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Existence of Positive Solution for a Nonlinear Weighted Bi-Harmonic System of Elliptic Partial Differential Equations via Fixed-Point Argument

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ABSTRACT

In this paper, we establish an existence criterion of positive solution for a nonlinear weighted bi-harmonic system of elliptic partial differential equations in the unit ball in $\mathbb{R}^n$ (n dimensional euclidean space). The analysis of this paper is based on a topological method (a fixed-point argument). Initially, we establish a priori solution estimates, and then use a fixed-point theorem for deducing the existence of positive solutions. Finally, we prove a non-existence criterion as the complement of existence criterion.

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1. Introduction

The nonlinear partial differential equations (PDEs for short) have proved to be valuable tools for the modeling of many physical, chemical and biological phenomena, see for instance [27, 32, 34] and references therein. In the last few decays, there has been a noticeable interest on the study of existence of solution to nonlinear elliptic

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systems, especially when the nonlinear term appears as a source in the equation with the Dirichlet's or Neumann's boundary conditions, see for instance [13, 28, 32] and references therein. Nonlinear systems are divided into two broad classes, first one is with a variational structure, namely Hamiltonian or gradients systems, see for instance [1, 16, 18] and second one is the class of non-variational problems, which can be maintained through the topological methods (fixed-point arguments), see for instance [2, 4, 9, 10]. In this paper, we deal with the existence of non-variational boundary value problem (BVP for short) for a given system using a topological method (a fixed-point argument).

The single fourth order nonlinear PDEs arise in various physical phenomenon such as study of travelling waves in suspension bridges [20], micro electro mechanical systems [24], radar imaging [3], bending behaviour of a thin elastic rectangular plate [31], geometric and functional design [5, 6, 29] etc. In this paper, we consider the following nonlinear weighted bi-harmonic system of elliptic PDEs

$$\begin{cases} \Delta^2 u = \lambda a(x) g(v),\ v > 0 \ \text{ in } B, \\ \Delta^2 v = \lambda b(x) f(u),\ u > 0 \ \text{ in } B, \\ u = 0 = v,\ \dfrac{\partial u}{\partial \nu} = 0 = \dfrac{\partial v}{\partial \nu} \ \text{ on } \partial B, \end{cases} \tag{1.1}$$

where B denotes the unit ball in $\mathbb{R}^N, (N > 4)$ with boundary ∂B, Ω is open, smooth and bounded subset of $\mathbb{R}^N$, λ is a positive parameter, $a, b: \Omega \to \mathbb{R}$ (set of real numbers) are sign changing weights, $f, g: 0, \infty) \to \mathbb{R}$ are continuous nonlinearities with $f(0) = 0, g(0) = 0$.

The system designated by (1.1) is omnipresent in physics and chemistry where steady-states are answers to problematic questions in a great diversity of systems of reaction-diffusion equations. This system interacts everywhere in nature and this interaction takes place in such unequal phenomena as the proliferation of virile mutants over a substantially wide habitat, the dispersion of fire flames in roomy forests, in combustion chambers, or in nuclear reactors where neutron populations evolve and develop. Lions [21] studied the existence of a positive solution to the following Dirichlet problem

$$\begin{cases} -\Delta u = \lambda a(x) f(u) \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

with weight function and nonlinearity satisfy $a \geq 0,\ f \geq 0,$ respectively. Problem with indefinite weights was studied by Brown *et al.* [7], Brown and Tertikas [8], Cac *et al.* [11], Hai [19] and their references. Dalmasso [14] studied the following BVP

$$\begin{cases} -\Delta u = \lambda a(x) f(v) \ \text{ in } \Omega, \\ -\Delta v = \lambda b(x) g(u) \ \text{ in } \Omega, \\ u = 0 = v \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

for $a(x) = 1,\ b(x) = 1,\ \lambda = 1,$ and established the existence of a positive solution to the BVP (1.3) using Schauder fixed-point theorem [30]. After Dalmasso [14], several authors studied the BVP (1.3) using different techniques and theorems for different values of $a(x),\ b(x),\ \lambda = 1,$ see for instance Chen [12] and their references. In 2014 Dwivedi [13] studied the existence of positive solutions to BVP (1.1) by using Leray-Schauder fixed-point theorem. Recently, Soltani and Yazidi [4] established the existence and non-existence of positive solutions to the following system of bi-Laplacian equations

$$\begin{cases} \Delta^2 u = g(v),\ v > 0 \ \text{ in } B, \\ \Delta^2 v = f(u),\ u > 0 \ \text{ in } B, \\ u = 0 = v,\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 \ \text{ on } \partial B, \end{cases} \tag{1.4}$$

applying a topological method (fixed-point argument) [15, 25].

Inspired by above mentioned works, in the present paper we study the existence and non-existence of positive solutions to (1.1) by using a topological method (fixed-point argument) [15, 25]. The priori estimates of positive solutions for elliptic partial differential equations gives a sufficient information about the existence of positive solutions, see for instance [9, 22, 23]. From this context, first we establish a priori estimates of solutions to our problem given by (1.1) and then we prove the existence and non-existence criteria of positive solutions depending on that priori estimate. The rest of this work is furnished as follows:
In Section 2, we establish some preliminary facts which will be needed to prove our main results. Section 3 is devoted to state and prove the existence and non-existence criteria of solutions to the nonlinear biharmonic system of elliptic PDEs given by (1.1). In the end of Section 3 we provide two illustrative examples to support our analytic proof.

2. Preliminary Notes

In this section, we recall some preliminary results related to the bi-Laplacian problem and sate a fixed-point theorem which will help us to prove our main results.

Let us consider the problem (1.1) for radially symmetric solutions, and let $r=|x|$, $u=u(r)$, $v=v(r)$,

$$\begin{cases} u^{(4)}+\dfrac{2(N-1)}{r}u'''+\dfrac{(N-1)(N-3)}{r^2}u''-\dfrac{(N-1)(N-3)}{r^3}u'=\lambda a(x)g(v),\ v>0 \text{ for } r\in(0,1), \\ v^{(4)}+\dfrac{2(N-1)}{r}v'''+\dfrac{(N-1)(N-3)}{r^2}v''-\dfrac{(N-1)(N-3)}{r^3}v'=\lambda b(x)f(u),\ u>0 \text{ for } r\in(0,1), \\ u'(0)=0=v'(0),\ u'''(0)=0=v'''(0),\ u'(1)=0=v'(1),\ u(1)=0=v(1). \end{cases} \tag{2.1}$$

Then any solution $(u(r), v(r))\in C^4(0,1)\times C^4(0,1)$ of (2.1) is a radial symmetric solution of (1.1).

Now, we recall the following lemma from [23, Lemma 2], which gives more information regarding the eigenvalue problem for the operator Δ^2.

