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ON THE USE OF PIECEWISE STANDARD POLYNOMIALS IN THE NUMERICAL SOLUTIONS OF FOURTH ORDER BOUNDARY VALUE PROBLEMS

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ABSTRACT

This paper is devoted to find the numerical solutions of the fourth order linear and nonlinear differential equations using piecewise continuous and differentiable polynomials, such as Bernstein, Bernoulli and Legendre polynomials with specified boundary conditions. We derive rigorous matrix formulations for solving linear and non-linear fourth order BVP and special care is taken about how the polynomials satisfy the given boundary conditions. The linear combinations of each polynomial are exploited in the Galerkin weighted residual approximation. The derived formulation is illustrated through various numerical examples. Our approximate solutions are compared with the exact solutions, and also with the solutions of the existing methods. The approximate solutions converge to the exact solutions monotonically even with desired large significant digits.

Keywords: Galerkin method, Fourth order linear and nonlinear BVP, Bernstein, Bernoulli and Legendre polynomials.

1. Introduction

In the literature of numerical analysis, there are many fourth order linear and nonlinear boundary value problems arising in science and engineering which are solved either analytically or numerically. For this, many authors have attempted to solving fourth order boundary value problem (BVP) to obtain high accuracy rapidly by using a numerous methods, such as least square method, finite difference method, Sinc-Galerkin method, and also some other methods using polynomial and nonpolynomial spline functions.

Since the piecewise polynomials can be differentiated and integrated easily, and can be approximated any function to any accuracy desired. So Bernstein polynomials have been studied by many authors [1 – 3], spline functions [6 – 10] have been studied extensively for solving only linear BVP. Recently Loghmani and Alavizadeh [7] has attempted to solve both linear and nonlinear BVP using least square method with B-splines. Special nonlinear boundary value problems have been studied by Twizell and Tirmizi [13] using multiderivatives with Pade` approximation method, also by El-Gamel *et al* [11] and only linear BVP by Smith *et al* [12] by the technique of Sinc-Galerkin methods.

Very recently, Bhatti and Bracken [1] used Bernstein polynomials for solving two point BVP by the Galerkin method, but it is limited only to second order linear BVP and to first order nonlinear IVP. Besides spline functions and Bernstein polynomials, there are another types of piecewise continuous polynomials, namely Bernoulli polynomials and Legendre polynomials, which are available in the book [Atkinson, 4]. But none has attempted, to the knowledge of the present authors, to solve the fourth order BVP using these polynomials. Therefore, the purpose of this paper is devoted to use three kinds of piece wise polynomials: Bernstein, Bernoulli and Legendre polynomials widely for solving linear and nonlinear fourth order BVP exploiting Galerkin weighted residual method.

However, in this paper, we give a short description on Bernstein, Bernoulli and Legendre polynomials and their properties first in section 2. Then we discuss in section 3, the formulation for solving linear fourth order BVP by Galerkin weighted residual method, using Bernstein, Bernoulli and Legendre polynomials as basis functions in the approximation, in details. Then we deduce similar formulation for nonlinear problems in the next section. Numerical examples, for both linear and nonlinear boundary value problems, are considered to verify the proposed formulation, and the obtained results are compared as well.

2. Some special polynomials

(a) Bernstein polynomials

The general form of the Bernstein polynomials of *n*th degree over the interval $[a,b]$ is defined by [1 – 3]

$$B_{i,n}(x) = \binom{n}{i} \frac{(x-a)^i (b-x)^{n-i}}{(b-a)^n}, \qquad a \le x \le b \qquad i = 0,1,2,\dots,n. \tag{1}$$

For example, the first 11 Bernstein polynomials of degree 10 over the interval $[0, 1]$ are given bellow:

$B_0(x) = (1-x)^{10}$

$B_1(x) = 10(1-x)^9 x$

$B_2(x) = 45(1-x)^8 x^2$

$B_3(x) = 120(1-x)^7 x^3$

$B_4(x) = 210(1-x)^6 x^4$

$B_5(x) = 252(1-x)^5 x^5$

$B_6(x) = 210(1-x)^4 x^6$

$B_7(x) = 120(1-x)^3 x^7$

$B_8(x) = 45(1-x)^2 x^8$

$B_9(x) = 10(1-x)x^9$

$B_{10}(x) = x^{10}$

Note that each of these $n+1$ polynomials having degree n satisfies the following properties:

(i) $B_{i,n}(x) = 0$ if $i < 0$ or $i > n$.

