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HAUSDORFF'S GRUNDZŰGE DER MENGENLEHRE

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Felix Hausdorff (1868 – 1942)'s Grundzűge der Mengenlehre (Outlines of set Theory) [2] ranks among the most influential mathematical works of all times. Its impact was immediate and decisive. The title is not however not fully indicative of its contents. Hausdorff presents in the first six chapters a thorough and elegant account of Cantor's set theory (the book is dedicated to Georg Cantor (1845 – 1918), "the creator of set theory, in thankful reverence") and goes on in the seventh chapter to apply set-theoretic concepts and methods to point-sets in general spaces. Hausdorff begins with an illuminating discussion of various possible approaches to the study of point-sets in general spaces, pointing out that any of three concepts **-** distance, neighbourhood, limit **-** can be taken as a starting point for building up a general theory. Hausdorff mentions that with the help of distance one can define neighbourhood and limit; with the help of neighbourhood one can define limit but not in general distance; with the help of limit one can define in general neither neighbourhood nor distance. "For various reasons, we prefer to build up the fundamental considerations of this chapter on neighbourhoods and bring up the other two concepts only later. However in order to give the reader at once the feeling of a complete picture we begin with the special type of neighbourhoods defined by distance", says Hausdorff [2, p. 213]. Hausdorff then defines a metric space in the now familiar fashion, denoting the distance between two points x and y by $\overline{xy}$. The usual notation is now $d(x, y)$.

Given a point x in a metric space E, a neighbourhood U_x of the point x is the set of all points y of E which satisfy the inequality $d(x,y) < \varepsilon$, where ε is some pre-assigned positive real number. The set of all these neighbourhoods U_x as ε runs through all positive real numbers, is called the neighbourhood system at x. The neighbourhood systems of points in any metric space E have the properties:

α. Every U_x contains x and is contained in E .

β. For two neighbourhoods U_x, U_x' of the same point x, $U_x \subseteq U_x'$ or $U_x' \subseteq U_x$ holds.

y. If y lies in U_x then there is a neighbourhood U_y of y which is contained in U_x.

δ. If x and y are two distinct points, then there are two neighbourhoods U_x, U_y without common point.

We now let Hausdorff speak for himself [2, p. 213] :

"These spherical neighbourhoods, as we shall call them, have a series of properties of which only a very few are needed at first. As indicated above, we now change our standpoint by disregarding the distance by means of which we defined neighbourhoods, and put those properties at the head as axioms.

By a topological space we mean a set in which certain subsets U_x which we call neighbourhoods, are assigned to the elements (points), and in fact according to the following:

Neighbourhood Axioms

A. To each point x there corresponds at least one neighbourhood U_x; every neighbourhood U_x has x as an element.

B. If U_x and V_x are two neighbourhood of the same point x, then there exists a neighbourhood W_x of x which is a subset of both U_x and V_x.

C. If y belongs to U_x then there exists a neighbourhood U_y of y which is a subset of U_x.

D. For any two distinct points *x*, *y* of *E*, there exists two neighbourhoods U_x, U_y without common point.

It is not hard to show that the spherical neighbourhood in a metric space do satisfy Hausdorff's neighbourhood axioms; so every metric space may be regarded as a topological space, but the latter concept is more general. Despite the greater generality, much of the theory of metric spaces and therefore of the distance related aspects of many specific spaces, can be carried over almost intact into the new setting."

Mathematicians soon realized that by adopting Axiom D, Hausdorff had restricted topological spaces rather severely. So later mathematicians (especially Bourbaki) dispensed with Axiom D when defining a topological space. Axiom D is called the Hausdorff separation axiom; a topological space (in the new sense) satisfying Axiom D is called a Hausdorff space. Every metric space is a Hausdorff space, but the converse is not true in general.

Hausdorff's great credit was to suitably modify Property β of metric spaces in making the transition from metric spaces to topological spaces. This was accomplished by replacing Property β with Axiom B. The transition from β to B was neither automatic nor obvious. Had Hausdorff simply adopted the properties α to δ as the defining axioms of neighbourhood systems of a topological space, the result would have disappointing; for we know that neighbourhood systems of many topological spaces do not satisfy β but do satisfy the weaker requirement B as formulated by Hausdorff.

The fact that **-** in analogy with metric spaces – Hausdorff required that the neighbourhood systems of a topological space should satisfy Axiom D, distracts nothing from Hausdorff being regarded as the creator of the theory of topological spaces. It is astonishing that prior to the publication of the Grundzűge, Hausdorff had not published anything at all on what is unquestionably the most original and influential part of the work, the theory of metric and of topological spaces (Chapters 7, 8, 9).

Interestingly enough the Preface of the Grundzűge reveals Hausdorff's own views on his labours. Expressing the hope that while the book might profitably be used by students of middle semesters (that is, advanced undergraduates), he stated that the book would have failed its purpose if it did not offer the professional colleagues something new, at least in methodical and formal aspects. Saying that one should logically and systematically organize scattered facts. remove unnecessarily special or complicating assumptions from previous results, that one should strive for attaining simplicity and generality, are surely the minimum that can be demanded of an author dealing with material already treated by others, Hausdorff expressed his belief of having reasonably fulfilled this demand and of having opened some new lines of inquiry. He points out especially that by axiomatizing point-set topology, many theorems on point-sets on the real line, more generally in $\mathbf{R}^n$ have been so transformed, generalized, decomposed, and tied up in another context that a mere reference to the existing literature would give no correct picture. Thus, despite these rather modest words, there is no doubt that Hausdorff was well aware of what he had accomplished with this book. So, while he was no doubt pleased with the impact created by the Grundzűge on the subsequent development, he was hardly surprised.

The definition of a topological space by means of a set of axioms on neighbourhoods is, beyond question, Hausdorff's greatest contribution. The concept of a neighborhood as such was nothing new; Hausdorff's great credit was, as Bourbaki has observed [1, p. 166], to make it the point of departure for an axiomatic development, abstract in form but adapted in advance to applications, and to have found the right set of axioms. The Grundzűge was the source to which the spectacular rise of point-set topology in the nineteen twenties and thirties is due. So many of the basic notions and concepts of the subject are to be found here, even the name metric space have come down to us unaltered, that one has to envy Hausdorff for his singular acumen and extraordinary foresight.

See [3] for a more detailed account of the emergence of point-set topology and of Hausdorff's unusual career. Noteworthy is the following result, now known as Hausdorff's Maximality Principle: Every partially ordered set contains a maximal linearly (totally) ordered subset. Hausdorff proved it using Zermelo's Well-ordering Theorem (1908) : Every set can be well-ordered The Maximally Principle is logically equivalent to the Axiom of Choice. Kelly [4] emphasizes Hausdorff's Maximality Principle and uses it profusely in his book. Willard's book [5] is a comprehensive and very well-written treatise which every serious student of point-set topology should look into.

On the occasion of the centenary of the publication of the Grundzűge we extend a grateful thought to the memory of its author, the extraordinary man and mathematician Hausdorff.

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