2) \wedge (t_1, t_2)$, (as (t_1, t_2) is standard in S).

This implies $(x \wedge (y \vee t_1), t_2) = ((x \wedge y) \vee (x \wedge t_1), t_2)$, Then $x \wedge (y \vee t_1) = (x \wedge y) \vee (x \wedge t_1)$ and so t_1 is standard in *A*.

Thus t_1 is central in *A*.

Similarly t_2 is central in B.

Conversely, Let t_1, t_2 be central in A and B.

Let $(p_1, p_2) \leq (t_1, t_2) \leq (q_1, q_2)$.

This implies $p_1 \leq t_1 \leq q_1$ and $p_2 \leq t_2 \leq q_2$.

So there exists $r_1 \in A, r_2 \in B$, such that,

$p_1 = r_1 \wedge t_1, p_2 = r_2 \wedge t_2$, $q_1 = r_1 \vee t_1$ and $q_2 = r_2 \vee t_2$

Hence $(t_1, t_2) \wedge (r_1, r_2) = (p_1, p_2)$ and $(t_1, t_2) \vee (r_1, r_2) = (q_1, q_2)$

Therefore (r_1, r_2) is the relative complement of (t_1, t_2) in $[(p_1, p_2), (q_1, q_2)]$.

Again for $(x, y), (p, q) \in S$,

$$(x, y) \wedge [(p, q) \vee (t_1, t_2)] = (x \wedge (p \vee t_1), y \wedge (q \vee t_2))$$
$$= ((x \wedge p) \vee (x \wedge t_1), (y \wedge q) \vee (y \wedge t_2)) \text{ (as } t_1, t_2 \text{ are standard),}$$
$$= (x \wedge p, y \wedge q) \vee (x \wedge t_1, y \wedge t_2)$$
$$= ((x, y) \wedge (p, q)) \vee ((x, y) \wedge (t_1, t_2))$$

This implies (t_1, t_2) is standard in *S* and hence it is central □

Thus we have the following result .

Corollary 1.7. *For lattices A and B in* $L = A \times B$ *an element* $t = (t_1, t_2) \in A \times B$ *is central if and only if* t_1 *and* t_2 *are central in A and B respectively* □

Lemma 1.8. *For a central element n of a nearlattice S,* $S_n \cong (n]^d \times [n)$.

Proof. Consider the map $\varphi : S_n \to (n]^d \times [n)$, defined by $\varphi(a) = (a \wedge n, a \vee n)$

Suppose $a \leq b$ in S_n. Then $a = (a \wedge b) \vee (a \wedge n) \vee (b \wedge n)$

and so $b \wedge n \leq a \wedge n \leq a \vee n \leq b \vee n$.

Thus $a \wedge n \leq_d b \wedge n$ in $(n]^d$ and $a \vee n \leq b \vee n$ in $[n)$.

Hence $(a \wedge n, a \vee n) \leq (b \wedge n, b \vee n)$ in $(n]^d \times [n)$

Therefore φ is isotone (order preserving).

Now let $a, b \in S_n$ are such that $\varphi(a) \leq \varphi(b)$

That is $(a \wedge n, a \vee n) \leq (b \wedge n, b \vee n)$ in $(n]^d \times [n)$.

Then $a \wedge n \leq_d b \wedge n$ in $(n]^d$ and $a \vee n \leq b \vee n$ in $[n)$.

This implies $b \wedge n \leq a \wedge n \leq a \vee n \leq b \vee n$ in S.

So $\langle a \rangle_n \subseteq \langle b \rangle_n$.

Thus by [2, Theorem 2.1], $a \subseteq b$ in S_n

and this says that φ is order isomorphism.

Finally, Let $(t_1, t_2) \in (n]^d \times [n)$.

Then $t_1 \leq n \leq t_2$ since n is central,

so there exists $c \in S$, such that $t_1 = c \wedge n, t_2 = c \vee n$ implies

$(t_1, t_2) = (c \wedge n, c \vee n) = \varphi(c)$, implies φ is onto.

Therefore φ is isomorphism □

Now we include another characterization of a central element in a nearlattice.

Lemma 1.9. *Let n be a neutral and upper element of a nearlattice S.Define $\varphi : S \to (n] \times [n)$ by $\varphi(a) = (a \wedge n, a \vee n)$.*

Then the following conditions are equivalent:

(i) n is central;

(ii) φ is an isomorphism.

Proof. $(i) \Rightarrow (ii)$ $\varphi(a) = (a \wedge n, a \vee n)$

$$\begin{aligned}\varphi(a \wedge b) &= ((a \wedge b) \wedge n, (a \wedge b) \vee n) \\ &= ((a \wedge n) \wedge (b \wedge n), (a \vee n) \wedge (b \vee n)) \\ &= (a \wedge n, a \vee n) \wedge (b \wedge n, b \vee n) \\ &= \varphi(a) \wedge \varphi(b)\end{aligned}$$

Similarly $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$

So φ is homomorphism.

Now suppose $\varphi(a) = \varphi(b)$. This implies

$(a \wedge n, a \vee n) = (b \wedge n, b \vee n)$, and so,

$a \wedge n = b \wedge n$ and $a \vee n = b \vee n$.

This implies $a = b$

Therefore φ is one-one.

Let $t \in (n] \times [n)$.