Lemma 2.1 [23]**:** There is a $\mu>0$ such that the problem

$$\Delta^2 v=\mu v \text{ in } B,\ v=\frac{\partial v}{\partial v} \text{ on } \partial B$$

possesses a positive, radial symmetric solution $\varphi(x)$ which satisfies for some positive constants c_1 and c_2,

$$c_1(1-|x|)^2\le\varphi(x)\le c_2(1-|x|)^2,\quad x\in\overline{B}. \tag{2.2}$$

From [23] as well as [17], we recall the Green's function $G(r,s)$ for the corresponding linear problem of (2.1) is given by

$$G(r,s)=\begin{cases} A_N(s)+r^2B_N(s), & \text{for } 0\le r\le s\le 1 \\ (s/r)^{N-1}\left(A_N(r)+s^2B_N(r)\right), & \text{for } 0\le s\le r\le 1, \end{cases} \tag{2.3}$$

where

$$A_N(t)=\frac{t^3}{4(N-2)(N-4)}\left[2+(N-4)t^{N-2}-(N-2)t^{N-4}\right] \tag{2.4}$$

and

$$B_N(t) = \frac{t}{4N(N-2)}\left[Nt^{N-2} - (N-2)t^N - 2\right]. \tag{2.5}$$

From [23] we also observed that the Green's function $G(r,s)$ has the following properties:
There exists a positive constant d such that

$$0 \le G(r,s) \le ds^{N-1}(1-s)^2\left(\max(r,s)\right)^{4-N}, \tag{2.6}$$

$$\frac{\partial}{\partial r} G(r,s) \le 0, \tag{2.7}$$

and

$$\left.\frac{\partial^2}{\partial r^2} G(r,s)\right|_{r=1} = \frac{1}{2} s^{N-1}\left(1-s^2\right). \tag{2.8}$$

Hence, the BVP (2.1) is transformed into the following integral equations:

$$\begin{cases} u(r) = \int_0^1 G(r,s)\lambda a(s) g\left(v(s)\right) ds, \\ v(r) = \int_0^1 G(r,s)\lambda b(s) f\left(u(s)\right) ds. \end{cases} \tag{2.9}$$

It is well established that the BVP (2.1) and the problem (2.9) are equivalent. For the study of BVP (1.1), we need the following eigenvalue problem:

$$\begin{cases} \Delta^2 \phi = \lambda_2 a(x)\psi \text{ in } B, \\ \Delta^2 \psi = \lambda_1 b(x)\phi \text{ in } B, \\ \phi = 0 = \psi, \ \dfrac{\partial \phi}{\partial \nu} = \dfrac{\partial \psi}{\partial \nu} = 0 \text{ on } \partial B, \end{cases} \tag{2.10}$$

where $\lambda_1, \lambda_2 > 0$.

Now we establish following lemma to comment on the solution of the eigenvalue problem (2.10).

Lemma 2.2: Let φ_1 be the corresponding eigenfunction of μ which is the first eigenvalue of Δ^2 on the unit ball B and $\lambda_1\lambda_2 = \mu^2$. Then the eigenvalue problem (2.10) has a positive solution (ϕ, ψ) satisfying (modulo a constant) $\phi = \frac{1}{\sqrt{\lambda_1}}\varphi_1$ and $\psi = \frac{1}{\sqrt{\lambda_2}}\varphi_1$.

Proof. Using the idea established in [33] for a Laplacian eigenvalue problem, we define

$$w_1 = \sqrt{\lambda_1}\phi, \tag{2.11}$$

$$w_2 = \sqrt{\lambda_2}\psi. \tag{2.12}$$

Combining (2.11), (2.12) and the eigenvalue problem (2.10), we obtain

$$\begin{cases} \Delta^2 w_1 = \sqrt{\lambda_1\lambda_2}\, a(x) w_2 \text{ in } B, \\ \Delta^2 w_2 = \sqrt{\lambda_1\lambda_2}\, b(x) w_1 \text{ in } B, \\ w_1 = 0 = w_2, \ \dfrac{\partial w_1}{\partial \nu} = \dfrac{\partial w_2}{\partial \nu} = 0 \text{ on } \partial B, \end{cases} \tag{2.13}$$

Adding and subtracting the equations and the equation of boundary conditions of the problem (2.13), we have

$$\begin{cases} \Delta^2(w_1+w_2)=\sqrt{\lambda_1\lambda_2}\left(a(x)w_2+b(x)w_1\right) \text{ in } B, \\ w_1+w_2=0,\ \dfrac{\partial(w_1+w_2)}{\partial\nu}=0 \text{ on } \partial B. \end{cases} \tag{2.14}$$

and

$$\begin{cases} \Delta^2(w_1-w_2)=\sqrt{\lambda_1\lambda_2}\left(a(x)w_2-b(x)w_1\right) \text{ in } B, \\ w_1-w_2=0,\ \dfrac{\partial(w_1-w_2)}{\partial\nu}=0 \text{ on } \partial B. \end{cases} \tag{2.15}$$

Multiply both sides of (2.15) by w_1-w_2 and take integration by parts, we obtain

$$\int_B |\Delta(w_1-w_2)|^2\,dx=-\sqrt{\lambda_1\lambda_2}\int_B \left(a(x)w_2-b(x)w_1\right)(w_1-w_2)\,dx\,. \tag{2.16}$$

Since $a(x)$ and $b(x)$ are sign changing weights, then (2.16) can be written as

$$\int_B |\Delta(w_1-w_2)|^2\,dx=-\sqrt{\lambda_1\lambda_2}\int_B |w_1-w_2|^2\,dx. \tag{2.17}$$

which prove that $w_1=w_2$ in $\overline{B}$. Since $\sqrt{\lambda_1\lambda_2}=\mu$ and by the properties of eigenvalue problem for the bi-Laplacian, we obtain that the BVP (2.14) has the first eigenfunction φ_1 as the only solution. Then for any positive constant $c,$ we have $w_1=w_2=c\varphi_1.$ Thus $\phi=c\dfrac{1}{\sqrt{\lambda_1}}\varphi_1$ and $\psi=c\dfrac{1}{\sqrt{\lambda_2}}\varphi_1$.

This completes the proof.

Lemma 2.3: Let F and G be primitives of f and g respectively with $F(0)=0$ and $G(0)=0.$ Suppose that (u,v) be a solution of the system given by (1.1) and $\alpha,\ \beta$ are some positive constants. Then we have the following identity:

$$\begin{aligned} \int_{\partial B}(\Delta u,\ \Delta v)(x,v)\,d\sigma_x &= \int_B \left(NF(u)+NG(v)-\alpha\lambda auf(u)-\beta\lambda bvg(v)\right)dx \\ &\quad +\left(N-4-(\alpha+\beta)\right)\int_B (\Delta u,\ \Delta v)(x,v)\,dx. \end{aligned} \tag{2.18}$$

Proof. Using [26, proposition 4], [33, theorem 2.1] and after some easy computations, we obtain the following identity:

$$\begin{aligned} &\frac{\partial}{\partial x_i}\left[x_i L-\left(x_k\frac{\partial u_l}{\partial x_k}+a_l u_l\right)\left(L-p_i-\frac{\partial}{\partial x_j}L_{r_{ij}}\right)-\frac{\partial}{\partial x_j}\left(x_k\frac{\partial u_l}{\partial x_k}+a_l u_l\right)L_{r_{ij}}\right] \\ &= NL+x_iL_{x_i}-a_lu_lL_{u_l}-(a_l+1)\frac{\partial u_l}{\partial x_i}L_{p_i}-(a_l+2)\frac{\partial^2 u_l}{\partial x_i\partial x_j}L_{r_{ij}}, \end{aligned} \tag{2.19}$$

where $L=L(x,U,p,r)$ is a Lagrangian with $U=(u_1,u_2),\ p=\left(p_i^k\right),\ p_i^k=\dfrac{\partial u_k}{\partial x_i},\ r=\left(r_{ij}\right),\quad i=1,2,\cdots,N$ and a_1,a_2 are some constants.

