THREE DISTINCT NON-HEREDITARY RADICALS WHICH COINCIDE WITH THE CLASSICAL RADICAL FOR RINGS WITH D.C.C.

Subrata Majumdar and Kalyan Kumar Dey[1]
Department of Mathematics, Rajshahi University, Rajshahi-6205
[1]Corresponding author: kkdmath@yahoo.com

ABSTRACT

Majumdar and Paul [3] introduced and studied a new radical E defined as the upper radical determined by the class of all rings each of whose ideals is idempotent. In this paper the authors continue the study further and also study the join radical and the intersection radical (due to Leavitt) obtained from E and the Jacobson radical J. These have been denoted by $E + J$ and EJ respectively. The radical and the semisimple rings corresponding to $E + J$ and EJ have been obtained. Both of these radicals coincide with the classical nil radical for Artinian rings. Important properties of these radicals and their position among the well-known special radicals have been investigated. It has been proved that E, EJ and $E + J$ are non-hereditary. It has also been proved as an independent result that the nil radical N is not dual, i.e., $N \neq N_\varphi$.

1. Introduction

In [3] a radical E was introduced and studied. In this paper a further study of E has been made and certain unsolved problems regarding this radical have been settled. Moreover, two new radicals R_1 and R_2 obtained with the help of E and J using Leavitt's construction [4], have been studied. In particular it has been shown that E, R_1, and R_2 are three distinct non hereditary radicals each of which coincides with the classical radicals for rings with d.c.c. on left on right ideals.

Also it has been shown that N is not a dual radical. Here J is the Jacobson radical, N the nil radical of Koethe. We have followed the terminology and the notations of Divinsky [1] except that we write radical for radical property.

*There is an unfortunate mistake in the title of [3]. It has been corrected in the reference. The first author expresses regret for the mistake.

2. Let C denote the class of rings A such that, for every ideal I of A. $I^2 = I$, and let, P denote the class of all prime rings. Let $D = C \cap P$. It was shown in [3] that C and D determine the same upper radical which we denote by E.

It was shown in Proposition 3.1 ([3]) that $L \subseteq E \subseteq G$, where L and G are the Levitzki radical and the Brown-McCoy radical respectively. Hence E coincides with the classical radical rings with d.c.c. (see Theorem 13, [1]). It was also shown that

(i) $L \neq E$,

(ii) $G \neq E$,

(iii) $B_\varphi \subseteq E$, (B is the Baer lower radical)

(iv) $E \subseteq V$, (V is the upper radical determined by the class of all simple non-nilpotent rings)

(v) $E \not\subset N$, (its proof was not, however, complete)

(vi) E and J are independent of each other, J being the Jacobson radical.

There we could not decide whether

(a) E is hereditary,

(b) $B_\varphi < E$,

(c) $E < V$,

(d) $E < N$,

In this paper, we shall settle the first two of these questions. Also we shall give a complete proof of (v) (mentioned above) in [3].

We consider the following example:

Example 2.1

Let F be a field and Let A be a vector space on F with a basis $\{\alpha, \beta\}$. Define multiplication in A by the table:

.	α	β
α	α	α
β	α	α

Then A is a ring whose ideals are A, F_α and 0. Since $A^2 = F_\alpha \neq A$, and $(A/F_\alpha)^2 = 0 \neq A/F_\alpha$, A is E-radical. But F_α is a simple ring with $(F_\alpha)^2 = F_\alpha$. So F_α is E-semisimple. Hence E is nonhereditary. Since B_φ and N are special radicals both of these are hereditary, and so, $E \neq B_\varphi$ and $E \neq N$. Since A is N-semisimple, $E \not\subset N$.

Thus we have thus proved:

Theorem 2.1

(i) E is non-hereditary,

(ii) $B_\varphi < E$,

(iii) $E \not\subset N$.

This theorem answers the questions (a) and (b) of section 2 in the affirmative and gives a complete proof of (v).

