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AN INDUCTIVE PROOF OF BERTRAND'S POSTULATE

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ABSTRACT

In this paper, we are going to prove a famous problem concerning the prime numbers called Bertrand's postulate. It states that there is always at least one prime, p between n and $2n$, means, there exists $n < p < 2n$ where $n > 1$. It is not a newer theorem to be proven. It was first conjectured by Joseph Bertrand in 1845. He did not find a proof of this problem but made important numerical evidence for the large values of n. Eventually, it was successfully proven by Pafnuty Chebyshev in 1852. That is why it is also called Bertrand-Chebyshev theorem. Though it does not give very strong idea about the prime distribution like Prime Number Theorem (PNT) does, the beauty of Bertrand's postulate lies on its simple yet elegant definition. Historically, Bertrand's postulate is also very important. After Euclid's proof that there are infinite prime numbers, there was no significant development in the prime number distribution. Peter Dirichlet stated the standard form of Prime Number Theorem (PNT) in 1838 but it was merely a conjecture that time and beyond the scope of proof to the then mathematicians. Bertrand's postulate was a simply stated problem but powerful enough, easy to prove and could lead many more strong assumptions about the prime number distribution. Illustrious Indian mathematician, Srinivasa Ramanujan gave a shorter but elegant proof using the concept of Chebyshev functions of prime, $\upsilon(x)$, $\Psi(x)$and Gamma function, $\Gamma(x)$ in 1919 which led to the concept of Ramanujan Prime. Later Paul Erdős published another proof using the concept of Primorial function, $p\#$ in 1932. The elegance of our proof lies on not using Gamma function yet finding the better approximations of Chebyshev functions of prime. The proof technique is very similar the way Ramanujan proved it but instead of using the Stirling's approximation to the binomial coefficients, we are proving similar results using well-known proving technique the mathematical induction and they lead to somewhat stronger than Ramanujan's approximation of Chebyshev functions of prime.

Keyword: *log* (Natural Logarithm)

MSC2010: 11N05 (Multiplicative Number Theory, Distribution of Primes)

1. Definition

We define function $\upsilon(x)$ and $\Psi(x)$ conventionally [1] as:

$$\upsilon(x) = \sum_{p \le x} \log(p) = \log \prod_{p \le x} p$$

$$\Psi(x) = \sum_{p^m \le x} \log(p)$$

Since $p^2 \leq x, p^3 \leq x, \ldots$ are equivalent to $p \leq x^{1/2}, p \leq x^{1/3}, \ldots$ we have [1], [2]:

$$\Psi(x) = \upsilon(x) + \upsilon(x^{1/2}) + \upsilon(x^{1/3}) + \ldots = \sum_{m>0} \upsilon(x)^{1/m}$$

and so $\Psi(2n) = \upsilon(2n) + \upsilon((2n)^{1/2}) + \upsilon((2n)^{1/3}) + \ldots = \sum_{m>0} \upsilon((2n)^{1/m})$

2. Proof of Postulate

We know [2], $\log((2n)!) = \Psi(2n) + \Psi(n) + \Psi(\frac{2n}{3}) + \ldots,$ (1)

From relation of $\upsilon(x)$ and $\Psi(x)$, we get [2]:

$$\Psi(2n) - 2\Psi(\sqrt{2n}) = \upsilon(2n) - \upsilon((2n)^{1/2}) + \upsilon((2n)^{1/3}) \quad (2)$$

Let $N_n = \frac{(2n)!}{n!n!}$, then from (1):

$$\log(N_n) = \Psi(2n) - \Psi(n) + \Psi(\tfrac{2n}{3}) \quad (3)$$

As $\upsilon(x)$ and $\Psi(x)$ are steadily increasing function, we find from (2) and (3) that:

$$\Psi(2n) - 2\Psi(\sqrt{2n}) \leq \upsilon(2n) \leq \Psi(2n), \quad (4)$$

and $\Psi(2n) - \Psi(n) \leq \log(N_n) \leq \Psi(2n) - \Psi(n) + \Psi(\frac{2n}{3}),$ (5)

