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Some Characterizations on Actions of Generalized Derivations AsHomomorphisms And Anti-Homomorphisms In SemiprimeGamma Rings

Sujoy Chakraborty*[a], andAkhil Chandra Paul[b]

[a]*Department of Mathematics, Shahjalal University of Science and Technology, Sylhet, Bangladesh*
[b]*Department of Mathematics, University of Rajshahi, Rajshahi-6205, Bangladesh*

ABSTRACT

Let I be a non-zero left ideal of a Γ-ring M satisfying the condition $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$and $\alpha, \beta \in \Gamma$. We show that M contains a non-trivial central ideal if M is semiprime which admits an appropriate non-zero derivation on I, and also that M is commutative if M is prime admitting a non-zero centralizing derivation on I. We next give some characterizations when non-zero generalized derivations act as homomorphisms and as anti-homomorphisms on some non-zero left or two-sided ideals of semiprime gamma rings, somewhere of prime gamma rings also, satisfying the above condition.

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1. Introduction

To begin with the definition of a Γ-ring, consider M and Γ as additive abelian groups. If there is a map $(a, \alpha, b) \to a\alpha b$ of $M \times \Gamma \times M \to M$ satisfying the conditions $(a + b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha + \beta)b = a\alpha b + a\beta b$, $a\alpha(b + c) = a\alpha b + a\alpha c$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, then M is called a Γ-ring. We now recall some useful definitions, identities and consequences in the theory of Γ-rings as follows.

The set $Z(M) = \{a \in M: a\alpha m = m\alpha a$ for all $\alpha \in \Gamma$and $m \in M\}$ is called the center of M. A Γ-ring M is said to be 2-torsion free if $2a = 0$with $a \in M$, then $a = 0$. A Γ-ring M is called commutative if $a\alpha b = b\alpha a$ holds for all $a, b \in M$and $\alpha \in \Gamma$. The symbol $[a, b]_\alpha$ stands for the commutator $a\alpha b - b\alpha a$ (for any $a, b \in M$and $\alpha \in \Gamma$). Two basic commutator identities are

$$[a\beta b, c]_\alpha = a\beta[b, c]_\alpha + a[\beta, \alpha]_c b + [a, c]_\alpha \beta b$$
$$\text{and } [a, b\beta c]_\alpha = b\beta[a, c]_\alpha + b[\beta, \alpha]_a c + [a, b]_\alpha \beta c,$$

* Corresponding author: Sujoy Chakraborty,*E-mail address: sujoy_chbty@yahoo.com*

where$[\alpha,\beta]_a = \alpha a\beta - \beta a\alpha$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. In this article, we'll consider the condition

$$(*)\ a\alpha b\beta c = a\beta b\alpha c$$

for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. With this condition (*), the above identities become

$$[a\beta b, c]_\alpha = a\beta[b, c]_\alpha + [a, c]_\alpha \beta b$$

$$\text{and } [a, b\beta c]_\alpha = b\beta[a, c]_\alpha + [a, b]_\alpha \beta c.$$

An element $x \in M$ called nilpotent if $(x\gamma)^n x = 0$, for all $\gamma \in \Gamma$, is satisfied for some positive integer n. An additive subgroup U of M is said to be a left (or, right) ideal of M if $M\Gamma U \subset U$ (or, $U\Gamma M \subset U$), whereas U is called a (two-sided) ideal of M if U is a left as well as a right ideal of M. A Γ-ring M is is said to be prime if, for any $a, b \in M$, $a\Gamma M\Gamma b = 0$ implies $a = 0$ or $b = 0$. And, a Γ-ring M is called semiprime if $a\Gamma M\Gamma a = 0$ with $a \in M$implies $a = 0$.

We now enunciate some well-known facts about prime and semiprimeΓ-rings: (i) a semiprime Γ-ring contains no non-zero central nilpotent elements; (ii) every annihilator ideal of a semiprime Γ-ring is invariant under all derivations of M; (iii) the centre of a non-zero ideal of a semiprime Γ-ring is contained in $Z(M)$.

