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# NUMERICAL SOLUTIONS OF VOLTERRA INTEGRAL EQUATIONS USING LEGENDRE POLYNOMIALS

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## ABSTRACT

In this paper, Legendre piecewise polynomials are used to approximate the solutions of linear Volterra integral equations. Both second and first kind integral equations with regular as well as weakly singular kernels are considered. A matrix formulation is given for linear Volterra integral equations by the technique of Galerkin method. Numerical examples are considered to verify the accuracy of the proposed derivations, and the numerical solutions in this paper are also compared with the existing methods in the published literature.

**Keywords:** Volterra integral equation, Galerkin method, Legendre polynomials

## 1. Introduction

In order to find out the numerical solutions of Integral solutions we have seen that there are many methods to solve analytically but a few methods for solving numerically various classes of integral equations [1] are available. Continuous or piecewise polynomials are incredibly useful as mathematical tools since they are precisely defined. They can be differentiated and integrated without difficulty. Bernsteinôs approximation were used in [2] by Maleknejad *et al* to find out the Numerical solution of Volterra integral equations. Bhattacharya and Mandal in [3] and Shirin and Islam in [4] studied on Volterra and Fredholm Integral Equations Using Bernstein Polynomials to find out their numerical solutions. Shahsavaran in [5] to solve Volterra integral equations of first and second kind Using Block ï Pulse Functions and Taylor expansion by Collocation Method. However, in this paper, we have solved Volterra integral equations of first and second kind numerically by the technique of very well-known Galerkin method [6] and Legendre piecewise polynomials [7] are used as trial function in the basis.

## 2. Legendre Polynomials

The general form of the Legendre polynomials [7] of *n*th degree is defined by

$$P_n(x) = \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{2^n\, r!\,(n-2r)!(n-r)!} x^{n-2r} \tag{1}$$

where, $[n/2]=\begin{cases} n/2 & if\ n\ is\ even \\ (n-1)/2 & if\ n\ is\ odd \end{cases}$

The first few Legendre polynomials are given below :

$$P_0(x)=1, \qquad P_1(x)=x, \qquad P_2(x)=\frac{1}{2}(3x^2-1), \qquad P_3(x)=\frac{1}{2}(5x^3-3x)$$

$$P_4(x)=\frac{1}{8}(35x^4-30x^2+3), \quad P_5(x)=\frac{1}{8}(63x^5-70x^3+15x)$$

$$P_6(x)=\frac{1}{16}(231x^6-315x^4+105x^2-5), \quad P_7(x)=\frac{1}{16}(429x^7-693x^5+315x^3-35x)$$

$$P_8(x)=\frac{1}{128}(6435x^8-12012x^6+6930x^4-1260x^2+35)$$

$$P_9(x)=\frac{1}{128}(12155x^9-25740x^7+18018x^5-4620x^3+315x)$$

$$P_{10}(x)=\frac{1}{256}(46189x^{10}-109395x^8+90090x^6-30030x^4+3465x^2-63)$$

Now the first six Legendre polynomials over the interval [-1, 1] are shown in Fig. 1(a), and the remaining six Legendre polynomials are shown in Fig. 1(b).

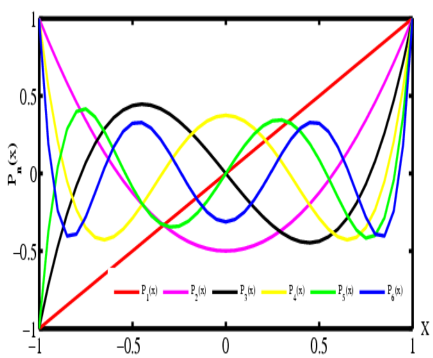


Fig. 1(a). Graph of first 6 Legendre polynomials over the interval [-1,1]