(ii) $\sum_{i=0}^{n} B_{i,n}(x) = 1$

$(iii)\ B_{i,n}(a) = B_{i,n}(b) = 0, \quad i = 1,2,\ldots,n-1$

For these properties, Bernstein polynomials are used in the trail functions satisfying the corresponding homogeneous form of the *essential* boundary conditions in the Galerkin method to solve a BVP.

(b) Bernoulli Polynomials

The Bernoulli polynomials [4] of degree n can be defined over the interval [0, 1] implicitly by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} b_k x^{n-k} \tag{2a}$$

where, b_k are Bernoulli numbers given by

$$b_0 = 1 \text{ and } b_k = -\int_0^1 B_k(x)\,dx \;\; k \ge 1. \tag{2b}$$

Also eqns. (2a) can be written explicitly as

$$B_0(x) = 1$$

$$B_m(x) = \sum_{n=0}^{m} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x+k)^m - \sum_{n=0}^{m} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^m, m \ge 1 \tag{3}$$

The first 10 Bernoulli polynomials are given bellow:

$$B_0(x) = 1 \qquad B_5(x) = -\frac{x}{6} + \frac{5x^3}{3} - \frac{5x^4}{2} + x^5$$

$$B_1(x) = x \qquad B_6(x) = -\frac{x^2}{2} + \frac{5x^4}{2} - 3x^5 + x^6$$

$$B_2(x) = -x + x^2 \qquad B_7(x) = \frac{x}{6} - \frac{7x^3}{6} + \frac{7x^5}{2} - \frac{7x^6}{2} + x^7$$

$$B_3(x) = \frac{x}{2} - \frac{3x^2}{2} + x^3 \qquad B_8(x) = \frac{2x^2}{3} - \frac{7x^4}{3} + \frac{14x^6}{3} - 4x^7 + x^8$$

$$B_4(x) = x^2 - 2x^3 + x^4 \qquad B_9(x) = -\frac{3x}{10} + 2x^3 - \frac{21x^5}{5} + 6x^7 - \frac{9x^8}{2} + x^9$$

(c) Legendre Polynomials

The general form of the Legendre polynomials [4] of degree n is defined by

$$P_n(x) = \frac{(-1)^n}{2^n(n!)}\frac{d^n}{dx^n}[(1-x^2)^n], \qquad n \ge 1. \tag{4}$$

The first few Legendre polynomials are given below :

$$P_0(x) = 1, \; P_1(x) = x, \; P_2(x) = \frac{1}{2}(3x^2 - 1), \; P_3(x) = \frac{1}{2}(5x^3 - 3x),$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \qquad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$$

$$P_7(x) = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$$

$$P_8(x) = \frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)$$

$$P_9(x) = \frac{1}{128}(12155x^9 - 25740x^7 + 18018x^5 - 4620x^3 + 315x)$$

$$P_{10}(x) = \frac{1}{256}(46189x^{10} - 10939x^8 + 90090x^6 - 30030x^4 + 3465x^2 - 63)$$

$$P_{11}(x) = \frac{1}{256}(88179x^{11} - 230945x^9 + 218790x^7 - 90090x^5 + 15015x^3 - 693x)$$

3. Galerkin Weighted Residual Formulation of 4th order BVP

In this section we first obtain the rigorous formulation for fourth order linear BVP and then we extend our idea for solving nonlinear BVP. For this, we consider a linear fourth order differential equation

$$\frac{d^2}{dx^2}\left(p(x)\frac{d^2u}{dx^2}\right) + r(x)u = s(x), \; a \le x \le b, \tag{5a}$$

with Dirichlet boundary conditions

$$u(a) = a_1, \qquad u(b) = a_2 \tag{5b}$$

and with Neumann boundary conditions

$$u''(a) = c_1, \qquad u''(b) = c_2, \tag{5c}$$

where $p(x)$, $s(x)$ and $r(x)$ are specified continuous functions, and a_1, a_2, c_1, c_2 are specified numbers. We want to solve the BVP of the form (5) by Galerkin method [5] using the polynomials, described in section 2, as trial functions.