An additive mapping $d: M \to M$ is called a derivation if $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$ for all $a, b \in M$and $\alpha \in \Gamma$. An additive mapping $f: M \to M$ is said to be a generalized derivation if there is a derivation $d: M \to M$ such that $f(a\alpha b) = f(a)\alpha b + a\alpha d(b)$ holds for all $a, b \in M$and $\alpha \in \Gamma$. Note that every derivation is a generalized derivation. If $d = 0$, then f is a left multiplier mapping of M. Thus, the concept of generalized derivation is the generalization of both the concepts of derivation and left multiplier mapping.

An additive mapping $\phi: M \to M$ is said to be a homomorphism if $\phi(a\alpha b) = \phi(a)\alpha\phi(b)$ for all $a, b \in M$and $\alpha \in \Gamma$. An additive mapping $\psi: M \to M$ is called an anti-homomorphism if $\psi(a\alpha b) = \psi(b)\alpha\psi(a)$ for all $a, b \in M$and $\alpha \in \Gamma$. Finally, a generalized derivation f of M is said to act as a homomorphism [resp. as an anti-homomorphism] on a subset S of M if $f(a\alpha b) = f(a)\alpha f(b)$ [resp. $f(a\alpha b) = f(b)\alpha f(a)$] for all $a, b \in S$ and $\alpha \in \Gamma$.

In [3], Bell and Martindale showed that a semiprime ring must have a non-trivial central ideal if it admits an appropriate derivation which is centralizing on some non-trivial one-sided ideal, and also prove the commutativity in case of prime rings under similar hypotheses. Our first goal is to establish the extension of these results in case of semiprime and prime gamma rings, respectively. Considering *I* as a non-zero left ideal of a Γ-ring M satisfying the condition $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$,we show here that M contains a non-trivial central ideal if M is semiprime which admits an appropriate non-zero derivation on I, and also that M is commutative if M is prime admitting a non-zero centralizing derivation on I.

B. Dhara proved some characterizations of generalized derivations which are acting as homomorphisms and anti-homomorphisms on some non-zero left ideals of semiprime rings in [5]. As an extensive work following this article, we next establish some analogous characterizations when non-zero generalized derivations are acting as homomorphisms and anti-homomorphisms on some non-zero left or two-sided ideals of semiprime gamma rings, somewhere of prime gamma rings also, satisfying the above mentioned condition.

2. Main Results

We proceed with the following lemmas.

Lemma 2.1 Let M be a prime Γ-ring and let $I \neq 0$ be an ideal of M satisfying the condition (*) such that $I \subseteq Z(M)$. Then M is commutative.

Proof. Let $m \in M$, $x \in I$ and $\alpha \in \Gamma$. Then we have $m\alpha x \in I$.
Since $I \subseteq Z(M)$, we get $[m\alpha x, s]_\beta = 0$ for all $s \in M$ and $\beta \in \Gamma$.
It follows that $0 = [m, s]_\beta \alpha x + m\alpha[x, s]_\beta = [m, s]_\beta \alpha x$.
Thus, $[m, s]_\beta \Gamma I = 0$ for all $m, s \in M$and $\beta \in \Gamma$.
As $M\Gamma I \subseteq I$, we obtain $[m, s]_\beta \Gamma M\Gamma I = 0$.
Since $I \neq 0$ and M is prime, we have $[m, s]_\beta = 0$ for all $m, s \in M$and $\beta \in \Gamma$.

This shows that M is commutative. □

Lemma 2.2 Let M be a 2-torsion free semiprime Γ-ring that satisfies (*) and let $I \neq 0$ be an ideal of M. If a in Mcentralizes $[I, I]_\Gamma$, then a centralizes I.

