GANIT *J. Bangladesh Math. Soc.* (ISSN 1606-3694) **37** (2017) 63-71

ON R_1 SPACE IN L-TOPOLOGICAL SPACES

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Received 27.12.2016 Accepted 30.07.2017

ABSTRACT

In this paper, R1 space in L-topological spaces are defined and studied. We give seven definitions of R_1 space in L-topological spaces and discuss certain relationship among them. We show that all of these satisfy 'good extension' property. Moreover, some of their other properties are obtained.

Keywords: L-fuzzy set, L-topology, Hereditary, projective and productive.

1. Introduction

The concept of R_1-property first defined by Yang [19] and there after Murdeshwar and Naimpally [15], Dorsett [6], Dude [7], Srivastava [17], Petricevic [16] and Candil [11]. Chaldas *et al* [4] and Ekici [8] defined and studied many characterizations of R_1-properties. Later, this concept was generalized to 'fuzzy R_1-propertise' by Ali and Azam [2, 3] and many other fuzzy topologists. In this paper we defined seven notions of R_1 space in L-topological spaces and we also showed that this space possesses many nice properties which are hereditary productive and projective.

2. Preliminaries

In this section, we recall some basic definitions and known results in L-fuzzy sets and L-fuzzy topology.

Definition 2.1. [20] Let X be a non-empty set and $I = [0, 1]$. A fuzzy set in X is a function $u: X \to I$ which assigns to each element $x \in X$, a degree of membership, $u(x) \in I$.

Definition 2.2. [9] Let X be a non-empty set and L be a complete distributive lattice with 0 and 1. An L-fuzzy set in X is a function $\alpha: X \to L$ which assigns to each element $x \in X$, a degree of membership, $\alpha(x) \in L$.

Definition 2.3. [14] An L-fuzzy point p in X is a special L-fuzzy sets with membership function

$p(x) = r \;\; if \; x = x_0$
$p(x) = 0 \;\; if \; x \neq x_0$ where $r \in L$.

Definition 2.4. [14] An L-fuzzy point p is said to belong to an L-fuzzy set α in X ($p \in \alpha$) if and only if $p(x) < \alpha(x)$ and $p(y) \leq \alpha(y)$. That is $x_r \in \alpha \; implies \; r < \alpha(x)$.

Definition 2.5. [10] Let X be a non-empty set and L be a complete distributive lattice with 0 and 1. Suppose that τ be the sub collection of all mappings from X to L $i.e. \tau \subseteq L^X$.Then τ is called L-topology on X if it satisfies the following conditions:

(i) $0^*, 1^* \in \tau$

(ii) If $u_1, u_2 \in \tau$ then $u_1 \cap u_2 \in \tau$

(iii) If $u_i \in \tau$ for each $i \in \Delta$ then $\cup_{i\in\Delta} u_i \in \tau$.

Then the pair (X,τ) is called an L-topological space (lts, for short) and the members of τ are called open L-fuzzy sets. An L-fuzzy sets v is called a closed L-fuzzy set if $1 - v \in \tau$.

Definition 2.6. [20] An L-fuzzy singleton in X is an L-fuzzy set in X which is zero everywhere except at one point say x, where it takes a value say r with $0 < r \leq 1$ and $r \in L$. The authors denote it by x_r and $x_r \in \alpha$ iff $r \leq \alpha(x)$.

Definition 2.7. [14] An L-fuzzy singleton x_r is said to be quasi-coincident (q-coincident, in short) with an L-fuzzy set α in X, denoted by $x_r q\alpha$ iff $r + \alpha(x) > 1$. Similarly, an L-fuzzy set α in X is said to be q-coincident with an L-fuzzy set β in X, denoted by $\alpha q\beta$ if and only if $\alpha(x) + \beta(x) > 1$ for some $x \in X$. Therefore $\alpha\bar{q}\,\beta$ iff $\alpha(x) + \beta(x) \leq 1$ for all $x \in X$, where $\alpha\bar{q}\,\beta$ denote an L-fuzzy set α in X is said to be not q-coincident with an L-fuzzy set β in X.