Applying the identity (2.19) to the Lagrangian associate with problem (1.1), we get

$$L = L(x,U,\nabla U,\Delta U) = (\Delta u,\Delta v) + F(u) + G(v), a_1 = \alpha,\ a_2 = \beta.$$

Integrating (2.19) over the unit ball B and using the conditions $u = 0 = v,\ \frac{\partial u}{\partial \nu} = 0 = \frac{\partial v}{\partial \nu}$ on ∂B, we get (2.18).

This completes the proof.

Remark 2.1: Putting $\alpha + \beta = N - 4$ in (2.18), we obtain the critical conditions on f and g are $NF(u) - \alpha uf(u) = 0$ and $NG(v) - (N - 4 - \alpha)vg(v) = 0,$ then we have

$$\frac{f(u)}{F(u)} = \frac{N/\alpha}{u} \text{ and } \frac{g(v)}{G(v)} = \frac{N/(N-4-\alpha)}{v}. \tag{2.20}$$

Hence, for some positive constants c_1 and c_2, we obtain

$$f(u) = c_1 u^{\frac{N}{\alpha}-1} \text{ and } g(v) = c_2 v^{\frac{N}{N-4-\alpha}-1}. \tag{2.21}$$

Definition 2.1 ([35])**:** Let $(X, \|.\|)$ be a real Banach space and C be a nonempty closed convex subset of X. This subset C is called a cone of X if it satisfies the following conditions:

(i) $x \in C,\ \mu > 0$ implies $\mu x \in C$; (ii) $x \in C, -x \in C$ implies $x = 0$.

Now, we state a fixed-point theorem due to Figueiredo *et al.* [15] and Peletier and van der Vorst [25], which will be used as the main tool to prove our main results.

Theorem 2.1 ([15, 25])**:** Let C be a cone in a Banach space X and $T : X \to X$ be a compact map such that $T(0) = 0$. Assume that there exist the numbers $0 < r < R$ such that

(a) $x \neq \delta T(x)$ for $\delta \in [0,1]$ and $\|x\| = r,$

(b) there exists a compact map $S : \overline{B_R} \times [0,\infty) \to C$ such that

$$S(x,0) = T(x),\ \textit{if}\ \|x\| = R,$$

$$S(x,\mu) \neq x,\ \textit{if}\ \|x\| = R \text{ and } 0 \le \mu < \infty,$$

$$S(x,\mu) \neq x,\ \textit{if}\ x \in \overline{B_R} \text{ and } \mu \ge \mu_0.$$

Then if $U = \{x \in C : r < \|x\| < R\}$ and $B_\rho = \{x \in C : \|x\| < \rho\},$ we have

$$i_C(T, B_R) = 0,\quad i_C(T, B_r) = 1,\quad i_C(T, U) = -1,$$

where $i_C(T, \Omega)$ denotes the index of T with respect to Ω. In particular, T has a fixed point in U.

3. Results and Discussions

This section is devoted to establish the existence and non-existence criterion of positive solutions to nonlinear weighted bi-harmonic system of elliptic PDEs given by (1.1).

Assume that the nonlinearities f and g satisfies the following hypothesis:

$$(H_1)\quad \begin{array}{l} \liminf_{s\to\infty} f(s)/s > \lambda_1,\ \limsup_{s\to 0} f(s)/s < \lambda_1, \\ \liminf_{s\to\infty} g(s)/s > \lambda_2,\ \limsup_{s\to 0} g(s)/s < \lambda_2, \end{array}$$

(H_2) $NF(s)-\alpha sf(s)\ge\theta_1 sf(s),\ s>0$ for some $\theta_1\ge 0,$ $NG(s)-\beta sg(s)\ge\theta_2\, sg(s),\ s>0$ for some $\theta_2\ge 0,$ and α and β are positive real numbers satisfying $\alpha+\beta=N-4$.

To establish our main results, we need the following priori estimates:

(H_3) there exists a constant $d>0$ such that for every positive solution (u,v) of the problem given (1) verifies $\|u\|_\infty\le d$ and $\|v\|_\infty\le d$.

Now we are in position to present and prove our main results.

Theorem 3.1: If the hypothesis $(H_1),(H_2)$ and (H_3) hold, then the nonlinear weighted bi-harmonic system of elliptic PDEs given by (1.1) has a positive solution.

Poof. We prove this theorem by applying theorem 2.1. Consider a Banach space $X=C^*(0,1)\times C^*(0,1),$ where $C^*(0,1)$ denote the space of continuous bounded functions defined on $(0,1)$, endowed with the norm $\|u\|=\sup_{t\in(0,1)}|u(t)|$. We define a cone C on X by

$$C=\{w\in X: w(t)\ge 0, \text{ for all } t\in(0,1)\},$$

where $w=(y,z)\ge 0$ means that $y\ge 0$ and $z\ge 0$. We also define a compact map $T:X\to X$ by

$$T(w)(r)=\int_0^1 G(r,s)\lambda h(w(s))ds, \text{ where } h(w)=\big(a(x)g(v),\, b(x)f(u)\big). \tag{3.1}$$

It is clear that a fixed point of T is a solution of the integral equations given by (2.9) and hence a fixed point of T is a solution of our problem (1.1).

Now, we will prove that all the conditions of theorem 2.1 for hypothesis $(H_1),(H_2)$ and (H_3).

From the hypothesis (H_1), we have there exists positive constants $q_1<1$ and $q_2<1$ such that $f(u(x))\le q_1\lambda_1 u(x)$ and $g(v(x))\le q_2\lambda_2 v(x)$. Then we have

$$\lambda_2\int va\psi dx=\int v\Delta^2\phi dx=\int\Delta^2 v\phi dx=\int\lambda bf(u)\phi dx\le\lambda q_1\lambda_1\int ub\phi dx. \tag{3.2}$$

On the other hand,

$$\lambda_1\int ub\phi dx=\int u\Delta^2\psi dx=\int\Delta^2 u\psi dx=\int\lambda ag(v)\psi dx\le\lambda q_2\lambda_2\int va\psi dx. \tag{3.3}$$

Combining the inequalities (3.2) and (3.3), we get

$$\lambda_2\int va\psi dx\le\lambda^2 q_1q_2\lambda_2\int va\psi dx, \tag{3.4}$$

$$\lambda_1\int ub\phi dx\le\lambda^2 q_1q_2\lambda_1\int ub\phi dx. \tag{3.5}$$

Since λ is a positive parameter, then from (3.4) and (3.5), we obtain $q_1q_2<1$. Hence the inequalities (3.4) and (3.5) give a contradiction as because the integrals are nonzero. Moreover, if u and v are replaced by δu and δv respectively, where $\delta\in[0,1]$, then the inequalities (3.4) and (3.5) also give a contradiction.

