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EXISTENCE OF WEAK SOLUTIONS FOR CAPUTO'S FRACTIONAL DERIVATIVES IN BANACH SPACES

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ABSTRACT

The objective of this project is to represent the existence of solutions for Caputo's fractional derivatives in Banach spaces. The result is based on some well-known fixed point theorems. To show the efficiency of the stated result some examples will be demonstrated.

Keywords: Caputo fractional derivative, Integral boundary, Fixed point theorems.

1. Introduction

Fractional calculus is the field of mathematical analysis which involves theinvestigation and applications of integrals and derivatives of arbitrary order. Nowadays, fractional differential equations have found numerous applications in various fields of physics and engineering. It is important to mention that most of the books and papers on fractional calculus are related to the solvability of initial value problems for differential equations of fractional order. But the theory of boundary value problems for nonlinear fractional differential equations has received attention of the researchers and many aspects of this theory need to be explored.The existence of positive solutions for three-point fractional boundary value problem was discussed in [1]. In [2] Sudsutad, W, Tariboon, J, studied the boundary value problem of fractional order differential equations with three-point fractional integral boundary. Recently researchers achieved great interest in boundary value problems of fractional differential equations involving Caputo Fractional derivative and Riemann-Liouville fractional integral. For more details see [3,4,5,6,7,8] and reference therein.Boundary value problems with nonlocal conditions are also a topic where researchers paid a large attention. As this is an effective tool to the modeling of physics, chemistry, biology, biophysics, blood flow phenomenon, wave propagation, fitting of experimental data, economics , etc. For examples and details see [10,11,12]. To see the recent development studies of fractional boundary value problems, we refer readers to see [13-21].

Inspired by the cited works on this field, the aim of the paper is to examine the existence of solutions of the following Caputo's fractional derivative with new boundary conditions in Banach spaces

$$^cD^\alpha x(t) = f\big(t, x(t)\big), \text{ for each } t \in [0,1],\ 1 < \alpha < 2 \tag{1.1}$$

$$x(0) = 0, \quad ax'(\xi_1) + bx'(\xi_2) = \beta \int_0^\eta x(s)ds \tag{1.2}$$

Where $^cD^\alpha$ is the Caputo fractional derivative of order α, $f: [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function and $a, b, \beta \in \mathbb{R}$. The boundary conditions in (1.2)implies that the linear combination of the values of the derivative of the unknown function at nonlocal positionsξ_1 andξ_2 is proportional to the continuous distribution of the unknown function over a strip of an arbitrary length η.

The remainder of the paper is organized as follows. Section 2 consists of some preliminary concepts of fractional calculus and auxiliary lemmas related to the linear variant of problem (1.1)–(1.2). Section 3 contains the derivation of the existence and uniqueness results for the given problem via some standard fixed point theorem. Examples are also included for illustration of the main results in section 4.

2. Preliminaries

Let us recall some basic definitions on fractional calculus.

Definition 2.1 The Riemann-Liouville factional integer of order α for a continuous function x is defined as

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{x(s)}{(t-s)^{\alpha-1}} ds, \quad \alpha > 0$$

provided the integral exists.

Definition 2.2 For at least nth continuously differentiable function $f: [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order α is defined as

$$^cD^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1}\, x^n(s)ds, \quad n-1 < \alpha < n, n = [\alpha] + 1,$$

where$[\alpha]$ denotes the integer part of the real number α.

Lemma 2.1[10]For $\alpha > 0$, the general solution of the fractional differential equation $^cD^\alpha x(t) = 0$ is given by

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots \ldots \ldots + c_{n-1} t^{n-1},$$

For some$c_i \in \mathbb{R}, i = 0,1, \ldots, n-1 (n = [\alpha] + 1)$.