Definition 2.8. [3] Let $f: X \to Y$ be a function and u be an L-fuzzy set in X. Then the image $f(u)$ is an L-fuzzy set in Y whose membership function is defined by

$(f(u))(y) = \{\sup\left(u\,(x)\right) \big| f(x) = y\}$ if $f^{-1}(y) \neq \emptyset, x \in X$

$(f(u))(y) = 0$ if $f^{-1}(y) = \emptyset, x \in X$.

Definition 2.9. [2] Let f be a real-valued function on an L-topological space. If $\{x: f(x) > \alpha\}$ is open for every real α, then f is called lower-semi continuous function (lsc, in short).

Definition 2.10. [14] Let (X,τ) and (Y,s) be two L-topological space and f be a mapping from (X,τ) into (Y,s) $i.e.\ f:(X,\tau) \to (Y,s)$. Then f is called

(i) Continuous iff for each open L-fuzzy set $u \in s \Longrightarrow f^{-1}(u) \in \tau$.

(ii) Open iff $f(\mu) \in s$ for each open L-fuzzy set $\mu \in \tau$.

(iii) Closed iff $f(\lambda)$ is s-closed for each $\lambda \in \tau^c$ where τ^c is closed L-fuzzy set in X.

(iv) Homeomorphism iff f is bijective and both f and f^{-1} are continuous..

Definition 2.11. [14] Let X be a nonempty set and T be a topology on X. Let $\tau = \omega(T)$ be the set of all lower semi continuous (lsc) functions from (X, T) to L (with usual topology). Thus $\omega(T) = \{u \in L^X: u^{-1}(\alpha, 1] \in T\}$ for each $\alpha \in L$. It can be shown that $\omega(T)$ is a L-topology on X. Let "P" be the property of a topological space (X, T) and LP be its L-topological analogue. Then LP is called a "good extension" of P "if the statement (X, T) has P iff $(X, \omega(T))$ has LP" holds good for every topological space (X, T).

Definition 2.12. [18] Let (X_i, τ_i) be a family of L-topological spaces. Then the space $(\Pi X_i, \Pi\tau_i)$ is called the product L-topological space of the family of L-topological space $\{(X_i, \tau_i): i \in \Delta\}$ where $\Pi\tau_i$ denote the usual product of L-topologies of the families $\{\tau_i: i \in \Delta\}$ of L-topologies on X.

An L-topological property 'P' is called productive if the product L-topological space of a family of L-topological space, each having property 'P' also has property 'P'.

A property 'P' in an L-topological space is called projective if for a family of L-topological space $\{(X_i, \tau_i): i \in \Delta\}$, the product L-topological space $(\Pi X_i, \Pi\tau_i)$ has property 'P' implies that each coordinate space has property 'P'.

Definition 2.13. [1] Let (X, τ) be an L-topological space and $A \subseteq X$. we define $\tau_A = \{u|A: u \in \tau\}$ the subspace L-topologies on A induced by τ. Then (A, τ_A) is called the subspace of (X, τ) with the underlying set A.

An L-topological property 'P' is called hereditary if each subspace of an L-topological space with property 'P' also has property 'P'.

3. R_1-property in L-Topological Spaces

We now give the following definitions of R_1-property in L-topological spaces.

Definition 3.1. An lts (X, τ) is called

(a) $L - R_1(i)$ if $\forall\, x, y \in X, x \neq y$ whenever $\exists\, w \in \tau$ with $w(x) \neq w(y)$then $\exists\, u, v \in \tau$ such that $u(x) = 1, u(y) = 0, v(x) = 0,\ v(y) = 1$ and $u \cap v = 0$.