Therefore, we have

$$w(t)\ne\delta T(w(t)) \text{ with } \delta\in[0,1],\ \|w\|=r,\ w\in C.$$

Hence the condition (a) of theorem 2.1 holds.

Now, define a compact map $S:C\times[0,\infty)\to C$ by

$$S(w,\gamma)(r)=T(w+\gamma)(r). \tag{3.6}$$

Then it is easy to see that $S(w,0)=T(w)$. This prove the first condition of (b) of theorem 2.1.

Again, the hypothesis (H_1), gives that there exists positive constants $k_1>\lambda_1$, $k_2>\lambda_2$ and $\mu_0>0$ such that $f(y+\mu)\le k_1 y$ and $g(z+\mu)\le k_2 z$, for all $\mu\ge\mu_0$ and $(y,z)\ge(0,0)$. Then we have

$$\lambda_2\int va\psi dx=\int v\Delta^2\phi dx=\int\Delta^2 v\phi dx=\int\lambda bf(u)\phi dx\ge\lambda k_1\int ub\phi dx\ge\lambda\lambda_1\int ub\phi dx\,. \tag{3.7}$$

Or

$$\lambda_1\int ub\phi dx=\int u\Delta^2\psi dx=\int\Delta^2 u\psi dx=\int\lambda ag(v)\psi dx\ge\lambda k_2\int va\psi dx\,. \tag{3.8}$$

Combining the inequalities (3.7) and (3.8), we get

$$\lambda_2 a\int v\psi dx\ge\lambda^2 k_2 a\int v\psi dx\,. \tag{3.9}$$

Similarly, we can prove that

$$\lambda_1 b\int u\phi dx\ge\lambda^2 k_1 b\int u\phi dx\,. \tag{3.10}$$

Since λ is a positive parameter, a and b are sign changing weights, the integrals $\int v\psi dx$ and $\int u\phi dx$ are nonzero and $k_1>\lambda_1$, $k_2>\lambda_2$, then the inequalities (3.4) and (3.5) give a contradiction. Therefore, there exists a constant $\mu_0>0$ such that

$$S(w,\mu)(t)\ne w(t),\ \text{for all } w\in C \text{ and } \mu\ge\mu_0. \tag{3.11}$$

This proves the third condition of (b) of theorem 2.1. Now, we prove the second condition of (b) of theorem 2.1 and to prove it, we consider the family of nonlinearities $\left(f(y+\mu),\,g(z+\mu)\right)$ for $\mu\in[0,\mu_0]$, using a priori estimates hypothesis (H_3) which does not depend on μ. Hence, for $R>r$ large enough, we get

$$S(w,\mu)(r)\ne w(r),\ \text{for all } w\in C \text{ and } \mu\in[0,\mu_0],\ \|w\|=R. \tag{3.12}$$

Therefore, the relations (3.11) and (3.12) prove the second condition of (b) of theorem 2.1. That is all the conditions of theorem 2.1 are satisfied. Thus, by theorem 2.1 and hypothesis (H_2), we conclude that the problem (2.9) exists a nontrivial positive solution and hence the nonlinear weighted bi-harmonic system of elliptic PDEs given by (1.1) has a positive solution.

This completes the proof.

Now, we introduce the following critical exponents associated to the nonlinear weighted bi-harmonic system (1.1) by

$$p^*=\frac{N-\alpha}{\alpha},\ q^*=\frac{4+\alpha}{N-4-\alpha},\ \text{where } \alpha\in\left((N-4)/2,N/2\right). \tag{3.13}$$

Then it is clear that

$$\frac{1}{p^*+1}+\frac{1}{q^*+1}=\frac{N-4}{4}. \tag{3.14}$$

Remark 3.1: From the hypothesis (H_1) and (H_2), we have

$$\lim_{t\to\infty} f(t)/t^{p^*}=0 \text{ and } \lim_{t\to\infty} g(t)/t^{q^*}=0.$$

Certainly, from (H_1), we have there exists $t_0>0$ such that $f(t)>0$ and $g(t)>0$ for $t>t_0$. Hence, for $t>t_0$, from (H_2) we can write

$$NF(t)\ge-\theta_1+\eta tf(t) \text{ and } NG(t)\ge-\theta_2+\mu tg(t), \tag{3.15}$$

where $\eta=\alpha+\theta_1$ and $\mu=\beta+\theta_2$. Therefore,

$$F'(t)-\frac{N}{\eta t}F(t)\le\frac{\theta_1}{\eta t} \text{ and } G'(t)-\frac{N}{\mu t}G(t)\le\frac{\theta_2}{\mu t}. \tag{3.16}$$

Multiplying the inequalities of (3.16) by $t^{-\frac{N}{\eta}}$ and $t^{-\frac{N}{\mu}}$ respectively, we get

$$\frac{d}{dt}\left(t^{-\frac{N}{\eta}}F(t)\right)\le\frac{\theta_1}{\eta}t^{-1-\frac{N}{\eta}} \text{ and } \frac{d}{dt}\left(t^{-\frac{N}{\mu}}G(t)\right)\le\frac{\theta_2}{\mu}t^{-1-\frac{N}{\mu}}.$$

Hence for some positive constants d_1 and d_2, we deduce that

$$F(t)\le d_1t^{N/\eta} \text{ and } G(t)\le d_2t^{N/\mu}. \tag{3.17}$$

For large enough t and for some positive constants D and $\overline{D}$, from (3.15) and (3.17), we have

$$f(t)\le Dt^{\frac{N}{\eta}-1} \text{ and } g(t)\le \overline{D}t^{\frac{N}{\mu}-1}. \tag{3.18}$$

Or since $\eta=\alpha+\theta_1$, $\mu=\beta+\theta_2$, $\theta_1,\theta_2>0$, and $\alpha+\beta=N-4$, then we have

$$\eta+\mu>N-4.$$

The following theorem verify the priori estimates hypothesis given by (H_3).

Theorem 3.2: Under the hypothesis (H_1) and (H_2) on nonlinearities f and g, the priori estimates hypothesis (H_3) is satisfied, namely, every positive solution of nonlinear weighted bi-harmonic system given by (1.1) is bounded in L^∞.

Proof. We prove this theorem using following four steps.

Step 1. In this step we claim that there exist four positive constants c_1,c_2,c_3,c_4 such that

$$\int_B\lambda bf(u)\phi dx\le c_1,\quad \int_B\lambda ag(v)\psi dx\le c_2, \tag{3.19}$$

$$\int_B\lambda bu\phi dx\le c_3,\quad \int_B\lambda av\psi dx\le c_4. \tag{3.20}$$

Using the first two equations of our problem (1.1), we have

$$\int_B\lambda bf(u)\phi dx=\int_B\Delta^2v\phi dx=\int_B v\Delta^2\phi dx=\lambda_2\int_B va\psi dx. \tag{3.21}$$

According to the hypothesis (H_1), there exist $k_2>\lambda_2$ and $A>0$ such that $g(v)\ge k_2v-A$. Hence, for a positive constant c, from (3.21) we have

$$\int_B\lambda bf(u)\phi dx=\lambda_2\int_B va\psi dx\le c+\frac{\lambda_2}{k_2}\int_B ag(v)\psi dx. \tag{3.22}$$

Similarly, using hypothesis (H_1) and for a positive constant d, we obtain

$$\int_B\lambda ag(v)\psi dx=\lambda_1\int_B bu\phi dx\le d+\frac{\lambda_1}{k_1}\int_B bf(u)\phi dx. \tag{3.23}$$

Combining (3.22) and (3.23), and for positive constants m_1 and m_2, we have

$$\begin{cases} \int_B \lambda bf(u)\phi dx \le m_1 + \dfrac{\lambda_1\lambda_2}{k_1k_2}\int_B bf(u)\phi dx, \\ \int_B \lambda ag(v)\psi dx \le m_2 + \dfrac{\lambda_1\lambda_2}{k_1k_2}\int_B ag(v)\psi dx. \end{cases} \tag{3.24}$$

Since $\frac{\lambda_1\lambda_2}{k_1k_2} < 1,$ hence we deduce (3.19) form (3.24). In the similar way by using the hypothesis (H_1), we can easily obtain (3.20).