Lemma 2.2 For any $h \in C[0,1]$, the unique solution of the linear fractional boundary value problem

$$^cD^\alpha x(t) = h(t), t \in [0,1],\ 1 < \alpha < 2 \tag{2.1}$$

$$x(0) = 0,\ ax'(\xi_1) + bx'(\xi_2) = \beta \int_0^\eta x(s)ds \tag{2.2}$$

is

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds$$

$$+ \frac{t}{B}\left\{\int_0^{\xi_1} \frac{(\xi_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + \int_0^{\xi_2} \frac{(\xi_2-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds - \beta \int_0^{\eta} \left(\int_0^{s} \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} h(u)du\right) ds\right\} \quad (2.3)$$

where $B = (\beta \frac{\eta^2}{2} - a\xi_1 - b\xi_2) \neq 0.$ (2.4)

Proof **:**The general solution of the fractional differential equations (2.1) and (2.2) can be written as

$$x(t) = c_0 + c_1 t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \quad (2.5)$$

where$c_0, c_1 \in \mathbb{R}$ are arbitrary constants.

Applying the given boundary conditions, we find that $c_0 = 0$ and

$$c_1 = \frac{1}{B}\left\{\int_0^{\xi_1} \frac{(\xi_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds + \int_0^{\xi_2} \frac{(\xi_2-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)ds - \beta \int_0^{\eta} \left(\int_0^{s} \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} h(u)du\right) ds\right\}$$

Substituting the values of c_0 and c_1 in (2.5), we get (2.3). This completes the proof of the lemma 2.2.

3. Main results

Let $\mathcal{L} = C([0,1], \mathbb{R})$ denote the Banach space of all continuous function from [0,1] to $\mathbb{R}$ with the norm $\|x\| = \sup\{|x(t)|, t \in [0,1]\}$.

In lemma 2.2, we define an operator $\mathcal{M}: \mathcal{L} \to \mathcal{L}$ as

$$(\mathcal{M}x)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s, x(s)\big)ds + \frac{t}{B}\left\{\int_0^{\xi_1} \frac{(\xi_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, x(s)ds + \int_0^{\xi_2} \frac{(\xi_2-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, x(s)ds - \beta \int_0^{\eta} \left(\int_0^{s} \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f(u, x(u))du\right) ds\right\} \quad (3.1)$$

where$B$ is given by (2.4). Observe that the problem (1.1) has solutions if and only if the operator $\mathcal{M}$ has fixed points. For convenience we set,

$$\Theta = \frac{1}{\Gamma(\alpha+1)} + \frac{1}{|B|}\left[\frac{{\xi_1}^{\alpha-1}}{\Gamma(\alpha)} + \frac{{\xi_2}^{\alpha-1}}{\Gamma(\alpha)} - \beta \frac{\eta^{\alpha+1}}{\Gamma(\alpha+2)}\right] \quad (3.2)$$

Theorem 3.1 Let$f: [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the Lipschitz condition:

$(A_1)|f(t,x) - f(t,y)| \leq k|x-y|, k > 0 \ \forall t \in [0,1], x, y \in \mathbb{R}.$

Then problem (1.1) has a unique solution on [0,1] provided that $\Theta k < 1$, where Θ is given by (3.2).

Proof: Consider$sup_{t\in[0,1]}|f(t,0)| = \lambda$ and show that the operator$\mathcal{M}B_r \subset B_r$,where$B_r = \{x \in \mathcal{M}: \|x\| \leq r\}$ and $r \geq \Theta\lambda(1-\Theta k)^{-1}$.

For $x \in B_r$, $t \in [0,1]$ using assumption (A_1) we get ,

$$|f(t,x(t))| = \left|f\big(t,x(t)\big) - f(t,0) + f(t,0)\right|$$
$$\leq \left|f\big(t,x(t)\big) - f(t,0)\right| + |f(t,0)|$$
$$\leq k\|x\| + \lambda \leq kr + \lambda \tag{3.3}$$