(b) $L - R_1(ii)$ if $\forall\, x, y \in X, x \neq y$ whenever $\exists\, w \in \tau$ with $w(x) \neq w(y)$ then for any pair of distinct L-fuzzy points $x_r, y_s \in S(X)$ and $\exists\, u, v \in \tau$ such that $x_r \in u, y_s \notin u$ and $x_r \notin v, y_s \in v$, $u \cap v = 0$.

(c) $L - R_1(iii)$ if $\forall\, x, y \in X, x \neq y$ whenever $\exists\, w \in \tau$ with $w(x) \neq w(y)$ then for all pairs of distinct L-fuzzy singletons $x_r, y_s \in S(X), x_r\bar{q}y_s$ and $\exists\, u, v \in \tau$ such that $x_r \subseteq u, y_s\bar{q}u$ and $y_s \subseteq v, x_r\bar{q}v$ and $u \cap v = 0$.

(d) $L - R_1(iv)$ if $\forall\, x, y \in X, x \neq y$ whenever $\exists\, w \in \tau$ with $w(x) \neq w(y)$ then for any pair of distinct L-fuzzy points $x_r, y_s \in S(X)$ and $\exists\, u, v \in \tau$ such that $x_r \in u, u\bar{q}y_s$ and $y_s \in v, v\bar{q}x_r$ and $u \cap v = 0$.

(e) $L - R_1(v)$ if $\forall\, x, y \in X, x \neq y$ whenever $\exists\, w \in \tau$ with $w(x) \neq w(y)$ and for any pair of distinct L-fuzzy points $x_r, y_s \in S(X)$ and $\exists\, u, v \in \tau$ such that $x_r \in u \subseteq coy_s$, $y_s \in v \subseteq cox_r$ and $u \subseteq cov$.

(f) $L - R_1(vi)$ if $\forall\, x, y \in X, x \neq y$ whenever $\exists\, w \in \tau$ with $w(x) \neq w(y)$ then $\exists\, u, v \in \tau$ such that $u(x) > 0, u(y) = 0$ and $v(x) = 0, v(y) > 0$.

(g) $L - R_1(vii)$ if $\forall\, x, y \in X, x \neq y$ whenever $\exists\, w \in \tau$ with $w(x) \neq w(y)$ then $\exists\, u, v \in \tau$ such that $u(x) > u(y)$ and $v(y) > v(x)$.

Here, we established a complete comparison of the definitions

$L-R_1(ii), L-R_1(iii),\ L-R_1(iv), L-R_1(v), L-R_1(vi)$ and $L-R_1(vii)$ with $L-R_1(i)$.

Theorem 3.2. Let $(X\ ,\tau)$ be an lts. Then we have the following implications:

$$\begin{tikzcd}
 & & L-R_1(iv) \arrow[dl] & & L-R_1(ii) \\
L-R_1(vii) \arrow[r] & L-R_1(vi) & & L-R_1(i) \arrow[ll] \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \\
 & & L-R_1(v) \arrow[ul] & & L-R_1(iii)
\end{tikzcd}$$

The reverse implications are not true in general except $L-R_1(vi)$ and $L-R_1(vii)$.

Proof: $L-R_1(i) \Rightarrow L-R_1(ii)$, $L-R_1(i) \Rightarrow L-R_1(iii)$can be proved easily. Now $L-R_1(i) \Rightarrow L-R_1(iv)$ and $L-R_1(i) \Rightarrow L-R_1(v)$, since $L-R_1(ii) \Leftrightarrow L-R_1(iv)$and $L-R_1(iv) \Leftrightarrow L-R_1(v)$. $L-R_1(i) \Rightarrow L-R_1(vi)$; It is obvious. $L-R_1(i) \Rightarrow L-R_1(vii)$, since $L-R_1(vi) \Rightarrow L-R_1(vii)$.