Proof. Let $a \in M$centralizes $[I, I]_\Gamma$. Then for all $b, c \in I$and $\alpha \in \Gamma$, we have

$$a\alpha[b, b\beta c]_\alpha = [b, b\beta c]_\alpha \alpha a,$$

which yields that

$$a\alpha b\beta[b, c]_\alpha + a\alpha[b, b]_\alpha \beta c = b\beta[b, c]_\alpha \alpha a + [b, b]_\alpha \beta c\alpha a;$$
$$\Rightarrow a\alpha b\beta[b, c]_\alpha = b\beta[b, c]_\alpha \alpha a.$$

This implies,

$$0 = a\alpha b\beta[b, c]_\alpha - b\beta[b, c]_\alpha \alpha a = [a, b\beta[b, c]_\alpha]_\alpha = [a, b]_\alpha \beta[b, c]_\alpha + b\beta[a, [b, c]_\alpha]_\alpha.$$

So, $[a, b]_\alpha \beta[b, c]_\alpha = 0$ for all $b, c \in I$and $\alpha, \beta \in \Gamma$.
Putting $c\delta a$for c, we obtain

$$0 = [a, b]_\alpha \beta[b, c\delta a]_\alpha = [a, b]_\alpha \beta c\delta[b, a]_\alpha + [a, b]_\alpha \beta[b, c]_\alpha \delta a = [a, b]_\alpha \beta c\delta[b, a]_\alpha.$$

Thus, $[a, b]_\alpha \beta c\delta[a, b]_\alpha = 0$ for all $b, c \in I$and $\alpha, \beta, \delta \in \Gamma$.
This gives that $[a, b]_\alpha \Gamma I\Gamma[a, b]_\alpha = 0$ for all $b \in I$and $\alpha \in \Gamma$.
Since I an ideal of M, it yields

$$[a, b]_\alpha \Gamma M\Gamma I\Gamma[a, b]_\alpha = 0 = [a, b]_\alpha \Gamma I\Gamma M\Gamma[a, b]_\alpha.$$

Hence we obtain

$$I\Gamma[a, b]_\alpha \Gamma M\Gamma I\Gamma[a, b]_\alpha = 0 = [a, b]_\alpha \Gamma I\Gamma M\Gamma[a, b]_\alpha \Gamma I.$$

Since M is semiprime, we have $I\Gamma[a, b]_\alpha = 0 = [a, b]_\alpha \Gamma I$.
Thus, $[[a, b]_\alpha, b]_\alpha = 0$ for all $b \in I$and $\alpha \in \Gamma$.
Now, we have

$$[[a, b]_\alpha, b]_\alpha = -[b, [b, a]_\alpha]_\alpha = 0 \text{for all } b \in I \text{ and } \alpha \in \Gamma.$$

Since $a \in M$, we replace a by $a\Gamma y$, for $y \in M$ and $\gamma \in \Gamma$, and we obtain

$$\begin{aligned} 0 &= [b, [b, a\Gamma y]_\alpha]_\alpha = [b, a\Gamma[b, y]_\alpha + [b, a]_\alpha \Gamma y]_\alpha \\ &= a\Gamma[b, [b, y]_\alpha]_\alpha + [b, a]_\alpha \Gamma[b, y]_\alpha + [b, [b, a]_\alpha]_\alpha \Gamma y + [b, a]_\alpha \Gamma[b, y]_\alpha \\ &= 2\,[b, a]_\alpha \Gamma[b, y]_\alpha. \end{aligned}$$

By the 2-torsion freeness of M, we obtain $[b, a]_\alpha \Gamma[b, y]_\alpha = 0$.
Putting $y\beta a$ in place of y, we have

$$0 = [b, a]_\alpha \Gamma[b, y\beta a]_\alpha = [b, a]_\alpha \Gamma y\beta[b, a]_\alpha + [b, a]_\alpha \Gamma[b, y]_\alpha \beta a = [b, a]_\alpha \Gamma y\beta[b, a]_\alpha.$$

It gives, $[b, a]_\alpha \Gamma M\Gamma[b, a]_\alpha = 0$ for all $b \in I$and $\alpha \in \Gamma$.
As M is semiprime, we get $[b, a]_\alpha = 0$.
This yields that $a$centralizes I. □

Lemma 2.3 Let $I \neq 0$ be a left ideal of a prime Γ-ring M and $d \neq 0$ a derivation of M. Then $d \neq 0$on I.