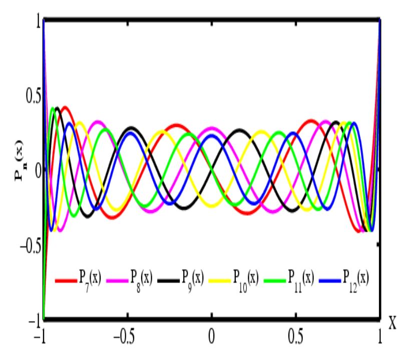


Fig. 1(b). Graph of last 6 Legendre polynomials over the interval [-1,1]

## 3. Formulation of Integral Equation in Matrix Form

We consider the Volterra integral equation (VIE) of the first kind [1] given by

$$\int_a^x k(x,t)\varphi(t)\,dt = f(x), \qquad a \le x \le b \tag{2}$$

where, $\varphi(x)$ is the unknown function, to be determined, $k(x,t)$ the kernel, is a continuous or discontinuous and square integrable function $f(x)$ being the known function satisfying $f(a)=0$.

Now we use the technique of Galerkin method, [Lewis, 6], to find an approximate solution $\tilde{\varphi}(x)$ of (2). For this, we assume that

$$\tilde{\varphi}(x)=\sum_{i=0}^{n} a_i P_i(x) \tag{3}$$

where, $P_i(x)$ are Legendre polynomials of degree $i$ defined in equation (1) and $a_i$ are unknown parameters, to be determined. Substituting (3) into (2), we get

$$\sum_{i=0}^{n} a_i \int_a^x k(x,t)\, P_i(t)\, dt = f(x),\ a \le x \le b \tag{4}$$

Then the Galerkin equations are obtained by multiplying both sides of (4) by $P_j(x)$ and then integrating with respect to $x$ from a to b, we have

$$\sum_{i=0}^{n} a_i \int_a^b \left[ \int_a^x k(x,t)\, P_i(t)\, dt \right] P_j(x) dx = \int_a^b P_j(x) f(x) dx,\ \ j=0,1,\ldots\ldots,n \tag{5}$$

Since in each equation, there are two integrals. The inner integrand of the left side is a function of $x$, and $t$ and is integrated with respect to $t$ form $a$ *to* $t$. As a result the outer integrand becomes a function of $x$ only and integration with respect to $x$ from $a$ *to* $b$ yields a constant. Thus for each $j=0,1,\ldots\ldots,n$, we have a linear equation with $n+1$ unknowns $a_i$, $i=0,1,\ldots\ldots,n$.

Finally, equation (5) represents the system of $n+1$ linear equations in $n+1$ unknowns, are given by

$$\sum_{i=0}^{n} a_i K_{i,j} = F_j ; i, j = 0,1,2,\cdots\cdots,n \tag{6a}$$

where, $$K_{i,j} = \int_a^b \left[ \int_a^x k(x,t)\, P_i(t)\, dt \right] P_j(x) dx,\ i,j=0,1,\ldots\ldots,n \tag{6b}$$

$$F_j = \int_a^b P_j(x) f(x) dx,\ \ j=0,1,\ldots\ldots,n \tag{6c}$$

Now the unknown parameters $a_i$ are determined by solving the system of equations (6) and substituting these values of parameters in (3), we get the approximate solution $\tilde{\varphi}(x)$ of the integral equation (2).

Now, we consider the Volterra integral equation (VIE) of the second kind [1] given by

$$c(x)\varphi(x)+\lambda \int_a^x k(x,t)\, \varphi(t)\, dt = f(x), \quad a \le x \le b \tag{7}$$

where $\varphi(x)$ is the unknown function to be determined, $k(x,t)$, the kernel, is a continuous or discontinuous and square integrable function, $f(x)$ and $c(x)$ being the known function and $\lambda$ is the constant. Then applying the same procedure as described above, we obtain

$$\sum_{i=0}^{n} a_i K_{i,j} = F_j ; i, j = 0,1,2,\cdots\cdots,n \tag{8a}$$

$$\text{where, } K_{i,j} = \left[\int_a^b \left[ c(x)P_i(x) + \lambda \int_a^x k(x,t)\,P_i(t)\,dt \right] P_j(x)dx \right],\ i, j = 0,1,\ldots\ldots,n \tag{8b}$$

$$F_j = \int_a^b P_j(x) f(x)dx,\ \ j = 0,1,\ldots\ldots,n \tag{8c}$$

Now the unknown parameters $a_i$ are determined by solving the system of equations (8) and substituting these values of parameters in (3), we get the approximate solution $\tilde{\varphi}(x)$ of the integral equation (7). The absolute relative error for this formulation is defined by

$$\text{Absolute Relative Error } = \left| \frac{\varphi(x) - \tilde{\varphi}(x)}{\varphi(x)} \right|$$