The reverse implications are not true in general except $L-R_1(vi)$ and $L-R_1(vii)$, it can be seen through the following counter examples:

Example-1: Let $X=\{x,y\}$, τ be the L-topology on X generated by $\{\alpha\colon \alpha \in L\} \cup \{u,v,w\}$ where $w(x)=0.6, w(y)=0.7, u(x)=0.5, u(y)=0, v(x)=0, v(y)=0.6$

$L=\{0,0.05,0.1,0.15, \dots \dots \dots 0.95,1\}$ and $r=0.4, s=0.3$.

Example-2: Let $X=\{x,y\}$, τ be the L-topology on X generated by $\{\alpha\colon \alpha \in L\} \cup \{u,v,w\}$ where $w(x)=0.8, w(y)=0.9, u(x)=0.5, u(y)=0, v(x)=0, v(y)=0.4$

$L=\{0,0.05,0.1,0.15, \dots \dots \dots 0.95,1\}$ and $r=0.5, s=0.4$.

Proof: $L-R_1(ii) \not\Rightarrow L-R_1(i)$: From example-1, we see that the lts (X,τ) is clearly $L-R_1(ii)$ but it is not $L-R_1(i)$. Since there is no L-fuzzy set in τ which grade of membership is 1.

$L-R_1(iii) \not\Rightarrow L-R_1(i)$: From example-2, we see the lts (X,τ) is clearly $L-R_1(iii)$ but it is not $L-R_1(ii)$. Since $L-R_1(iii) \not\Rightarrow L-R_1(ii)$ and $L-R_1(ii) \not\Rightarrow L-R_1(i)$ so $L-R_1(iii) \not\Rightarrow L-R_1(i)$.

$L-R_1(iv) \not\Rightarrow L-R_1(i)$: This follows automatically from the fact that

$L-R_1(ii) \Leftrightarrow L-R_1(iv)$ and it has already been shown that $L-R_1(ii) \not\Rightarrow$

$L-R_1(i)$ so $L-R_1(iv) \not\Rightarrow L-R_1(i)$.

$L-R_1(v) \not\Rightarrow L-R_1(i)$: Since $L-R_1(iv) \Leftrightarrow L-R_1(v)$and $L-R_1(iv) \not\Rightarrow L-R_1(i)$ so $L-R_1(v) \not\Rightarrow L-R_1(i)$. But $L-R_1(vii) \Rightarrow L-R_1(vi) \Rightarrow$

$L-R_1(i)$ is obvious.

4. Good extension, Hereditary, Productive and Projective Properties in L-Topology

We show that all definitions $L - R_1(i), L - R_1(ii), L - R_1(iii)$,

$L - R_1(iv), L - R_1(v)$, $L - R_1(vi)$ and $L - R_1(vii)$ are 'good extensions' of $R_1 -$ property, as shown below:

Theorem 4.1. Let (X, T) be a topological space. Then (X, T) is R_1 iff $(X, \omega(T))$ is $L - R_1(j)$, where $j = i, ii, iii, iv, v, vi, vii$.

Proof: Let (X, T) be R_1. Choose $x, y \in X$, $x \neq y$. Whenever $\exists\, W \in T$ with $x \in W, y \notin W$ or $x \notin W, y \in W$ then $\exists\, U, V \in T$ such that $x \in U, y \notin U$ and $y \in V, x \notin V$ and $U \cap V = \emptyset$. Suppose $x \in W, y \notin W$ since $W \in T$ then $1_w \in \omega(T)$ with $1_w(x) \neq 1_w(y)$. Also consider the lower semi continuous function $1_U, 1_V$, then $1_U, 1_V \in \omega(T)$ such that $1_U(x) = 1, 1_U(y) = 0$ and $1_V(x) = 0, 1_V(y) = 1$ and so that $1_U \cap 1_V = 0$ as $U \cap V = \emptyset$. Thus $(X, \omega(T))$ is $L - R_1(i)$.