Proof. Suppose $d = 0$on I. Then, for any $a \in I$, $m \in M$ and $\alpha \in \Gamma$, it follows that

$$0 = d(a) = d(m\alpha a) = d(m)\alpha a + m\alpha d(a) = d(m)\alpha a.$$

Thus, we have $d(m)\alpha a = 0$ for all $a \in I$, $m \in M$ and $\alpha \in \Gamma$.
Replacing a by $s\beta a$ (for $s \in M$and $\beta \in \Gamma$) here, we obtain $d(m)\alpha s\beta a = 0$.
Therefore, $d(m)\Gamma M\Gamma I = 0$ for all $m \in M$.
The primeness of M shows that $d(m) = 0$ for all $m \in M$ (since $I \neq 0$), completing the proof. □

Lemma 2.4 Let $I \neq 0$ be a left ideal of a semiprime Γ-ring M that satisfies the condition (*) and let d be a derivation of M such that $[d(a), a]_\alpha \in Z(M)$ for all $a \in I$and $\alpha \in \Gamma$. Then $[d(a), a]_\alpha = 0$ for all $a \in I$and $\alpha \in \Gamma$.

Proof. For arbitrary $a \in I$, $a\beta a \in I$ for all $\beta \in \Gamma$.
Therefore, we have $[d(a\beta a), a\beta a]_\alpha \in Z(M)$.
This shows that

$$\begin{aligned} [d(a)\beta a + a\beta d(a), a\beta a]_\alpha &= [2a\beta d(a) - a\beta d(a) + d(a)\beta a, a\beta a]_\alpha \\ &= \left[2a\beta d(a) + [d(a), a]_\beta, a\beta a\right]_\alpha \\ &= [2a\beta d(a), a\beta a]_\alpha + \left[[d(a), a]_\beta, a\beta a\right]_\alpha \end{aligned}$$

$$= 2a\beta[d(a), a\beta a]_\alpha + 2\,[a, a\beta a]_\alpha \beta d(a)$$
$$= 2a\beta a\beta[d(a), a]_\alpha + 2a\beta[d(a), a]_\alpha \beta a$$
$$= 4a\beta a\beta[d(a), a]_\alpha \in Z(M).$$

Thus, for $\alpha = \beta$, we obtain $4[a\alpha a\alpha[d(a), a]_\alpha, d(a)]_\alpha = 0$,
which yields that $8[a\alpha[d(a), a]_\alpha \alpha[d(a), a]_\alpha = 0$.
Hence we get $8[d(a), a]_\alpha \alpha[d(a), a]_\alpha \alpha[d(a), a]_\alpha = 0$.
That is, $(2[d(a), a]_\alpha \alpha)^2 2[d(a), a]_\alpha = 0$.
Since a semiprimeΓ-ring contains no non-zero central nilpotent elements (as we know), for all $a \in I$and $\alpha \in \Gamma$, we have

$$2[d(a), a]_\alpha = 0 \ldots\ldots\ldots \quad (1)$$

For all $a \in I$and $\alpha \in \Gamma$, it follows that

$$2[d(a), a\alpha a]_\alpha = 0. \ldots\ldots\ldots \quad (2)$$

Putting $a + b$ for a in (1), and in the hypothesis $[d(a), a]_\alpha \in Z(M)$, we get

$$2[d(a), b]_\alpha + 2[d(b), a]_\alpha = 0 \ldots\ldots\ldots \quad (3)$$

and

$$[d(a), b]_\alpha + [d(b), a]_\alpha \in Z(M), \ldots\ldots\ldots \quad (4)$$

respectively, for all $a, b \in I$ and $\alpha \in \Gamma$.
By combining these two results with (4), we find, for all $a, b \in I$and $\alpha \in \Gamma$, that

$$[d(x), a\alpha b + b\alpha a]_\alpha + [d(y), x\alpha x]_\alpha = 0. \ldots\ldots\ldots \quad (5)$$