Step 2. In this step we claim that there exist four positive constants d_1, d_2, d_3, d_4 such that

$$u(r) \le d_1,\ v(r) \le d_2, \text{ for } \frac{2}{3} \le r \le 1, \tag{3.25}$$

$$u''(1) \le d_3,\ v''(1) \le d_4. \tag{3.26}$$

Using (2.6) and (2.7), we observed that $r \to G(r,s)$ is decreasing, hence we deduce that $u(r)$ and $v(r)$ are decreasing in r and for arbitrary $\frac{2}{3} \le r \le 1,$ we have

$$\begin{aligned} u(r) &\le u\left(\frac{2}{3}\right) \le 3\int_{1/3}^{2/3} \lambda a(s)u(s)ds \le d\int_0^1 s^{N-1}(1-s)^2 \lambda b(s)u(s)ds \\ &\le d + \int_0^1 s^{N-1}(1-s)^2 \lambda b(s)u(s)ds, \end{aligned} \tag{3.27}$$

where d is a positive constant.

From (2.2) and lemma 2.2 and (3.27), we obtain

$$u(r) \le d\left(1 + \int_0^1 s^{N-1}(1-s)^2 \lambda b(s)u(s)ds\right) \le d\left(1 + \int_B \lambda b\phi u dx\right).$$

Thus by (3.20), we yield that $u(r) \le d_1,$ for $\frac{2}{3} \le r \le 1,$ and in the similar way we can prove that $v(r) \le d_2,$ for $\frac{2}{3} \le r \le 1$.

Now, differentiating the relations of (2.9) two times with respect to r, we have

$$u''(r) = \int_0^1 \frac{\partial^2 G(r,s)}{\partial r^2} \lambda a(s) g(v(s))ds, \quad v''(r) = \int_0^1 \frac{\partial^2 G(r,s)}{\partial r^2} \lambda b(s) f(u(s))ds. \tag{3.28}$$

Since the integrals of (3.28) converge, then taking limit as $r \to 1$ and we obtain

$$u''(1) = \int_0^1 \left.\frac{\partial^2 G(r,s)}{\partial r^2}\right|_{r=1} \lambda a(s) g(v(s))ds, \quad v''(1) = \int_0^1 \left.\frac{\partial^2 G(r,s)}{\partial r^2}\right|_{r=1} \lambda b(s) f(u(s))ds.$$

Using (2.8), we get

$$u''(1) = \frac{1}{2}\int_0^1 s^{N-1}(1-s^2)\lambda a(s) g(v(s))ds, \quad v''(1) = \frac{1}{2}\int_0^1 s^{N-1}(1-s^2)\lambda b(s) f(u(s))ds. \tag{3.29}$$

Now, for some positive constant d, from (2.2) and lemma 2.2 and (3.29), we obtain that

$$u''(1) \le d\int_B \lambda a\psi g(v)dx, \quad v''(1) \le d\int_B \lambda b\phi f(u)dx. \tag{3.30}$$

Combining (3.15) and (3.30), we have

$$u''(1) \le d_3,\ v''(1) \le d_4.$$

This proves (3.26).

Step 3. In this step we claim that for a small number $0 < \kappa < 1$, there exists positive constants e_1, e_2, e_3, e_4 such that

$$\int_0^\kappa s^{N-1} \lambda b(s) f(u(s)) ds \le e_1, \qquad \int_0^\kappa s^{N-1} \lambda a(s) g(v(s)) ds \le e_2, \tag{3.31}$$

$$\int_B \lambda buf(u) dx \le e_3, \quad \int_B \lambda avg(v) dx \le e_4. \tag{3.32}$$

For proving (3.31), we follow the proof of (3.19) and (3.20), and using lemma 2.1 and lemma 2.2 and for the small number $0 < \kappa < 1$, we get

$$\begin{aligned}
&\int_0^\kappa s^{N-1} \lambda b(s) f(u(s)) ds \\
&\le \int_0^\kappa s^{N-1} \frac{(1-s)^2}{(1-\kappa)^2} \lambda b(s) f(u(s)) ds \le \frac{1}{(1-\kappa)^2} \int_0^\kappa s^{N-1} (1-s)^2 \lambda b(s) f(u(s)) ds \\
&\le d \int_0^\kappa s^{N-1} \lambda b(s) \phi(s) f(u(s)) ds = d \int_B \lambda b \phi f(u) dx \le dc_1 = e_1, \\
&\int_0^\kappa s^{N-1} \lambda a(s) g(v(s)) ds \\
&\le \int_0^\kappa s^{N-1} \frac{(1-s)^2}{(1-\kappa)^2} \lambda a(s) g(v(s)) ds \le \frac{1}{(1-\kappa)^2} \int_0^\kappa s^{N-1} (1-s)^2 \lambda a(s) g(v(s)) ds \\
&\le \bar{d} \int_0^\kappa s^{N-1} \lambda a(s) \psi(s) g(v(s)) ds = \bar{d} \int_B \lambda a \psi g(v) dx \le \bar{d} c_2 = e_2,
\end{aligned}$$

where $d, \bar{d}$ are some constants. These prove (3.31).

For proving (3.32), rewrite the identity (2.18) of lemma 2.3 and considering the fact that $\alpha + \beta = N - 4$, we have

$$\int_B \big(NF(u) - \alpha \lambda buf(u)\big) dx + \int_B \big(NG(v) - \beta \lambda avg(v)\big) dx = \int_{\partial B} (\Delta u, \Delta v)(x, v) d\sigma_x. \tag{3.33}$$

Using the hypothesis (H_2) in the left-hand side of (3.33) and after easy computation, we get

$$\theta_1 \int_B \lambda buf(u) dx + \theta_2 \int_B \lambda avg(v) dx \le eu''(1) v''(1), \tag{3.34}$$

where e is generic constant and θ_1 and θ_2 are constants given in hypothesis (H_2). Thus (3.34) can be written as

$$\theta_1 \int_B \lambda buf(u) dx + \theta_2 \int_B \lambda avg(v) dx \le e. \tag{3.35}$$

Since the two integrals of (3.35) are positive separately, hence we obtain (3.32) directly from (3.35).