Using (3.2) and (3.3), we get

$$\begin{aligned}\|(\mathcal{M}x)\| \leq sup_{t\in[0,1]} \Bigg\{ & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds \\ & + \frac{t}{B}\left\{\int_0^{\xi_1} \frac{(\xi_1 - s)^{\alpha-2}}{\Gamma\,(\alpha-1)} f(s,x(s)ds + \int_0^{\xi_2} \frac{(\xi_2 - s)^{\alpha-2}}{\Gamma\,(\alpha-1)} f(s,x(s)ds \right. \\ & \left. - \beta \int_0^{\eta}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f(u,x(u))du\right)ds\right\}\Bigg\}\\ \leq (kr+\lambda) sup_{t\in[0,1]} & \left\{\frac{1}{\Gamma(\alpha+1)} + \frac{1}{|B|}\left[\frac{{\xi_1}^{\alpha-1}}{\Gamma(\alpha)} + \frac{{\xi_2}^{\alpha-1}}{\Gamma(\alpha)} - \beta\frac{\eta^{\alpha+1}}{\Gamma(\alpha+2)}\right]\right\}\end{aligned}$$

$$\leq (kr+\lambda)\Theta \leq \mathrm{r}.$$

This shows that$\mathcal{M}B_r \subset B_r$.

Now for any $x, y \in \mathcal{L}$and $t \in [0,1]$, we obtain

$$\begin{aligned}\|\mathcal{M}x - \mathcal{M}y\| \leq sup_{t\in[0,1]} \Bigg\{ & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left|f\big(s,x(s)\big) - f\big(s,y(s)\big)\right|ds \\ & + \frac{t}{B}\left\{\int_0^{\xi_1} \frac{(\xi_1 - s)^{\alpha-2}}{\Gamma\,(\alpha-1)} \left|f\big(s,x(s)\big) - f(s,y(s))\right|ds \right. \\ & + \int_0^{\xi_2} \frac{(\xi_2 - s)^{\alpha-2}}{\Gamma\,(\alpha-1)} \left|f\big(s,x(s)\big) - f(s,y(s))\right|ds \\ & \left. - \beta \int_0^{\eta}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} |f(u,x(u)) - f(u,y(u))|du\right)ds\right\}\Bigg\}\end{aligned}$$

$$\leq k\Theta|\mathrm{x}-\mathrm{y}|$$

Since $k\Theta < 1$, by the given condition, it shows that $\mathcal{M}$ is a contraction. Thus, by Banach contraction mapping principle, there exists a unique solution for the problem (1.1).

Our next result is based on Krasnoselskii's fixed point theorem [13]

Theorem 3.2 Let $f: [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous function satisfying (A_1) and (A_2) .

$$(A_2)|f(t,x)| \leq \tau(t),\ \forall\,(t,x) \in [0,1] \times \mathbb{R}, \text{ and } \tau \in C([0,1] \times \mathbb{R}^+).$$

Then problem (1.1)-(1.2) has at least one solution on [0, 1], if $k\left(\Theta - \frac{1}{\Gamma(\alpha+1)}\right) < 1$ where Θ is given by (3.2).

***Proof*:** consider $B_r = \{x \in \mathcal{M}: \|x\| \leq r\}$ with $r \geq \Theta\|\tau\|$. Define two operators $\mathcal{S}_1$ and $\mathcal{S}_2$ on B_r as follows

$$(\mathcal{S}_1 x)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))ds$$

$$(\mathcal{S}_2 x)(t) = \frac{t}{B}\left\{\int_0^{\xi_1} \frac{(\xi_1 - s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, x(s)ds + \int_0^{\xi_2} \frac{(\xi_2 - s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, x(s)ds \right.$$
$$\left. - \beta \int_0^{\eta}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f(u, x(u))du\right)ds\right\}$$

For $x, y \in B_r$, it is easy to show that $\|(\mathcal{S}_1 x) + (\mathcal{S}_2 x)\| \leq \Theta\|\tau\| \leq r$, which shows that $(\mathcal{S}_1 x) + (\mathcal{S}_2 x) \in B_r$.

Now,

$$\|(\mathcal{S}_2 x) - (\mathcal{S}_2 y)\| \leq sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f\big(s, x(s)\big) - f(s, y(s))\right|ds\right.$$
$$+ \frac{t}{B}\left\{\int_0^{\xi_1} \frac{(\xi_1 - s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|f\big(s, x(s)\big) - f(s, y(s))\right|ds\right.$$
$$+ \int_0^{\xi_2} \frac{(\xi_2 - s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|f\big(s, x(s)\big) - f(s, y(s))\right|ds$$
$$\left.\left. - \beta \int_0^{\eta}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}|f(u, x(u)) - f(u, y(u))|du\right)ds\right\}\right\}$$

$$\leq k\left(\Theta - \frac{1}{\Gamma(\alpha+1)}\right)\|x - y\|$$

This shows that $\mathcal{S}_2$ is a contraction as $k\left(\Theta - \frac{1}{\Gamma(\alpha+1)}\right) < 1$. The continuity of f implies that $\mathcal{S}_1$ is continuous. Also $\mathcal{S}_1$ is uniformly bounded on B_r as