Replacement of b by $b\alpha a$ yields

$$(a\alpha b + b\alpha a)[d(a), a]_\alpha + ([d(x), a\alpha b + b\alpha a]_\alpha$$
$$+[d(b), a\alpha a]_\alpha)\alpha a + b\alpha[d(x), a\alpha a]_\alpha + [y, x\alpha x]_\alpha \alpha d(x) = 0,$$

for all $a, b \in I$ and $\alpha \in \Gamma$.
By using (1), (2) and (5), we obtain

$$[a, b]_\alpha \alpha[d(x), x]_\alpha + [y, x\alpha x]_\alpha \alpha d(x) = 0.$$

Taking $b = d(x)\alpha x$ and using (2), we conclude that

$$[d(a), a]_\alpha \alpha a\alpha[d(a), a]_\alpha = 0.$$

It gives that $([d(a), a]_\alpha \alpha)^2[d(a), a]_\alpha = 0$,
and hence, we get $[d(a), a]_\alpha = 0$ for all $a \in I$ and $\alpha \in \Gamma$. □

Now we prove our main results consecutively.

Theorem 2.1 (Following [3]) Let$I \neq 0$ be a left ideal of a semiprime Γ-ring M satisfying the condition (*). If M admits a derivation d which is non-zero on I such that $[d(a), a]_\alpha \in Z(M)$ for all $a \in I$and $\alpha \in \Gamma$, then M contains a non-zero central ideal.

Proof. By Lemma 2.4, for all $a \in I$and $\alpha \in \Gamma$, we have

$$[d(a), a]_\alpha = 0. \ldots\ldots\ldots \quad (6)$$

By linearizing (6), we obtain

$$[d(a), b]_\alpha + [d(b), a]_\alpha = 0 \ldots\ldots\ldots \quad (7)$$

for all $a, b \in I$ and $\alpha \in \Gamma$.
Replacing b by $b\beta a$ in (7), we have

$$0 = [d(a), b\beta a]_\alpha + [d(b\beta a), a]_\alpha$$
$$= b\beta[d(a), a]_\alpha + [d(a), b]_\alpha \beta a + [d(b)\beta a + b\beta d(a), a]_\alpha$$
$$= [d(a), b]_\alpha \beta a + [d(b), a]_\alpha \beta a + [b, a]_\alpha \beta d(a) + b\beta[d(a), a]_\alpha$$
$$= [d(a), b]_\alpha \beta a + [d(b), a]_\alpha \beta a + [b, a]_\alpha \beta d(a)$$
$$= ([d(a), b]_\alpha + [d(b), a]_\alpha)\beta a + [b, a]_\alpha \beta d(a).$$

By using (7) in the above relation, for all $a, b \in I$and $\alpha, \beta \in \Gamma$, we get

$$[b, a]_\alpha \beta d(a) = 0. \ldots\ldots\ldots \quad (8)$$

Now, putting $c\gamma b$for b, where $c \in I$ and $\gamma \in \Gamma$, we have

$$0 = [c\gamma b, a]_\alpha \beta d(a) = c\gamma[b, a]_\alpha \beta d(a) + [c, a]_\alpha \gamma b\beta d(a) = [c, a]_\alpha \gamma b\beta d(a). \ldots \quad (9)$$

Since $b \in I$ and I is a left ideal of M, so $M\Gamma I \subseteq I$.
Hence, from (9), we get

$$[c, a]_\alpha \Gamma M\Gamma I\Gamma d(I) = 0. \ldots\ldots\ldots \quad (10)$$

Let $\mathcal{P} = \{P_i : i \in \Lambda\}$ be a family of prime ideals with $\bigcap_{i\in\Lambda} P_i = 0$.

From (10), it follows that

(i) $[c, a]_\alpha \in P_i$, for all $c, a \in I$ and $\alpha \in \Gamma$, or (ii) $I\Gamma d(I) \subseteq P_i$.

Call P_i a type-one prime if it satisfies (i), a type-two prime otherwise; let P_1 and P_2 be respectively the intersections of all type-one and type-two primes; then $P_1 \cap P_2 = 0$.

We now investigate a typical type-two prime $P = P_i$.