## 4. Numerical Examples

In this section, we explain both first and second kind Volterra integral equations, with regular and weakly singular kernels, which are available in the existing literature [1-3, 5] to verify the accuracy of our formulation presented in the previous section. The convergence of each linear Volterra integral equations is calculated by

$$E = \left| \tilde{\varphi}_{n+1}(x) - \tilde{\varphi}_n(x) \right| \prec \delta$$

where, $\tilde{\varphi}_n(x)$ denotes the approximate solution by the proposed method using $n$th degree polynomial approximation and $\delta$ varies from $10^{-6}$ for $n \geq 10$.

**Example 1:** Consider the VIE of first kind (regular kernels) [1, pp 20]

$$\int_0^x e^{x-t}\,\varphi(t)dt = x, \quad 0 \leq x \leq 1 \tag{9}$$

The exact solution is $\varphi(x) = 1 - x$. Using the formula derived in the previous section and solving the system (6) for $n \geq 1$, we get the approximate solution is $\tilde{\varphi}(x) = 1 - x$, which is the exact solution.

**Example 2:** Consider an Abelȏs integral equation (VIE of first kind with weakly singular kernels) of the form [2]

$$\int_0^x \frac{1}{\sqrt{(x-t)}}\varphi(t)dt = \frac{2}{105}\sqrt{x}\,(105-56x^2+48x^3) \quad 0 \le x \le 1 \tag{10}$$

The exact solution is $\varphi(x) = x^3 - x^2 + 1$. Results have been shown in Fig.2 for $n = 10$. The absolute relative errors are obtained in the order of $10^{-16}$ for $n = 10$. On the contrary, the accuracy is found nearly the order of $10^{-7}$ by Maleknejad *et al* in [2] for $n = 10$ using Bernstein approximation.

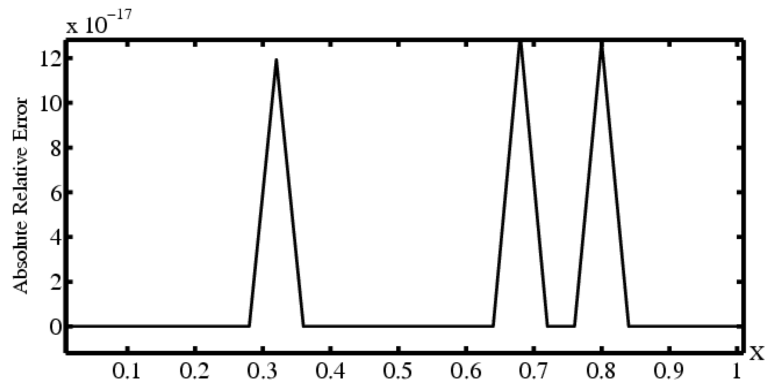


**Fig.2:** Absolute relative error of example 2 for $n = 10$

**Example 3:** Consider an Abelôs integral equation (VIE of first kind with weakly singular kernels) of the form [3]

$$\int_0^x \frac{1}{\sqrt{(x-t)}}\varphi(t)dt = x^5,\ 0 \le x \le 1 \tag{11}$$

The exact solution is $\varphi(x) = \frac{1280}{315\pi}x^{9/2}$. Results have been shown in Fig.3 $n = 10$. The absolute relative errors are found in the order of $10^{-7}$ for $n = 10$, while the accuracy were found in [3] by Bhattacharya and Mandal nearly the order of $10^{-7}$ for $n = 10$ using Bernstein polymomials.