Conversely, let $(X, \omega(T))$ be $L - R_1(i)$. To show that (X, T) is R_1. Choose $x, y \in X$ with $x \neq y$. Whenever $\exists\, w \in T$ with $w(x) \neq w(y)$ then $\exists\, u, v \in \omega(T)$ such that $u(x) = 1, u(y) = 0, v(x) = 0, v(y) = 1$ and $u \cap v = 0$. Since $w(x) \neq w(y)$, then either $w(x) < w(y)$ or $w(x) > w(y)$. Choose $w(x) < w(y)$, then $\exists\, s \in L$ such that $w(x) < s < w(y)$. So it is clear that $w^{-1}(s, 1] \in T$ and $x \notin w^{-1}(s, 1]$, $y \in w^{-1}(s, 1]$. Let $U = u^{-1}\{1\}$ *and* $V = v^{-1}\{1\}$, then U, $V \in T$ and is $x \in U, y \notin U$, $x \notin V$, $y \in V$, and $U \cap V = \emptyset$ as $u \cap v = 0$. Hence (X, T) is R_1.

Similarly, we can show that $L - R_1(ii), L - R_1(iii), L - R_1(iv)$,

$L - R_1(v), L - R_1(vi)$, $L - R_1(vii)$ are also hold 'good extension' property.

Theorem 4.2. Let (X, τ) be an lts, $A \subseteq X$ and $\tau_A = \{u|A : u \in \tau\}$, then

(a) (X, τ) is $L - R_1(i) \Rightarrow (A, \tau_A)$ is $L - R_1(i)$.
(b) (X, τ) is $L - R_1(ii) \Rightarrow (A, \tau_A)$ is $L - R_1(ii)$.
(c) (X, τ) is $L - R_1(iii) \Rightarrow (A, \tau_A)$ is $L - R_1(iii)$.
(d) (X, τ) is $L - R_1(iv) \Rightarrow (A, \tau_A)$ is $L - R_1(iv)$.
(e) (X, τ) is $L - R_1(v) \Rightarrow (A, \tau_A)$ is $L - R_1(v)$.
(f) (X, τ) is $L - R_1(vi) \Rightarrow (A, \tau_A)$ is $L - R_1(vi)$.
(g) (X, τ) is $L - R_1(vii) \Rightarrow (A, \tau_A)$ is $L - R_1(vii)$.

Proof: We prove only (a). Suppose (X, τ) is L-topological space and is also $L - R_1(i)$.We shall prove that (A, τ_A) is $L - R_1(i)$. Let $x, y \in A$ with $x \neq y$ and $\exists\, w \in \tau_A$ such that $w(x) \neq w(y)$, then $x, y \in X$ with $x \neq y$ as $A \subseteq X$. Consider m be the extension function of w on X, then $m(x) \neq m(y)$, Since (X, τ) is $L - R_1(i)$, $\exists\, u, v \in \tau$ such that $u(x) = 1, u(y) = 0, v(x) = 0, v(y) = 1$ and $u \cap v = 0$. For $A \subseteq X$, we find ,$u|A, v|A \in \tau_A$ and $u|A(x) = 1, u|A(y) = 0$ and $v|A(x) = 0, v|A(y) = 1$ and $u|A \cap v|A = (u \cap v)|A = 0$ as $x, y \in A$. Hence it is clear that the subspace (A, τ_A) is $L - R_1(i)$.

Similarly, (b), (c), (d), (e), (f), (g) can be proved.

So it is clear that $L-R_1(j), j=i, ii, \ldots, vi$ satisfy hereditary property.

Theorem 4.3. Given $\{(X_i,\tau_i): i \in \Lambda\}$ be a family of L-topological space. Then the product of L-topological space $(\Pi X_i, \Pi\, \tau_i)$ is $L-R_1(j)$ iff each coordinate space (X_i,τ_i) is $L-R_1(j)$, where $j=i, ii, iii, iv, v, vi, vii$.