From (ii) and the fact that $[d(a), a]_\alpha = 0$ for all $a \in I$and $\alpha \in \Gamma$, we have

$$d(a)\alpha a \in P \text{ and } a\alpha d(a) \in P \text{ for all } a \in I \text{ and } \alpha \in \Gamma.$$

Thus, $\big(d(a) + d(b)\big)\alpha(a + b) \in P$ and $(a + b)\alpha\big(d(a) + d(b)\big) \in P$ for all $a, b \in I$and $\alpha \in \Gamma$;

and consequently, $d(a)\alpha b + d(b)\alpha a \in P$ and $a\alpha d(b) + b\alpha d(a) \in P$.

Now we have $d(a\alpha b + b\alpha a) = d(a)\alpha b + a\alpha d(b) + d(b)\alpha a + b\alpha d(a) \in P$,

and so, for all $a, b \in I$ and $\alpha \in \Gamma$, we get

$$d(a\alpha b + b\alpha a) \in P. \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (11)$$

It follows that $d(c\beta(a\alpha b + b\alpha a) + (a\alpha b + b\alpha a)\beta c) \in P$ for all $a, b, c \in I$and $\alpha, \beta \in \Gamma$.

After calculation, we have

$$d(c)\beta(a\alpha b + b\alpha a) + c\beta d(a\alpha b + b\alpha a) + d(a\alpha b + b\alpha a)\beta c + (a\alpha b + b\alpha a)\beta d(c) \in P.$$

Noting that the last three summands are in P, by (ii) and (11), we have

$$d(c)\beta(a\alpha b + b\alpha a) \in P \text{ for all } a, b, c \in I \text{ and } \alpha, \beta \in \Gamma.$$

Putting $c\gamma a$for a, we find that $d(c)\beta(c\gamma a\alpha b + b\alpha c\gamma a) \in P$.

By using $d(c)\beta c \in P$, we get $d(c)\beta b\alpha c\gamma a \in P$ for all $a, b, c \in I$ and $\alpha, \beta \in \Gamma$.

Hence $d(c)\beta c\delta M\mu b\alpha c\gamma a \in P$ for all $a, b, c \in I$and $\alpha, \beta, \gamma, \delta, \mu \in \Gamma$.

Since P is a prime ideal, we have either $d(I) \subseteq P$or $I\Gamma I\Gamma I \subseteq P$.

But if $I\Gamma I\Gamma I \subseteq P$ holds, then we have $I \subseteq P$, and hence (i) holds for P, contradicting our definition of type-two prime; therefore, $d(I) \subseteq P$.

It now follows that, for $m \in M$, $a \in I$ and $\alpha \in \Gamma$,

$$d(m\alpha a) = d(m)\alpha a + m\alpha d(a) \text{ forces that } d(m)\alpha a = d(m\alpha a) - m\alpha d(a) \in P,$$

so that $M\Gamma d(M)\Gamma I \subseteq P$; and since $I \nsubseteq P$, we conclude that $M\Gamma d(M) \subseteq P$.

This being true for every type-two prime, we obtain

$$M\Gamma d(M) \subseteq P_2. \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (12)$$

Consider now the left ideal V generated by the set $d(M)\Gamma I$, we shall show that V is commutative, hence a two-sided central ideal.

A typical element of V is a sum of elements of the forms $d(m)\alpha a$and $s\beta d(m)\alpha a$, where $m, s \in M$, $a \in I$ and $\alpha, \beta \in \Gamma$.

Thus, we need only to show that the commutators of the forms

$$[d(m_1)\alpha a_1, d(m_2)\alpha a_2]_\gamma,$$
$$[s_1\beta d(m_1)\alpha a_1, d(m_2)\alpha a_2]_\gamma$$
$$\text{and} [s_1\beta d(m_1)\alpha a_1, s_2\beta d(m_2)\alpha a_2]_\gamma$$

are all trivial.

Clearly, all three types are in P_1 by (i), and they are all in P_2 by (12); so they all belong to $P_1 \cap P_2 = 0$.

If $V \neq 0$, our proof is finished.

Assume, therefore, that $V = 0$.

In this case, $d(M)\Gamma I = 0$.

Since $I\Gamma d(M)\Gamma I\Gamma d(M) = 0$, the left ideal $I\Gamma d(M)$ is nilpotent, so $I\Gamma d(M) = 0$.