3. In this section we digress and prove that the nil radical is not dual.

Theorem 3.1

$N < N_\varphi$, i.e. N is not a dual radical.

Proof: Let A be the ring of all rational numbers of the form $2x/(2y + 1)$ where x and y are integers with $(x, 2y + 1) = 1$ (see [1], Example 10). A is obviously N-semisimple.

Let I be any non-zero ideal of A and Let n' be the smallest positive integer such that $2^{n'}x/(2y + 1) \in I$, for some odd integer x. Let $i = 2^{n'}(2z + 1)/(2y + 1) \in I$, for some integers y and z such that $(2z + 1, 2y + 1) = 1$.

We note that $(2z + 1)2^{n'} = (2y+1)i \in I$ and so, $2.2^{n'} = [2/(2z + 1)](2z + 1).2^{n'} \in I$. Since $(2, 2z + 1) = 1$ there exist integers λ and μ such that $2\lambda + (2z + 1)\mu = 1$ and so $2^{n'} = 2\lambda.2^{n'} + (2z + 1)\mu.2^{n'} \in I$. Therefore, $2^{n'}A \subseteq I$.

Also, for each $i \in I$

$$i = 2^{n'}/(2u + 1), \text{ for some integers } u \text{ and } v \text{ with } (u, 2v + 1) = 1.$$
$$= 2^{n'-1}.2u/(2v + 1) \in 2^{n'-1}.A.$$

Thus, $2^{n'}.A \subseteq I \subseteq 2^{n'-1}.A$.

Also, for each non-negative integer n, $2^n A$ is an ideal of A. Therefore, it follows that A is subdirectly reducible and that every non-trivial homomorphic image of A is nil. Thus, A cannot be mapped homomorphically onto a subdirectly irreducible ring with a nil semisimple heart. So A is N_φ radical.

Hence $N \neq N_\varphi$ and the proof is complete.

4. Radicals EJ and $E + J$

It has been shown in [3] the radical E and J are independent of each other, i.e., neither of E and J contains the other. Hence both $E \cup J$ and $E \cap J$ are distinct and are different from both E and J. Thus the intersection radical EJ and the join radical, $L_{E \cup J}$, the lower radical determined by $E \cup J$, are two distinct radicals different from both E and J. We denote EJ by R_1 or EJ and $L_{E \cup J}$ by $E + J$ or R_2. We first note the following:

(i) EJ and $E + J$ coincide with the classical nil radical for rings with d.c.c. on left or right ideals.

Since both E and J lie between D and T, where D is the lower radical determined by all nilpotent rings which are nil radicals of rings with d.c.c. on left ideals, and T is the upper radical determined by all matrix rings over division rings, it follows that both EJ and $E + J$ too lie between D and T. Hence by Theorem 13 of [1] both EJ and $E + J$ coincide with the classical nil radical.

2. EJ and $E + J$ are non-hereditary

The ring A in example 2.1 has d.c.c on left ideals. Hence the radicals EJ and the $E + J$ radicals of A and F_α are the same as the $E =$ radicals of A and F_α respectively. It thus follows that just like E, both EJ and $E + J$ are non-hereditary.

The radical rings and the semisimple rings with respect to the radicals EJ and $E + J$ are described below:

1) *EJ-radical rings*

It is clear from the definition of *EJ* the radical rings are precisely the rings which cannot be mapped homomorphically onto a non-zero ring in C_P where P is the class of all primitive rings. Thus, the *EJ*-radical rings are precisely those whose non-zero homomorphic images are neither primitive nor have each ideal idempotent.

2) *EJ-semisimple rings*

It follows from the construction of the intersection radical ($EJ = E \cap J$) of a ring A (see Leavitt [2]) that A is *EJ*-semisimple if and only if $E(A) \cap J(A) = 0$.

To have an idea about the $(E + J)$-radical of A, we need a few results due to Majumdar ([3]).