Proof: Let each coordinate space $\{(X_i,\tau_i): i \in \Lambda\}$ be $L-R_1(i)$. Then we show that the product space is $L-R_1(i)$. Suppose $x\,, y \in X$ with $x \neq y$ and $w \in \Pi\, \tau_i$ with $w(x) \neq w(y)$, again suppose $x=\Pi x_i,\ y=\Pi y_i$ then $x_j \neq y_j$ for some $j \in \Lambda$.But we have $w(x)=\min\{w_i(x_i):\ i \in \Lambda\}$, and $w(y)=\min\{w_i(y_i):\ i \in \Lambda\}$. Hence we can find at least one $w_j \in \tau_j$ with $w_j(x_j) \neq w_j(y_j)$, since each $(X_i,\tau_i): i \in \Lambda$ be $L-R_1(i)$ then $\exists\, u_j, v_j \in \tau_j$ such that $u_j(x_j)=1, u_j(y_j)=0, v_j(x_j)=0, v_j(y_j)=1$ and $u_j \cap v_j=0$. Now take $u=\Pi u'_j, v=\Pi v'_j$ where $u'_j=u_j\,, v'_j=v_j$ and $u_i=v_i=1$ for $i \neq j$. Then $u, v \in \Pi\tau_i$ such that $u(x)=1, u(y)=0, v(x)=0, v(y)=1$and $u \cap v=0$. Hence the product of

L-topological space is also L-topological space and $(\Pi X_i, \Pi\, \tau_i)$ is $L-R_1(i)$.

Conversely, let the product L-topological space $(\Pi X_i, \Pi\, \tau_i)$ is $L-R_1(i)$. Take any coordinate space (X_j,τ_j), choose $x_j\,, y_j \in X_j\,, x_j \neq y_j$ and $w_i \in \Pi\tau_i$ with $w_i(x_i) \neq w_i(y_i)$. Now construct $x, y \in X$ such that $x=\Pi x'_i, y=\Pi y'_i$ where $x'_i=y'_i$ for $i \neq j$ and $x'_j=x_j, y'_j=y_j$. Then $x \neq y$ and using the product space $L-R_1(i)$, $\Pi w_i \in \Pi\tau_i$ with $\Pi w_i(x_i) \neq \Pi w_i(y_i)$, since $(\Pi X_i, \Pi\, \tau_i)$ is $L-R_1(i)$ then $\exists\, u, v \in \Pi\tau_i$ such that $u(x)=1, u(y)=0, v(x)=0,\ v(y)=1$ and $u \cap v=0$. Now choose any L-fuzzy point x_r in u. Then $\exists$ a basic open L-fuzzy set $\Pi u_j^r \in \Pi\tau_j$ such that $x_r \in \Pi u_j^r \subseteq u$ which implies that $r < \Pi u_j^r(x)$ or that $r < inf_j u_j^r(x'_j)$

and hence $r < \Pi u_j^r(x'_j) \forall\, j \in \Lambda \ldots \ldots (i)$ and

$$u(y)=0 \Rightarrow\ \Pi u_j(y)=0\ \ldots \ldots (ii).$$

Similarly, corresponding to a fuzzy point $y_s \in v$ there exists a basic fuzzy open set $\Pi v_j^s \in \Pi\tau_j$ such that $y_s \in \Pi v_j^s \subseteq v$ which implies that

$$s < v_j^s(j) \forall\, j \in \Lambda \ldots \ldots (iii)\ \text{ and}$$

$\Pi v_j^s(y)=0 \ldots \ldots (iv)$. Further, $\Pi u_j^r(y)=0 \Rightarrow u_i^r(y_i)=0$, since for $j \neq i, x'_j=y'_j$ and hence from (i), $u_j^r(y_j)=u_j^r(x_j) > r$. Similarly, $\Pi v_j^s(x)=0 \Rightarrow v_i^s(x_i)=0$ using (iii).