$\|(\mathcal{S}_1 x)\| \leq sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f\big(s, x(s)\big)\right|ds\right\} \leq \frac{\|\tau\|}{\Gamma(\alpha+1)}$.

Therefore $\mathcal{S}_1$ is uniformly bounded on B_r.

To show $\mathcal{S}_1$ is relatively compact on B_r let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $x \in B_r$. Then we get,

$$\|(\mathcal{S}_1 x)(t_2) - (\mathcal{S}_1 x)(t_1)\| \leq \frac{f_m}{\Gamma(\alpha+1)}(|t_2^{\alpha} - t_1^{\alpha}| + |t_2 - t_1|^{\alpha})$$

where $sup_{(t,x)\in[0,1]\times B_r}|f(t,x)| = f_m < \infty$. As $t_2 \to t_1$, $x \in B_r$. Hence by Arzela-Ascoli theorem $\mathcal{S}_1$ is compact on B_r. Thus all the conditions of Krasnoselskii's fixed point theorem are satisfied. This implies that problem (1.1) has at least one solution on $[0,1]$. This completes the proof of the theorem.

Next we establish another existence result based on Schaefer's fixed point theorem [13]

Theorem 3.3 Let X be a Banach space. Assume that $F: X \to X$ is a completely continuous operator and the set $V = \{u \in X: u = \varepsilon Fu, 0 < \varepsilon < 1\}$ is bounded. Then F has a fixed point in X.

Theorem 3.4 Assume that there exists a positive constant L_1 such that $|f(t, x)| \leq L_1$ for all $t \in [0, 1]$, $x \in \mathbb{R}$. Then there exists at least one solution for the problem (1.1) on [0,1].

***Proof*:** Firstly, we show the operator $\mathcal{M}$ is continuous. Let x_n be a sequence such that $x_n \to x$in $\mathcal{L}$. Then for each $t \in [0,1]$, we have

$$|(\mathcal{M}x_n)(t) - (\mathcal{M}x)(t)| \leq \left\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \frac{t}{B}\left\{\int_0^{\xi_1} \frac{(\xi_1 - s)^{\alpha-2}}{\Gamma\,(\alpha-1)} ds + \int_0^{\xi_2} \frac{(\xi_2 - s)^{\alpha-2}}{\Gamma\,(\alpha-1)} ds - \beta \int_0^{\eta}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} du\right) ds\right\}\right\} \times \|f(s, x_n) - f((s, x)\|$$

$$\leq \Theta\|f(s, x_n) - f((s, x)\|$$

Since f is continuous, then $\|\mathcal{M}x_n - \mathcal{M}x\| \to 0$as $n \to \infty$. Thus $\mathcal{M}$is continuous.

Now we will show that $\mathcal{M}$ maps bounded sets into bounded sets in $\mathcal{L}$. Let $B_\rho = \{x \in \mathcal{L}: \|x\| \leq \rho\}$ be a bounded set in $\mathcal{L}$ with $\rho > 0$. With the aid of assumption (A_2) , it is easy to establish $\|\mathcal{M}x\| \leq k\Theta = \lambda, x \in B_\rho$. Thus $\mathcal{M}$ is uniformly bounded on B_ρ.