We approximate the solution of the differential equation (5a) as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n-1} a_i B_i(x), \qquad n \geq 2 \tag{6}$$

Here $\theta_0(x)$ is specified by the essential boundary conditions and $B_i(a) = B_i(b) = 0,$ for each $i = 1,2,3,\ldots,n-1.$

Using (6) into Eq. (5a), the Galerkin weighted residual equations are:

$$\int_a^b \left[\frac{d^2}{dx^2}\left(p(x)\frac{d^2\tilde{u}}{dx^2} \right) + r(x)\tilde{u} - s(x) \right] B_j(x)dx = 0, \qquad j = 1,2,\ldots,n-1, \tag{7}$$

Now integrating the first part of (7) by parts, we have

$$\int_a^b \left[\frac{d^2}{dx^2}\left(p(x)\frac{d^2\tilde{u}}{dx^2} \right) \right] B_j(x)dx = \left[\frac{d}{dx}\left(p(x)\frac{d^2\tilde{u}}{dx^2} \right) B_j(x) \right]_a^b - \int_a^b \frac{d}{dx}\left(p(x)\frac{d^2\tilde{u}}{dx^2} \right)\frac{dB_j}{dx}dx$$

$$= \left[\frac{d}{dx}\left(p(x)\frac{d^2\tilde{u}}{dx^2} \right) B_j(x) - p(x)\frac{d^2\tilde{u}}{dx^2}\frac{dB_j}{dx} \right]_a^b + \int_a^b p(x)\frac{d^2\tilde{u}}{dx^2}\frac{d^2B_j}{dx^2}dx,$$

$$= \int_a^b p(x)\frac{d^2\tilde{u}}{dx^2}\frac{d^2B_j}{dx^2}dx - c_2 p(b)\frac{dB_j}{dx}(b) + c_1 p(a)\frac{dB_j}{dx}(a), \;\; j = 1,2,\ldots,n-1, \tag{8}$$

since $B_j(a) = B_j(b) = 0$ and using the boundary condition in Eq. (5c).

On using (8), the Eq. (7) leads us

$$\int_a^b \left[p(x)\frac{d^2\tilde{u}}{dx^2}\frac{d^2B_j}{dx^2} + r(x)\tilde{u}(x)B_j(x) \right] dx$$

$$= c_2 p(b)\frac{dB_j}{dx}(b) - c_1 p(a)\frac{dB_j}{dx}(a) + \int_a^b s(x)B_j(x)dx \quad j = 1,2,\ldots,n-1 \tag{9}$$

Since from (6), we have

$$\frac{d^2\tilde{u}}{dx^2} = \frac{d^2\theta_0}{dx^2} + \sum_{i=1}^{n-1} a_i \frac{d^2B_i}{dx^2}$$

then Eq. (9) becomes to

$$\sum_{i=1}^{n-1}\left[\int_a^b \left[p(x)\frac{d^2B_i}{dx^2}\frac{d^2B_j}{dx^2} + r(x)B_i(x)B_j(x) \right] dx \right] a_i$$

$$= \int_a^b s(x)B_j(x)dx - \int_a^b \left[p(x)\frac{d^2\theta_0}{dx^2}\frac{d^2B_j}{dx^2} + r(x)\theta_0(x)B_j(x) \right]dx$$