Step 4. In this step using the hypothesis (H_1) and (H_2), we claim that there exist two positive constants c_1 and c_2 such that, for any solution (u, v) of our problem given by (1.1),

$$\|u\|_\infty \le c_1, \quad \|v\|_\infty \le c_2. \tag{3.36}$$

For u, we have

$$\begin{aligned}\|u\|_\infty &\le u(0) \le \int_0^1 G(0,s)\lambda a(s) g(v(s))ds \le c\int_0^1 s^3(1-s)^2 \lambda a(s) g(v(s))ds \\ &\le c\int_0^1 s^3 \lambda a(s) g(v(s))ds \le c\int_0^t s^3 \lambda a(s) g(v(s))ds + c\int_t^1 s^3 \lambda a(s) g(v(s))ds,\end{aligned} \tag{3.37}$$

where $t \in (0,1)$ is arbitrary and c is a positive constant whose value may vary.

Let $g(m) = \max_{s\in[0,m]} g(s)$ for $m \in (0,\infty)$, and applying Hölder's inequality in (3.37), we get

$$\begin{aligned}\|u\|_\infty &\le c\lambda a t^4 g(\|v\|_\infty) + c\lambda a\left(\int_t^1 s^{\gamma_1(q^*+1)}ds\right)^{\frac{1}{q^*+1}}\left(\int_t^1 s^{N-1} g(v(s))ds\right)^{\frac{q^*}{q^*+1}} \\ &\le c\lambda a t^4 g(\|v\|_\infty) + c\lambda a\left(\int_t^1 s^{\gamma_1(q^*+1)}ds\right)^{\frac{1}{q^*+1}}\left(\int_t^1 s^{N-1} g(v(s))\left(g(v(s))\right)^{\frac{1}{q^*}} ds\right)^{\frac{q^*}{q^*+1}},\end{aligned}$$

where $\gamma_1 = 3-(N-1)\dfrac{q^*}{q^*+1}$.

From remark 3.1, we have a positive constant D such that

$$\begin{aligned} &g(s) < D(1+s)^{q^*}, \text{ for all } s \ge 0, \\ &f(s) < D(1+s)^{p^*}, \text{ for all } s \ge 0. \end{aligned} \tag{3.38}$$

Hence, we get

$$\begin{aligned}\|u\|_\infty &\le c\lambda a t^4 g(\|v\|_\infty) + D^{\frac{1}{q^*}} c\lambda a\left(\int_t^1 s^{\gamma_1(q^*+1)}ds\right)^{\frac{1}{q^*+1}}\left(\int_t^1 s^{N-1} g(v(s))(1+v(s))ds\right)^{\frac{q^*}{q^*+1}} \\ &\le c\lambda a t^4 g(\|v\|_\infty) + D^{\frac{1}{q^*}} c\lambda a\left(\int_t^1 s^{\gamma_1(q^*+1)}ds\right)^{\frac{1}{q^*+1}}\left(\int_B g(v)dx + \int_B g(v)v(x)dx\right)^{\frac{q^*}{q^*+1}}.\end{aligned} \tag{3.39}$$

Using (3.31) and (3.32) in (3.39), we get

$$\|u\|_\infty \le c\lambda a t^4 g(\|v\|_\infty) + c\lambda a\left(\int_t^1 s^{\gamma_1(q^*+1)}ds\right)^{\frac{1}{q^*+1}}. \tag{3.40}$$

Similarly, for v, we get

$$\|v\|_\infty \le c\lambda b t^4 f(\|u\|_\infty) + c\lambda b\left(\int_t^1 s^{\gamma_2(p^*+1)}ds\right)^{\frac{1}{p^*+1}}, \tag{3.41}$$

where $\gamma_2 = 3-(N-1)\dfrac{p^*}{p^*+1}$.

We consider that in all the next inequalities c will always represent a positive constant.

Now, after some easy computations from (3.40) and (3.41), we get

$$\|u\|_\infty \le c\lambda a t^4 g(\|v\|_\infty) + c\lambda a t^{\frac{4+(4-N)q^*}{q^*+1}}, \tag{3.42}$$

$$\|v\|_\infty \le c\lambda b t^4 f\left(\|u\|_\infty\right) + c\lambda b t^{\frac{4+(4-N)p^*}{p^*+1}}. \tag{3.43}$$

Note that if f and g are bounded then (3.36) comes directly from (3.42) and (3.43). But if g is not bounded then, by remark 3.1, we have there exists a positive constant K, as like (3.38) such that $g(r) \le K(r)^{q^*}$, for all $r \ge 1$. Thus, we can write (3.42) as

$$\|u\|_\infty \le c\lambda a t^4 \left(\|v\|_\infty\right)^{q^*} + c\lambda a t^{\frac{4+(4-N)q^*}{q^*+1}}. \tag{3.44}$$

Now, inserting (3.43) in (3.44), and using the inequality $(a+b)^n \le c_n\left(a^n + b^n\right)$ for $a, b, n \ge 0$, where c_n is a positive constant depending on n, we yield

$$\|u\|_\infty \le c\lambda a t^{4\left(q^*+1\right)} \left(f\left(\|u\|_\infty\right)\right)^{q^*} + c\lambda a t^{\frac{\left[4+(4-N)p^*\right]q^*}{p^*+1}+4} + c\lambda a t^{\frac{4+(4-N)q^*}{q^*+1}}. \tag{3.45}$$

But it is easy to show that

$$\frac{\left[4+(4-N)p^*\right]q^*}{p^*+1} + 4 = \frac{4+(4-N)q^*}{q^*+1} = -\frac{N}{p^*+1} = 4 - \frac{Nq^*}{q^*+1}. \tag{3.46}$$

Hence from (3.45), we have

$$\|u\|_\infty \le c\lambda a t^{4\left(q^*+1\right)} \left(f\left(\|u\|_\infty\right)\right)^{q^*} + c\lambda a t^{4-\frac{Nq^*}{q^*+1}} \tag{3.47}$$

Let $r = 4\left(q^*+1\right)$. Since $t \in (0,1)$, then from (3.47) we have

$$\|u\|_\infty \le c\lambda a t^r \left(f\left(\|u\|_\infty\right)\right)^{q^*} + c\lambda a t^{4-\frac{Nq^*}{q^*+1}}.$$

Now, if we let $h(t) = t^r \left(f\left(\|u\|_\infty\right)\right)^{q^*} + t^{4-\frac{Nq^*}{q^*+1}}$. Then the function h gives its minimum value at

$$t_0 = c\lambda a\left(f\left(\|u\|_\infty\right)\right)^{-\frac{q^*\left(q^*+1\right)}{(r-4)\left(q^*+1\right)+Nq^*}},$$

and we have

$$\|u\|_\infty \le c\lambda a\left(f\left(\|u\|_\infty\right)\right)^{\frac{-q^*\left(q^*+1\right)r}{(r-4)\left(q^*+1\right)+Nq^*}+q^*} + c\lambda a\left(f\left(\|u\|_\infty\right)\right)^{\frac{-q^*\left(q^*+1\right)}{(r-4)\left(q^*+1\right)+Nq^*}\left(4-\frac{Nq^*}{q^*+1}\right)}. \tag{3.48}$$

But it is easy to show that

$$\frac{-q^*\left(q^*+1\right)r}{(r-4)\left(q^*+1\right)+Nq^*} + q^* = \frac{-q^*\left(q^*+1\right)}{(r-4)\left(q^*+1\right)+Nq^*}\left(4 - \frac{Nq^*}{q^*+1}\right) = \frac{1}{p^*}.$$

Hence from (3.48), we get

$$\|u\|_\infty \le c\lambda a\left(f\left(\|u\|_\infty\right)\right)^{\frac{1}{p^*}}. \tag{3.49}$$

On the other hand, from remark 3.1, we have $f(x) = o\left(x^{p^*}\right)$ for $x \to \infty$, hence the inequality (3.49) becomes

$$\|u\|_\infty \le c\lambda a\left(1 + o\left(\|u\|_\infty\right)\right),$$

which proves that $\|u\|_\infty$ is bounded. Therefore, by (3.43) we have $\|v\|_\infty$ is bounded.