Then $t = (t_1, t_2)$ such that $t_1 \in (n], t_2 \in [n)$.

Thus $t_1 \leq n \leq t_2$.

Then there exists $r \in S$ such that $r \wedge n = t_1, r \vee n = t_2$.

This implies $t = (r \wedge n, r \vee n) = \varphi(r)$, and so φ is onto.

Hence $S \cong (n] \times [n)$

$(ii) \Rightarrow (i)$. Let $a \leq n \leq b$

Since φ is an isomorphism, so it is onto.

Then $(a,b) \in (n] \times [n)$.

Since φ is onto, so there exists $r \in S$ such that $\varphi(r) = (a,b)$.

Thus $(r \wedge n, r \vee n) = (a,b)$, and so $r \wedge n = a, r \vee n = b$.

That is *r* is the relative complement of *n* in $[a,b]$.

Therefore *n* is central □

Thus we have the following results:

Theorem 1.10. *Let n be a central element of a nearlattice S. Then any* $t \in S$ *is central if and only if it is central in* S_n .

Proof. By the above Lemma, $S \cong (n] \times [n)$

So $t \in S$ is central in *S* if and only if it is also central in $(n] \times [n)$.

Then by Lemma 1.5. and Lemma 1.6 , *t* is central in $(n]^d \times [n)$.

As *t* is central in $(n]^d \times [n)$ then by Lemma 1.5, it is central in S_n as $(n]^d \times [n) \cong S_n$ □

Corollary 1.11. *For a central element n of a lattice L, L is distributive and relatively complemented if and only if* L_n *is so* □

Theorem 1.12. *If n is neutral in L, then the following conditions are equivalent:*

(i) *n is central and L is bounded;*

(ii) *There exists a unique* n' *such that for all* $x \in L, x = m(n, x, n')$

(iii) $(L_n;\cap,\cup)$ *is a bounded lattice with* n' *as the largest element and for any* $x, y \in L_n, x \cup y = m(x, n', y)$.

Proof. $(i) \Rightarrow (iii)$ follows from Theorem 1.3.

$(iii) \Rightarrow (ii)$. Since $x \cup y = m(x, n', y)$ for all $x, y \in L$ (ii) clearly follows by choosing $y = n$.

Moreover n' is unique as it is the largest element of L_n.

$(ii) \Rightarrow (i)$. Clearly $n \wedge n' \le x \le n \vee n'$ for all $x \in L$.

Thus *L* is bounded. Since *n* is neutral, we already know by [2, Theorem 2.3.] that L_n is a nearlattice with *n* as the smallest element.

Now (*ii*) says that obviously n' is the largest element of L_n, and hence L_n is a bounded lattice. Thus by Theorem 1.2, *n* is central □

Following result is due to Goetz [4].

Theorem 1.13. *Let* $(B; \wedge, \vee, ', 0, 1)$ *be a Boolean algebra,* $n \in B$. *Then* B_n *is also a Boolean algebra with* $B_n = (B; \cap, \cup, ', n, n')$.

Also the complement of any element is invariant under the formation of isotopes. Moreover $B \cong B_n$ □

We conclude the paper with the following result which is an extension of the above result.

Theorem 1.14. *Let* n *be a central element of the distributive lattice* $B = (B;\wedge,\vee,0,1)$. *If the isotope* $B_n = (B;\cap,\cup,', n, n')$ *is the Boolean algebra, then B is also Boolean algebra with the same complement.*

Proof. By [1, Theorem 3.1.6], *B* is distributive as B_n is so.

Let a' be the complement of *a* in B_n.

Then $a \cap a' = n$ and $a \cup a' = n'$.

Then $(a \vee a') \wedge n = (a \wedge n) \vee (a' \wedge n)$

$= (a \cap a') \wedge n$

$= n \wedge n = n = 1 \wedge n$

$n' = a \cup a'$

$= m(a, n', a')$

$= (a \vee n') \wedge (a \vee a') \wedge (a' \wedge n')$

$1 = n' \vee n$

$= n \vee [(a \vee n') \wedge (a \vee a') \wedge (a' \vee n')]$

$= (a \vee n \vee n') \wedge (a \vee a' \vee n) \wedge (a' \vee n \vee n')$

$= 1 \wedge (a \vee a' \vee n) \wedge 1$

$= a \vee a' \vee n$

So, $a \vee a' \vee n = 1 = 1 \vee n$.

Thus $a \vee a' = 1$ (by the neutrality of *n*).

Again, $0 = (a \cap a') \wedge n' = [(a \wedge a') \vee (a \wedge n) \vee (a' \wedge n)] \wedge n'$

$= (a \wedge a' \wedge n') \vee (a \wedge n \wedge n') \vee (a' \wedge n \wedge n')$

$= (a \wedge a' \wedge n') \vee o \vee o = a \wedge a' \wedge n'$

Thus $a \wedge a' \wedge n' = o \wedge n'$

Again $a \cup a' = n'$

Thus $m(a, n', a') = n'$ implies $(a \wedge a') \vee (a \wedge n') \vee (a' \wedge n') = n'$

This implies $(a \wedge a') \vee n' = n' = o \vee n'$

Hence $a \wedge a' = o$.

This implies that a' is also the complement of a in $(B; \wedge, \vee, o, 1)$.

Therefore *B* is also Boolean □

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