$$+ c_2 p(b)\frac{dB_j}{dx}(b) - c_1 p(a)\frac{dB_j}{dx}(a)$$

or, equivalently

$$\sum_{i=1}^{n-1} D_{i,j}\ a_i = F_j, \qquad j = 1,2,\ldots,n-1 \tag{10a}$$

Where

$$D_{i,j} = \int_a^b \left[p(x)\frac{d^2B_i}{dx^2}\frac{d^2B_j}{dx^2} + r(x)B_i(x)B_j(x) \right]dx \tag{10b}$$

$$F_j = \int_a^b s(x)B_j(x)dx - \int_a^b \left[p(x)\frac{d^2\theta_0}{dx^2}\frac{d^2B_j}{dx^2} + r(x)\theta_0(x)B_j(x) \right]dx$$

$$+ c_2 p(b)\frac{dB_j}{dx}(b) - c_1 p(a)\frac{dB_j}{dx}(a), \qquad j = 1,2,\cdots\cdots,n-1 \tag{10c}$$

Solving the system (10a), we find the values of the parameters a_i, and then substituting into (6), we get the approximate solution of the desired BVP (5).

For nonlinear fourth order BVP, we first compute the initial values on neglecting the nonlinear terms and using the system (10a). Then using the iterative method we find the numerical approximations for desired nonlinear BVP. This formulation is described through the numerical examples in the next section.

4. Numerical Examples

In this section, we consider four linear and one nonlinear problems to verify the proposed formulation in section 3. For this, we give the results for linear problems in brief depending on prescribed boundary conditions, but the nonlinear problem is illustrated in details. All the computations are performed by *MATLAB*. Since the convergence of linear BVP is calculated by

$$E = \left| \tilde{u}_{n+1}(x) - \tilde{u}_n(x) \right| < \delta$$

where $\tilde{u}_n(x)$ denotes the approximate solution using *n* polynomials and δ depends on the problem which varies from 10^{-7}. In addition, the convergence of nonlinear BVP is assumed when the absolute error of two consecutive iterations, δ satisfies

$$\left| \tilde{u}_n^{N+1} - \tilde{u}_n^{N} \right| < \delta$$

where N is the Newton's iteration number and δ varies from 10^{-11}.

Example 1: We first consider the BVP [6]:

$$\frac{d^4u}{dx^4} - u = -4(2x\cos x + 3\sin x),\ 0 < x < 1, \tag{11a}$$

$$u(0) = u(1) = 0 \tag{11b}$$

$$u''(0) = 0,\quad u''(1) = 2\sin 1 + 4\cos 1 \tag{11c}$$

whose exact solution is, $u(x) = (x^2 - 1)\sin x$.

Using the method illustrated in section 3, we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n-1} a_i B_i(x), \qquad n \ge 2 \tag{12}$$

Here $\theta_0(x) = 0$ as specified by the essential boundary conditions of Eq. (11b).

Now the parameters $a_i (i = 1,2,.......,n-1)$ satisfy the linear system

$$\sum_{i=1}^{n-1} D_{i,j} a_i = F_j, \qquad j = 1,2,\ldots,n-1, \tag{13a}$$

where

$$D_{i,j} = \int_0^1 \left[\frac{d^2B_i}{dx^2}\frac{d^2B_j}{dx^2} - B_i(x)B_j(x) \right] dx \tag{13b}$$

$$F_j = \int_0^1 -4(2x\cos x + 3\sin x)B_j(x)dx + (2\sin 1 + 4\cos 1)\frac{dB_j}{dx}(1),\quad j = 1,2,\ldots,n-1 \tag{13c}$$

Solving the system (13a), we find the values of the parameters and then substituting these parameters in Eq.(12), we get the approximate solution of the BVP (11) for different values of n.

Table 1: Observed maximum errors for the example 1.