Thus we have $u_i^r(x_i) > r, u_i^r(y_i)=0$ and $v_i^s(y_i) > s, v_i^s(x_i)=0$. Now consider $sup_r u_i^r=u_i, sup_s v_i^s=v_i$, then $u_i(x_i)=1, u_i(y_i)=0, v_i(x_i)=0, v_i(y_i)=1$ and $u_i \cap v_i=0$, showing that (X_i,τ_i) is $L-R_1(i)$.

Moreover one can easily verify that

$(X_i,\tau_i), i \in \Lambda$ is $L-R_1(ii) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L-R_1(ii)$.

$(X_i,\tau_i), i \in \Lambda$ is $L-R_1(iii) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L-R_1(iii)$.

$(X_i,\tau_i), i \in \Lambda$ is $L-R_1(iv) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L-R_1(iv)$.

$(X_i, \tau_i), i \in \Lambda$ is $L - R_1(v) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - R_1(v)$.

$(X_i, \tau_i), i \in \Lambda$ is $L - R_1(vi) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - R_1(vi)$.

$(X_i, \tau_i), i \in \Lambda$ is $L - R_1(vii) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - R_1(vii)$.

Hence, we see that $L - R_1(i), L - R_1(ii), L - R_1(iii), L - R_1(iv)$,

$L - R_1(v), L - R_1(vi), L - R_1(vii)$ Properties are productive and projective.

5. Mapping in L-topological spaces

We show that $L - R_1(j)$ property is preserved under one-one, onto and continuous mapping for $j = i, ii, iii, iv, v, vi, vii$.

Theorem 5.1 Let (X, τ) and (Y, s) be two L-topological space and $f: (X, \tau) \to (Y, s)$ be one-one, onto L-continuous and L-open map, then

(a) $(X\,, \tau)$ is $L - R_1(i) \Rightarrow (Y, s)$ is $L - R_1(i)$.
(b) $(X\,, \tau)$ is $L - R_1(ii) \Rightarrow (Y, s)$ is $L - R_1(ii)$.
(c) $(X\,, \tau)$ is $L - R_1(iii) \Rightarrow (Y, s)$ is $L - R_1(iii)$.
(d) $(X\,, \tau)$ is $L - R_1(iv) \Rightarrow (Y, s)$ is $L - R_1(iv)$.
(e) $(X\,, \tau)$ is $L - R_1(v) \Rightarrow (Y, s)$ is $L - R_1(v)$.
(f) $(X\,, \tau)$ is $L - R_1(vi) \Rightarrow (Y, s)$ is $L - R_1(vi)$.
(g) $(X\,, \tau)$ is $L - R_1(vii) \Rightarrow (Y, s)$ is $L - R_1(vii)$.

Proof: Suppose (X, τ) is $L - R_1(i)$.We shall prove that (Y, s) is $L - R_1(i)$. Let $y_1\,, y_2 \in Y$ with $y_1 \neq y_2$ and $w \in s$ with $w(y_1) \neq w(y_2)$. Since f is onto then $\exists\, x_1,\ x_2\ \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$, also $x_1 \neq x_1$, as f is one-one. Now we have $f^{-1}(w) \in \tau$, Since f is L-continuous, also we have $f^{-1}(w)(x_1) = wf(x_1) = w(y_1)$ and $f^{-1}(w)(x_2) = wf(x_2) = w(y_2)$.Therefore $f^{-1}(w)(x_1) \neq f^{-1}(w)(x_2)$. Again since (X, τ) is $L - R_1(i)$ and $\exists f^{-1}(w) \in \tau$ with $f^{-1}(w)(x_1) \neq f^{-1}(w)(x_2)$ then $\exists\, u, v \in \tau$

such that $u(x_1) = 1,\ u(x_2) = 0, v(x_1) = 0, v(x_2) = 1$ and $u \cap v = 0$.