It has been shown by Majumdar and Saha ([4]) that, if a class C of rings satisfies certain properties called P_1 and P_2 which are described below, then the lower radical L_c and the corresponding radical $L_c(A)$ for a ring A can be expressed in a way similar to that for the radical $B(A)$ for A, where B is the Baer lower radical. Because we shall use this fact, we state here the relevant results without proof. We start with two definitions and a notation:

Definition: A non-empty class C of rings is said to have property P if C is homomorphically closed, and is said to have property P_2 if for every ring A and every nonzero ideal I of A, A has no non-zero C-ideal implies I has no non-zero C-ideals.

Notation: Let C be a non empty homomorphically closed class of ring, and let A be a ring. Let A_τ be the sum of all C-ideals of A. For an ordinal $\beta > 2$ and not a limit ordinal, define A_β to be the ideals of $A_\beta/A_{\beta-1}$ such that $A_\beta/A_{\beta-1}$ is the sum of all C-ideals of all C-ideals. of A/A_τ and if $\beta < 2$ is a limit ordinal, define $A_\beta = \cup A_\alpha$ ($\alpha < \beta$) Thus A_α exists for each ordinal α. Then there is a smallest ordinal τ such that $A_{\tau+1} = A_\tau$. We denote A_τ by A_C.

We next state the useful results:

(a) Theorem 4.1

Let C be a non-empty class of rings satisfying conditions P_1 and let A be a ring. Then A is the intersection of all ideals Q_i of A such that A/Q_i has no non-zero C-ideals.

(b) Theorem 4.2 (see [4])

Let C be a non-empty class of rings satisfying conditions P_1 and P_2. Then the class R of all rings such that $A = A_c$ for some rings A, is a radical.

Also $R_c(B) = B_c$ for every ring B with properties P_1 and P_2.

(c) Theorem 4.3 (see [4])

Let C a non-empty class of rings satisfying conditions P_1 and P_2, Then $R_c = L_c$ is the lower radical determined by C.

(d) Theorem 4.4

If C is a non-empty class of rings with conditions P_1 and P_2, then a ring A is L_c-semisimple if and only if A has no zero C-ideals.

The last theorem is a direct consequence of Theorem 4.2, Theorem 4.3 and the definition of A_c.

We are now ready to discuss the $(E + J)$-radical rings and the $(E + J)$-semisimple rings.

(3) $(E + J)$-radical rings

We readily see that if R_1 and R_2 are two radicals, then $R_1 \cup R_2 = R_{R1 \cup R2}$ and so, the $(L_{R1 \cup R2})$-radical of a ring A is $A_{R1 \cup R2}$. Thus the $(E + J)$-radical of a ring A is $A_{E \cup J}$ and so, the $(E + J)$-radical rings are precisely the rings A such that $A = A_{E \cup J}$.

(4) $(E + J)$-semisimple rings

By Corollary 4.4, a ring A is $L_{R1 \cup R2}$ semisimple if and only if A has no non-zero $R_1 \cup R_2$ ideals i.e., if and only if A is both R_1-semisimple and R_2-semisimple. Hence we have.

The $(E + J)$-semisimple ring are precisely the rings A which are both E-semisimple and J-semisimple i.e., $S(E + J) = S(E) \cap S(J) =$ The class of all those subdirect sums of primitive rings each non-zero ideal of which can be mapped onto a nonzero ring in D. (or equivalently in C). Here $S(R)$ denotes the semisimple class of a radical R.

REFERENCES

[1] Divinsky N. J., Rings and Radicals, University of Toronto Press, (1965).

[2] Leavitt W.G., Sets of radical classes, Publications Mathematicae, 14 (1967), 321-325

[3] Majumdar S. and Paul A. C., A non-hereditary (?) radical which coincides with the classical radical for rings with d.c.c., Ganit (J. Bangladesh Math Soc), 1(2) (1981), 85 - 89.

[4] Majumdar S. and Shaha S. K., On the lower radical determined by a class of rings, Ganit (J. Bangladesh Math Soc), 9(1) (1989), 39 - 44.