Number of Polynomial used	Bernstein	Bernoulli	Legendre	Reference results
3	1.14×10^{-3}	1.14×10^{-3}	1.13×10^{-3}	1.417×10^{-11}, [6]
5	2.85×10^{-6}	2.85×10^{-6}	2.85×10^{-6}	
7	4.43×10^{-9}	4.43×10^{-9}	4.43×10^{-9}	
9	1.36×10^{-12}	3.50×10^{-12}	3.50×10^{-12}	

The absolute errors, using different number of polynomials, with previous results obtained in [6], are summarized in Table 1.

Example 2: Consider the BVP [6, 7, 8, 9]

$$\frac{d^4u}{dx^4} + xu = -(8+7x+x^3)e^x,\ 0 < x < 1 \tag{14a}$$

$$u(0) = u(1) = 0\,, \tag{14b}$$

$$u''(0) = 0,\ \ u''(1) = -4e\,, \tag{14c}$$

with exact solution, $u(x) = x(1-x)e^x$.

Our absolute errors for different number of polynomials, shown in Table 2, to compare with existing methods.

Table 2: Observed maximum errors for the example 2.

Number of Polynomial used	Bernstein	Bernoulli	Legendre	Reference results
3	2.46×10^{-3}	2.47×10^{-3}	2.47×10^{-3}	1.42×10^{-11}, [6]
5	5.94×10^{-6}	5.94×10^{-6}	5.94×10^{-6}	3.72×10^{-11}, [7]
7	9.01×10^{-9}	9.01×10^{-9}	9.01×10^{-9}	4.96×10^{-8}, [8]
9	7.13×10^{-12}	7.12×10^{-12}	7.15×10^{-12}	5.37×10^{-6}, [9]

Example 3: We consider another BVP [8]

$$\frac{d^4x}{dx^4} - xu = -(11+9x+x^2-x^3)e^x,\ -1 \le x \le 1 \tag{15a}$$

$$u(-1) = u(1) = 0 \tag{15b}$$

$$u''(-1) = \frac{2}{e}, u''(1) = -6e\,, \tag{15c}$$

The exact solution of this BVP is, $u(x) = (1-x^2)e^x$.

Table 3: Observed maximum errors for the example 3.

Number of Polynomial used	Bernstein	Bernoulli	Legendre	Reference results
3	5.26×10^{-2}	5.26×10^{-2}	5.26×10^{-2}	1.84×10^{-5}, [8]
5	4.72×10^{-4}	4.72×10^{-4}	4.72×10^{-4}	
7	2.83×10^{-6}	2.87×10^{-6}	2.87×10^{-6}	
9	9.11×10^{-9}	9.12×10^{-9}	9.12×10^{-9}	

The absolute errors, shown in Table 3, are listed to compare with existing results obtained in [8].

Example 4: We consider linear BVP [9, 10]:

$$\frac{d^4u}{dx^4} + 4u = 1, \; -1 \le x \le 1 \tag{16a}$$

$$u(-1) = u(1) = 0 \, , u''(-1) = u''(1) = 0 \tag{16b}$$

whose exact solution is

$$u(x) = \frac{1}{4}\left[1 - 2\frac{\sin(1)\sinh(1)\sin(x)\sinh(x) + \cos(1)\cosh(1)\cos(x)\cosh(x)}{\cos(2)+\cosh(2)}\right] .$$

In Table 4, we list the absolute errors are obtained by the present method to compare with the results obtained so far.

Table 4: Observed maximum errors for the example 4.

Number of Polynomial used	Bernstein	Bernoulli	Legendre	Reference results
7	8.71×10^{-9}	8.71×10^{-9}	8.72×10^{-9}	6.22×10^{-6}, [9] 2.82×10^{-12}, [10]
9	2.912×10^{-11}	2.912×10^{-11}	2.912×10^{-11}	
12	8.50×10^{-14}	8.50×10^{-14}	8.50×10^{-14}	
15	8.33×10^{-17}	8.33×10^{-17}	5.55×10^{-17}	

Example 5: We consider the 4th order nonlinear BVP [11, 13]

$$\frac{d^4u}{dx^4} = 6e^{-4u} - 12(1+x)^{-4} \, , 0 \le x \le 1 \tag{17a}$$

$$u(0) = 0 \, , \;\; u(1) = \ln 2 \tag{17b}$$

$$u''(0) = -1, \;\; u''(1) = -\frac{1}{4} \tag{17c}$$

The exact solution of this BVP is $u(x) = \ln(1+x)$.

We approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n-1} a_i B_i(x), \qquad n \ge 2 \tag{18}$$

Here $\theta_0(x) = x\ln 2$ is specified by the essential boundary conditions in (17b). Also $B_i(0) = B_i(1) = 0$ for each $i = 1,2,3,\ldots,n-1$

Using (18) into Eqn. (17a), the Galerkin weighted residual equations are:

$$\int_0^1 \left[\frac{d^4\tilde{u}}{dx^4} - 6e^{-4\tilde{u}} + 12(1+x)^{-4} \right] B_j(x)dx = 0, \qquad j = 1,2,3,\ldots,n-1. \tag{19}$$

Using the method discussed in section 3 and with minor simplifications, the above equations (19) are equivalent to

$$D\,A = B + G \tag{20a}$$

where the elements of column matrix A are a_i, and the elements of the matrix D and the column matrices B and G are given by

$$d_{i,j} = \int_0^1 \frac{dB_i}{dx}\frac{dB_j}{dx}dx \tag{20b}$$

$$b_j = \int_0^1 6\exp[-4\theta_0]\exp[-4\sum_{k=1}^{n-1} a_k B_k(x)]B_j(x)dx \tag{20c}$$

$$g_j = -\int_0^1 12(1+x)^{-4} B_j(x)dx - \int_0^1 \frac{d^2\theta_0}{dx^2}\frac{d^2B_j}{dx^2}dx - \frac{1}{4}\frac{dB_j}{dx}(1) + \frac{dB_j}{dx}(0) \tag{20d}$$

Now the initial values of the coefficients a_i are obtained by applying the modified Galerkin method to the BVP neglecting the nonlinear term in (20a). That is, to find initial coefficients we will solve the system only

$$D\;A = G \tag{21a}$$

whose matrices are constructed from

$$d_{i,j} = \int_0^1 \frac{dB_i}{dx}\frac{dB_j}{dx}dx \tag{21b}$$

$$g_j = -\int_0^1 12(1+x)^{-4} B_j(x)dx - \int_0^1 \frac{d^2\theta_0}{dx^2}\frac{d^2B_j}{dx^2}dx - \frac{1}{4}\frac{dB_j}{dx}(1) + \frac{dB_j}{dx}(0) \tag{21c}$$

Once the initial values of the a_i are obtained from Eqn. (21a), they are substituted into Eqn. (20a) to obtain new estimates for the values of a_i. This iteration process continues until the converged values of the unknown parameters are obtained. Substituting the final values of the parameters in Eqn. (18), we obtain an approximate solution of the BVP (17).

Since the absolute errors, for different number of polynomials, are shown in Table 5 with 5 iterations.

Table 5: Observed maximum errors for the example 5 with five iterations.

Number of Polynomial used	Bernstein	Bernoulli	Legendre	Reference results
3	2.80×10^{-4}	2.80×10^{-4}	2.80×10^{-4}	2.2×10^{-8}, [11] 6.5×10^{-5}, [13]
5	4.71×10^{-6}	4.71×10^{-6}	4.59×10^{-6}	
7	9.17×10^{-8}	9.17×10^{-8}	9.17×10^{-8}	
9	1.10×10^{-9}	8.99×10^{-8}	3.10×10^{-9}	

5. Conclusion

In this paper first we have used Bernstein, Bernoulli and Legendre, the piecewise continuous and differentiable polynomials, for solving fourth order linear and nonlinear BVPs in the Galerkin method. The concentration has given not only on the performance of the results but also on the formulation. We may notice that the formulation of this paper is very easy to understand and may be implemented to solve for BVP whose analytical solution is not available. The results of each problem guarantee the convergence and stability. Some results confirm also with great accuracy than the results obtained by the previous methods so far.

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