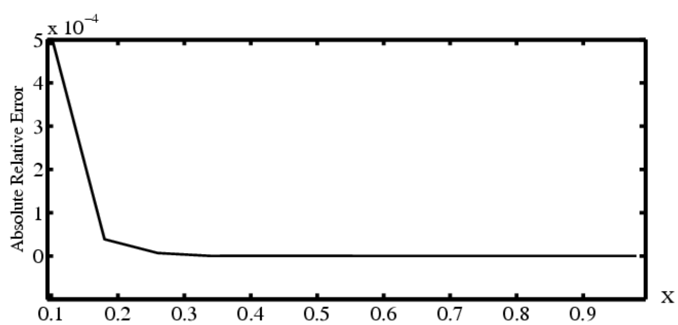


**Fig 3:** Absolute relative error of example 3 for $n = 10$

**Example 4:** Consider the weakly singular VIE of second kind [3]

$$\varphi(x)-\int_0^x \frac{1}{\sqrt{(x-t)}}\varphi(t)dt = x^7(1-\frac{4096}{6435}\sqrt{x}),\ 0\le x\le 1 \tag{12}$$

The exact solution is $\varphi(x)=x^7$. Results have been shown in Fig.4 for $n=10$.The absolute relative errors are obtained in the order of $10^{-16}$ for $n=10$.On the contrary, the accuracy is found nearly the order of $10^{-7}$ for $n=10$ in [3] by Bhattacharya and Mandal using Bernstein polynomials.

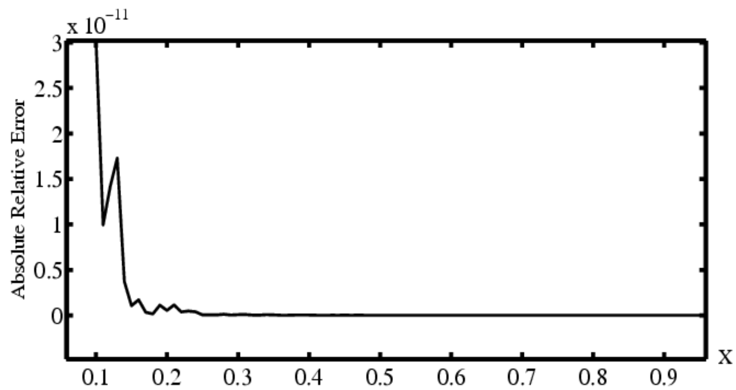


**Fig 4:** Absolute relative error of example 4 for $n=10$

**Example 5:** Consider the weakly singular VIE of second kind [5]

$$\varphi(x)+\int_0^x \frac{1}{\sqrt{(x-t)}}\varphi(t)dt = x^2+\frac{16}{15}x^{\frac{5}{2}},\ 0\le x\le 1 \tag{13}$$

The exact solution is $\varphi(x)=x^2$. Using the formula derived in the previous section and solving the system (8) for $n\ge 2$, we get the approximate solution is $\tilde{\varphi}(x)=x^2$, which is the exact solution. On the contrary, the accuracy is found nearly the order of $10^{-3}$ for $n=32$ in [5] by Shahsavaran using Block ï Pulse Functions and Taylor Expansion.

## 5. Conclusions

The objective of this paper is to present an efficient and accurate method to solve Volterra integral equation. In this paper, we have developed Galerkin method to approximate the solution of Volterra integral equation of first kind, second kind and also singular types of these equations. In this method we have used Legendre polynomials as trial functions in the basis. The proposed method is applied to solve a several number of Volterra integral equations both second and first kind with regular, as well as weakly singular kernels. The numerical results obtained by the proposed method are in good agreement with the exact solutions. In this paper, we may note that the numerical solutions coincide with the exact solutions even a few of the polynomials are used in the approximation, which are shown in examples 1 & 5. We also notice that the error terms are almost smaller than that of [2, 3, 5], which are shown in Fig. [2-4].

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