This completes the proof of this step.

Theorem 3.3: If the nonlinearities f and g satisfies the following conditions:

$$NF(t) - \alpha tf(t) \le 0 \text{ and } NG(t) - \beta tg(t) \le 0, \text{ for } t > 0,$$

where α and β are positive real numbers satisfying $\alpha + \beta = N - 4$, then the problem given by (1.1) has no any nontrivial solution $(u, v) \in C^4\left(\overline{B}\right) \times C^4\left(\overline{B}\right)$.

Proof. If we put $\alpha + \beta = N - 4$ in the identity (2.18), then by remark 2.1 and the boundary conditions of (1.1), we obtain that

$$(\Delta u, \Delta v) = \frac{\partial^2 u}{\partial v^2}\frac{\partial^2 v}{\partial v^2}.$$

Now, if $(u, v) \in C^4\left(\overline{B}\right) \times C^4\left(\overline{B}\right)$ is a nontrivial solution of (1.1) and since B is star-shaped domain about 0, then we have $u, v \ge 0$ on ∂B. In this case if we apply the given conditions $NF(t) - \alpha tf(t) \le 0$ and $NG(t) - \beta tg(t) \le 0,$ for $t > 0$, then the identity (2.18) raised a contradiction. Hence the problem given by (1.1) may not have any nontrivial solution $(u, v) \in C^4\left(\overline{B}\right) \times C^4\left(\overline{B}\right)$.

This completes the proof.

Now, we discuss two illustrative examples to verify our main results.

Example 3.1: Let B denotes the unit ball in $\mathbb{R}^n$. Consider the weight functions a and b are defined as

$$a(x) = |x|^2 \text{ and } b(x) = |x|^2, \text{ for all } x \in B,$$

and the nonlinear functions f and g are defined as

$$f(x) = x^3 \text{ and } g(x) = x^2, \text{ for all } x \in [0, \infty).$$

Then there exists a positive parameter λ such the following nonlinear weighted bi-harmonic system of elliptic PDEs

$$\begin{cases} \Delta^2 u = \lambda |x|^2 v^2, \ v > 0 \text{ in } B, \\ \Delta^2 v = \lambda |x|^2 u^3, \ u > 0 \text{ in } B, \\ u = 0 = v, \ \dfrac{\partial u}{\partial \nu} = 0 = \dfrac{\partial v}{\partial \nu} \text{ on } \partial B, \end{cases} \tag{3.50}$$

has a positive solution.

Proof. From the definition of f and g, it is clear that the hypothesis (H_1) holds. If we fix the dimension of the space $N = 5$, then the corresponding problem for radially symmetric solution is as follows:

$$\begin{cases} u^{(4)} + \dfrac{2(5-1)}{r}u''' + \dfrac{(5-1)(5-3)}{r^2}u'' - \dfrac{(5-1)(5-3)}{r^3}u' = \lambda|x|^2 v^2, \text{ for } r \in (0,1), \\ v^{(4)} + \dfrac{2(5-1)}{r}v''' + \dfrac{(5-1)(5-3)}{r^2}v'' - \dfrac{(5-1)(5-3)}{r^3}v' = \lambda|x|^2 u^3, \text{ for } r \in (0,1), \\ u'(0) = 0 = v'(0),\ u'''(0) = 0 = v'''(0),\ u'(1) = 0 = v'(1),\ u(1) = 0 = v(1). \end{cases} \tag{3.51}$$

Then any solution $(u(r), v(r)) \in C^4(0,1) \times C^4(0,1)$ of (3.51) is a radial symmetric solution of (3.50).

Now, it easy to verify that for a suitable choice of values of $\theta_1, \theta_2, \alpha$ and β, the hypothesis (H_2) and (H_3) are holds. Since, all the hypothesis of theorem 3.1 are satisfied, then an application of theorem 3.1 gives a positive solution of the problem given by (3.50).

Example 3.2: Let B denotes the unit ball in $\mathbb{R}^n$. Consider the weight functions a and b same as to example 3.1 and the nonlinear functions f and g are defined as

$$f(x) = x^3 + x^2 + x \text{ and } g(x) = x^2 + x, \text{ for all } x \in [0, \infty).$$

Then there exists a positive parameter λ such the following nonlinear weighted bi-harmonic system of elliptic PDEs

$$\begin{cases} \Delta^2 u = \lambda|x|^2 v^2 + v,\ v > 0 \text{ in } B, \\ \Delta^2 v = \lambda|x|^2 u^3 + u^2 + u,\ u > 0 \text{ in } B, \\ u = 0 = v,\ \dfrac{\partial u}{\partial \nu} = 0 = \dfrac{\partial v}{\partial \nu} \text{ on } \partial B, \end{cases} \tag{3.52}$$

has a positive solution.

Proof. From the definition of f and g, it is clear that the hypothesis (H_1) holds. If we fix the dimension of the space $N = 5$, then the corresponding problem for radially symmetric solution is as follows:

$$\begin{cases} u^{(4)} + \dfrac{2(5-1)}{r}u''' + \dfrac{(5-1)(5-3)}{r^2}u'' - \dfrac{(5-1)(5-3)}{r^3}u' = \lambda|x|^2 v^2 + v, \text{ for } r \in (0,1), \\ v^{(4)} + \dfrac{2(5-1)}{r}v''' + \dfrac{(5-1)(5-3)}{r^2}v'' - \dfrac{(5-1)(5-3)}{r^3}v' = \lambda|x|^2 u^3 + u^2 + u, \text{ for } r \in (0,1), \\ u'(0) = 0 = v'(0),\ u'''(0) = 0 = v'''(0),\ u'(1) = 0 = v'(1),\ u(1) = 0 = v(1). \end{cases} \tag{3.53}$$

Then any solution $(u(r), v(r)) \in C^4(0,1) \times C^4(0,1)$ of (3.53) is a radial symmetric solution of (3.52).

Now, it easy to verify that for a suitable choice of values of $\theta_1, \theta_2, \alpha$ and β, the hypothesis (H_2) and (H_3) are holds. Since, all the hypothesis of theorem 3.1 are satisfied, then an application of theorem 3.1 gives a positive solution of the problem given by (3.52).