Now, $N_{n+1} = \frac{(2(n+1))!}{(n+1)!(n+1)!} = 2 \times \frac{2n+1}{n+1} \times N_n$

for $n = 1, \frac{2n+1}{n+1} = \frac{3}{2}$ and $\lim_{n\to\infty} \frac{2n+1}{n+1} = 2$, which implies for $n \geq 1$,

$$3N_n \leq N_{n+1} \leq 4N_n, \quad (6)$$

We assume, $a = \log(3)$ and $b = \log(4)$

for $n = 1$, $N_1 = \frac{2!}{1!1!} = 2$; $\log(N_1) = \log(2) < b \times 1$

and $n = 5$, $N_5 = \frac{10!}{5!5!} = 252$; $a \times 5 < \log(252) = \log(N_5)$

We assume, $\log(N_n) < bn$ if $n \geq 1$, and $an < \log(N_n)$ if $n \geq 5$, (7)

It follows from (6) and (7) that: $\log(3) + \log(N_n) \leq \log(N_{n+1}) \leq \log(4) + \log(N_n)$ implies $a(n+1) < \log(N_{n+1}) < b(n+1)$, by induction (7) is proven.

It follows from (5) and (7) that:

$$\Psi(2n) - \Psi(n) < bn \text{ if } n \geq 1, \quad (8)$$

$$\Psi(2n) - \Psi(n) + \Psi(\tfrac{2n}{3}) > an \text{ if } n \geq 5, \quad (9)$$

Now, changing n to $\frac{n}{2}, \frac{n}{4}, \frac{n}{8}, \ldots$ in (8) and adding up all the results, we get:

$$\Psi(2n) < 2bn \text{ if } n \geq 1, \quad (10)$$

Finally, we have from (4) and (10):

$\Psi(2n) - \Psi(n) + \Psi(\frac{2n}{3}) \leq \upsilon(2n) + 2\Psi(\sqrt{2n})\text{-}\upsilon(n) + \Psi(\frac{2n}{3})$

or $\Psi(2n) - \Psi(n) + \Psi(\frac{2n}{3}) < \upsilon(2n) - \upsilon(n) + 2b\sqrt{2n} + \frac{2bn}{3}$, (11)

We conclude from (9) and (11) that:

$$\upsilon(2n) - \upsilon(n) > an - 2b\sqrt{2n} - \frac{2bn}{3}$$

By considering right hand side a quadratic equation,

$$\upsilon(2n) - \upsilon(n) > 0 \text{ if } n > 505$$

It is easy to verify using simple computer program there are primes for $1 < n \leq 505$, hence, we have proved Bertrand's postulate.

Let's assume $F(n) = \upsilon(2n) - \upsilon(n)$ then $\frac{d}{dn}(F(n)) > a - \frac{\sqrt{2}b}{\sqrt{n}} - \frac{2b}{3} > 0$ if $n > 126$

As $F(n)$ is an increasing function, we proved Erdős assumption that for any positive integer k, there is a natural number N such that for all $n > N$, there are at least k primes between n and $2n$.

3. Discussion

In Ramanujan's original paper [2], results (8) and (9) are proven using Stirling's approximation of Gamma function to binomial coefficients. We deduce similar results using only mathematical induction. Ramanujan showed (8) to be less than 3/2. We showed (8) to be less than $b = \log(4) < 3/2$. He also showed (9) to be greater than 4/3. For larger values of n, we can increase the value of a and prove by mathematical induction (9) so that $a > 4/3$. So, our method can lead to much better approximation of Chebyshev functions of prime.

REFERENCES

[1] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford University Press, NY, 340-341, 1979.

[2] Edited: G. H. Hardy, P. V. S. Aiyar and B. M. Wilson, Collected Papers of Srinivasa Ramanujan, Cambridge University Press, 208-209, 1927.