Now

$$f(u)(y_1) = \{sup u(x_1): f(x_1) = y_1\} = 1$$
$$f(u)(y_2) = \{sup u(x_2): f(x_2) = y_2\} = 0$$
$$f(v)(y_1) = \{sup v(x_1): f(x_1) = y_1\} = 0$$
$$f(v)(y_2) = \{sup v(x_2): f(x_2) = y_2\} = 1$$

And

$$f(u \cap v)(y_1) = \{\sup(u \cap v)(x_1): f(x_1) = y_1$$
$$f(u \cap v)(y_2) = \{\sup(u \cap v)(x_2): f(x_2) = y_2$$

Hence $f(u \cap v) = 0 \Rightarrow f(u) \cap f(v) = 0$

Since f is L-open, $f(u), f(v) \in s$. Now it is clear that $\exists\, f(u), f(v) \in s$ such that $(u)(y_1) = 1$, $f(u)(y_2) = 0, f(v)(y_1) = 0,\ f(v)(y_2) = 1$ and $f(u) \cap f(v) = 0$. Hence it is clear that the L-topological space (Y, s) is $L - R_1(i)$.

Similarly (b), (c), (d), (e), (f), (g) can be proved.

Theorem 5.2 Let (X, τ) and (Y, s) be two L-topological spaces and $f: (X, \tau) \to (Y, s)$ be L-continuous and one-one map, then

(a) $(Y\,, s)$ is $L - R_1(i) \Rightarrow (X, \tau)$ is $L - R_1(i)$.
(b) $(Y\,, s)$ is $L - R_1(ii) \Rightarrow (X, \tau)$ is $L - R_1(ii)$.
(c) $(Y\,, s)$ is $L - R_1(iii) \Rightarrow (X, \tau)$ is $L - R_1(iii)$.
(d) $(Y\,, s)$ is $L - R_1(iv) \Rightarrow (X, \tau)$ is $L - R_1(iv)$.
(e) $(Y\,, s)$ is $L - R_1(v) \Rightarrow (X, \tau)$ is $L - R_1(v)$.
(f) $(Y\,, s)$ is $L - R_1(vi) \Rightarrow (X, \tau)$ is $L - R_1(vi)$.
(g) $(Y\,, s)$ is $L - R_1(vii) \Rightarrow (X, \tau)$ is $L - R_1(vii)$.

Proof: Suppose (Y, s) is $L - R_1(i)$.We shall prove that (X, τ) is $L - R_1(i)$. Let $x_1\,, x_2 \in X$ with $x_1 \neq x_2$ and $w \in \tau$ with $w(x_1) \neq w(x_2), \Rightarrow f(x_1) \neq f(x_2)$ as f is one-one, also $f(w) \in s$ as f is L-open. We have $f(w)(f(x_1)) = \sup\{w(x_1)\}$ and $f(w)\big(f(x_2)\big) = \sup\{w(x_2)\}$ and $f(w)(f(x_1)) \neq f(w)\big(f(x_2)\big)$. Since (Y, s) is $L - R_1(i), \exists\, u, v \in s$ such that $u\big(f(x_1)\big) = 1, u(f(x_2)) = 0, v\big(f(x_1)\big) = 0, v(f(x_2)) = 1$ and $u \cap v = 0$. This implies that $f^{-1}(u)(x_1) = 1, f^{-1}(u)(x_2) = 0, f^{-1}(v)(x_1) = 0, f^{-1}(v)(x_2) = 1$ and $f^{-1}(u \cap v) = 0 \Rightarrow f^{-1}(u) \cap f^{-1}(v) = 0$.

Now it is clear that $\exists\, f^{-1}(u), f^{-1}(v) \in \tau$ such that $f^{-1}(u)(x_1) = 1,\ \ f^{-1}(u)(x_2) = 0$, $f^{-1}(v)(x_1) = 0, f^{-1}(v)(x_2) = 1$ and $f^{-1}(u) \cap f^{-1}(v) = 0$. Hence the L-topological space (X, τ) is $L - R_1(i)$.

Similarly (b), (c), (d), (e), (f), (g) can be proved.

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