4. Conclusion

In this study, we have proven the existence and non-existence criteria for the solution of nonlinear weighted bi-harmonic system of elliptic PDEs given by (1.1) applying a fixed-point theorem due to Figueiredo *et al.* [15]

and Peletier and van der Vorst [25]. A priori solution estimates of the considered problem have been established which is proved in theorem 3.2. Using theorem 3.1, one can easily be checked the existence of positive solution of nonlinear weighted bi-harmonic system of elliptic PDEs given by (1.1) and theorem 3.3 ensure the case of non-existence of solutions of that same system. The established results provide an easy and straightforward technique to cheek the existence and non-existence of solution to nonlinear weighted bi-harmonic system of elliptic PDEs given by (1.1). Furthermore, the results of this research extend the corresponding results of Soltani and Yazidi [28] and Dwivedi [13].

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REFERENCES

[1] Alves, C.O., de Morais Filho, D.C., Souto, M.A.S., (2000). On systems of elliptic equations involving subcritical or critical Sobolev exponents, Nonlinear Analysis 42(5), p. 771-787.
[2] Aydi, H., Nashine, K., Samet, B., Yazidi, H., (2011). Coincidence and common fixed-point results of Choudhury-Metiya type in partially ordered cone metric, Nonlinear Analysis: Theory Methods & Applications 74(18), p. 6814-6825.
[3] Andersson, L.E., Elfving, T., Golub, G.H., (1998). Solution of biharmonic equations with application to radar imaging, Journal of Computational and Applied Mathematics 94(2), p. 153-180.
[4] Boccardo, L., Figueiredo, D.G.D., (2002). Some remarks on a system of quasilinear elliptic equations, Nonlinear Differential Equations and Applications 9, p. 309-323.
[5] Bloor, M.I.G., Wilson, M.J., (2000). Method for efficient shape parametrization of fluid membranes and vesicles, Physical Review E 61(4), p. 4218-4229.
[6] Bloor, M.I.G., Wilson, M.J., (2006). An approximate analytic solution method for the biharmonic problem, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science 462(2068), p. 1107-1121.
[7] Brown, K.J., Stavrakakis, N., (1996). Global bifurcation results for a semilinear elliptic equation on all of R^n, Duke Mathematical Journal 85(1), p. 77-94.
[8] Brown, K.J., Tertikas, A., (1991). On the bifurcation of radially symmetric steady-state solutions arising in population genetics, SIAM Journal on Mathematical Analysis 22(2), p. 400-413.
[9] Clement, Ph., Figueiredo, D.G.D., Mitidieri, E., (1992). Positive solutions of semilinear elliptic systems, Communications in Partial Differential Equations 17, p. 923-940.
[10] Cosner, C., (1984). Positive solutions for superlinear elliptic systems without variational structure, Nonlinear Analysis: Theory Methods & Applications 8, p. 1427-1436.
[11] Cac, N.P., Fink, A.M., Gatica, J.A., (1995). Nonnegative solutions of the radial Laplacian with nonlinearity that changes sign, Proceedings of the American Mathematical Society 123(5), p. 1393-1398.
[12] Chen, R., (2011). Existence of positive solutions for semilinear elliptic systems with indefinite weight, Electronic Journal of Differential Equations 164, p. 1-8.
[13] Dwivedi, G., (2014). Existence of Solution for Biharmonic Systems with Indefinite Weights, Differential Equations and Applications 6(4), p. 495-516.
[14] Dalmasso, R., (2000). Existence and uniqueness of positive solutions of semilinear elliptic systems, Nonlinear Analysis: Theory Methods & Applications 39(5), p. 559-568.
[15] de Figueiredo, D.G., Lions, P.L., Nussbaum, R.D., (1982). A Priori estimates and existence of positive solutions of semilinear elliptic equations, Journal de Mathématiques Pures et Appliquées 61, p. 41-63.
[16] Gazzola, F., Sweers, G., (2008). On positivity for the biharmonic operator under Steklov boundary conditions, Archive for Rational Mechanics and Analysis 188, p. 339-427.

[17] Gazzola, F., Grunau, H.C., Sweers, G., (2010). Polyharmonic boundary value problems, Lecture notes in Mathematics, Springer-Verlag, Berlin, Heidelberg.

[18] Hadiji, R., Yazidi, H., (2014). A nonlinear general Neumann problem involving two critical exponents, Asymptotic Analysis 89, p. 123-150.

[19] Hai, D.D., (1998). Positive solutions to a class of elliptic boundary value problems, Journal of Mathematical Analysis and Applications 227(1), p. 195-199.

[20] Lazer, A.C., Mckenna, P.J., (1990). Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis, Siam Review 32(4), p. 537-578.

[21] Lions, P.L., (1982). On the existence of positive solutions of semilinear elliptic equations, SIAM review, 24(4), p. 441-467.

[22] Nussbaum, R.D., (1975). Positive solutions of nonlinear elliptic boundary value problems, Journal of Mathematical Analysis and Applications 51, p. 461-482.

[23] Oswald, P., (1985). On a priori estimates for positive solutions of a semilinear biharmonic equation in a Ball, Commentationes Mathematicae Universitatis Carolinae 26(3), p. 565-577.

[24] Pelesko, J.A., Bernstein, D.H., (2002). Modeling Mems and Nems, CRC press, Boca Raton.

[25] Peletier, L.A., van der Vorst, R.C.A.M., (1982). Existence and non-existence of positive solutions of non-linear elliptic systems and The Biharmonic equation, Differential and Integral Equations, 5(4), p. 441-467.

[26] Pucci, P., Serrin, J., (1986). A general variational identity, Indiana University Mathematics Journal 35, p. 681-703.

[27] Resat, H., Petzold, L., Pettigrew, M.F., (2009). Kinetic modeling of biological system, Methods in Molecular Biology 541(14), p. 311-335.

[28] Soltani, F., Yazidi, H., (2016). Existence of Solutions for a Non-Variational System of Elliptic PDE's via Topological Methods, Electronic Journal of Differential Equations 309, p. 1-13.

[29] Sevant, N.E., Bloor, M.I.G., Wilson, M.J., (2000). Aerodynamic design of a flying wing using response surface methodology, Journal of Aircraft 379(4), p. 562-569.

[30] Smart, D.R., (1974). Fixed point theorems, Cambridge Tracts in Mathematics, No. 66, Cambridge University Press, London, UK.

[31] Timoshenko, S., Sergius, W.K., (1959). Theory of plates and shells, McGraw-hill, 2nd Edition, New York.

[32] Vicenţiu D. Rădulescu, (2008). Qualitative Analysis of Nonlinear Elliptic Partial Differential Equations: Monotonicity, Analytic, and Variational Methods, Contemporary Mathematics and Its Applications 6, Hindawi Publishing Corporation New York, NY 10022, USA.

[33] van der Vorst, R.C.A.M., (1990). Variational identities and applications to differential systems, Archive for Rational Mechanics and Analysis, 116, p. 375-398.

[34] Wazwaz, A.M., (2009). Partial Differential Equations and Solitary Waves Theory, Nonlinear Physical Science, Springer-Verlag, Berlin, Heidelberg.

[35] Zhai, C., Song, R., Han, Q., (2011). The existence and the uniqueness of symmetric positive solutions for a fourth-order boundary value problem, Computers and Mathematics with Applications 62, p. 2639-2647.