To show the operator $\mathcal{M}$ is completely continuous let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$and $x \in B_\rho$.

$$|(\mathcal{M}x)(t_2) - (\mathcal{M}x)(t_1)| \leq \int_0^{t_1} \frac{(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} \left|f\big(s, x(s)\big)\right| ds + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} \left|f\big(s, x(s)\big)\right| ds$$
$$+ \frac{t_2 - t_1}{|B|}\left\{\int_0^{\xi_1} \frac{(\xi_1 - s)^{\alpha-2}}{\Gamma(\alpha-1)} \left|f\big(s, x(s)\big)\right| ds + \int_0^{\xi_2} \frac{(\xi_2 - s)^{\alpha-2}}{\Gamma(\alpha-1)} \left|f\big(s, x(s)\big)\right| d\,s - -\beta \int_0^{\eta}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} |f(u, x(u))| du\right) ds\right\}$$

$$\leq k\left\{\frac{1}{\Gamma(\alpha+1)}[(t_2^\alpha - t_1^\alpha) + 2(t_2 - t_1)^\alpha] + \frac{t_2 - t_1}{|B|}\left[\frac{{\xi_1}^{\alpha-1}}{\Gamma(\alpha)} + \frac{{\xi_2}^{\alpha-1}}{\Gamma(\alpha)} - \beta \frac{\eta^{\alpha+1}}{\Gamma(\alpha+2)}\right]\right\}$$

As $t_2 \to t_1$, the right hand side tends to zero independently of $x \in B_\rho$. Hence by the Arzela-Ascoli theorem, the operator $\mathcal{M}$ is completely continuous.

Next, consider the set $V = \{x \in \mathcal{L} : x = \varepsilon\mathcal{M}x, 0 < \varepsilon < 1\}$. To show that V is bounded, let $x \in V, t \in [0,1]$. Then

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds + \frac{t}{B}\left\{\int_0^{\xi_1} \frac{(\xi_1 - s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,x(s)ds + \int_0^{\xi_2} \frac{(\xi_2 - s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,x(s)ds - \beta\int_0^{\eta}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f(u,x(u))du\right)ds\right\}$$

And $|x(t)| = \varepsilon|(\mathcal{M}x)(t)| \le k\Theta = \lambda$. Thus V is bounded. Theorem 3.3 implies that (1.1) has at least one solution. This completes the proof of the theorem.

4. Examples

Example 4.1 Consider a fractional boundary value problem given by

$$^{c}D^{\frac{3}{2}}x(t) = \frac{e^{-t^2}}{\sqrt{t^2+81}} \cdot \frac{sin\, x}{1+|x|}, \qquad t \in [0,1]$$

$$x(0) = 0, \qquad 3x'\left(\frac{1}{4}\right) + 4x'\left(\frac{1}{2}\right) = 2\int_0^{\frac{1}{3}} x(s)ds$$

Here, $\alpha = \frac{3}{2}, a = 3, b = 4, \xi_1 = \frac{1}{4}$, $\xi_2 = \frac{1}{2}$, $\beta = 2$, $\eta = \frac{1}{3}$ and $f(t,x) = \frac{e^{-t^2}}{\sqrt{t^2+81}} \cdot \frac{sin\, x}{1+|x|}$.

With the given values $\Theta \cong 1.263$ and $k = \frac{1}{9}$ as $|f(t,x) - f(t,y)| \le \frac{1}{9}|x-y|$. Thus $k\Theta \cong 0.1403 < 1$. Therefore by theorem 3.1, there exists a unique solution for the given problem.

Example 4.2 Consider a fractional boundary value problem given by

$$^{c}D^{\frac{3}{2}}x(t) = \frac{e^{-sin^2 t}}{(14e^t+2)\sqrt{t+16}}\cos x + \frac{t^2}{t+1}, \qquad t \in [0,1]$$

$$x(0) = 0, \qquad 2x'\left(\frac{2}{5}\right) + 3x'\left(\frac{3}{7}\right) = 3\int_0^{\frac{1}{2}} x(s)ds$$

Here, $\alpha = \frac{3}{2}, a = 2, b = 3, \xi_1 = \frac{2}{5}$, $\xi_2 = \frac{3}{7}$, $\beta = 3$, $\eta = \frac{1}{2}$

and $f(t,x) = \frac{e^{-sin^2 t}}{(14e^t+2)\sqrt{t+16}}\cos x + \frac{t^2}{t+1}$

With the given data $\Theta \cong 1.507$ and $k\left(\Theta - \frac{1}{\Gamma(\alpha+1)}\right) \cong 0.01179 < 1$. Thus by theorem 3.2, example 4.2 has a solution in [0, 1].

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