Thus, $a\alpha d(m\beta s) = 0$ for all $a \in I$, $m, s \in M$ and $\alpha, \beta \in \Gamma$, so that $a\alpha d(m)\beta s + a\alpha m\beta d(s) = 0$,

and hence $a\alpha m\beta d(s) = 0$, since $a\alpha d(m) = 0$.

Thus, we find that

$$I\Gamma M\Gamma d(M) = 0. \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (13)$$

In particular, for each $a \in I$, $x \in M$ and $\alpha, \beta \in \Gamma$, we have $a\alpha x\beta d(a) = 0$,

and hence

$$0 = d(a\alpha x\beta d(a)) = d(a)\alpha x\beta d(a) + a\alpha d(x\beta d(a))$$
$$= d(a)\alpha x\beta d(a) + a\alpha d(x)\beta d(a) + a\alpha x\beta d^2(a).$$

Since $a\alpha d(x)\beta d(a) = 0$ and $a\alpha x\beta d^2(a) = 0$ by (13), we find

$$d(a)\alpha x\beta d(a) = 0 \text{ for all } a \in I,\ x \in M \text{ and } \alpha, \beta \in \Gamma.$$

This implies that $d(I)\Gamma M\Gamma d(I) = 0$.

Since M is semiprime, we find that $d(I) = 0$.

This contradicts our initial hypothesis, so the central ideal V is not zero. □

Theorem 2.2 (Following [3]) Let M be a prime Γ-ring satisfying the condition (*) and I a non-zero left ideal of M. If M admits a non-zero derivation which is centralizing on I, then M is commutative.

Proof. By Theorem 2.1, we find that M has a non-zero central ideal. By Lemma 2.1, we see that M is commutative. □

Theorem 2.3 Let M be a semiprime Γ-ring satisfying the condition (*), $I \neq 0$ a left ideal of M and f a non-zero generalized derivation of M with an associated derivation d of M. If $f(a\alpha b) = f(a)\alpha f(b)$ for all $a, b \in I$ and $\alpha \in \Gamma$, then $I\Gamma d(I) = 0$, $f(I) = f(I)\Gamma I$ and $I\Gamma[a, f(a)]_\beta = 0$ for all $a \in I$ and $\beta \in \Gamma$.

Proof. For all $a, b, c \in I$ and $\alpha, \beta \in \Gamma$, we have

$$f(a\alpha b\beta c) = f\big(a\alpha(b\beta c)\big) = f(a)\alpha f(b\beta c) = f(a)\alpha f(b)\beta c + f(a)\alpha b\beta d(c). \ldots\ldots \quad (14)$$

On the other hand, for all $a, b, c \in I$ and $\alpha, \beta \in \Gamma$, we also have

$$f\big((a\alpha b)\beta c\big) = f(a\alpha b)\beta c + a\alpha b\beta d(c) = f(a)\alpha f(b)\beta c + a\alpha b\beta d(c). \ldots \quad (15)$$

From (14) and (15), for all $a, b, c \in I$ and $\alpha, \beta \in \Gamma$, we obtain

$$(f(a) - a)\alpha b\beta d(c) = 0. \ldots \quad (16)$$

For each $m \in M$, $b \in I$ and $\gamma \in \Gamma$, $m\gamma b \in I$.
Replacing b by $m\gamma b$ in (16), we get $(f(a) - a)\alpha m\Gamma b\beta d(c) = 0$ for all $a, b, c \in I$, $m \in M$ and $\alpha, \beta, \gamma \in \Gamma$.
This yields that

$$f(b)\beta(f(a) - a)\alpha M\Gamma b\beta d(c) = 0 \ldots \quad (17)$$

for all $a, b, c \in I$ and $\alpha, \beta, \gamma \in \Gamma$.
Again, we have

$$f(b\beta a) = f(b)\beta a + b\beta d(a) = f(b)\beta f(a),$$

which implies that

$$f(b)\beta(f(a) - a) = b\beta d(a) \ldots \quad (18)$$

for all $a, b \in I$ and $\beta \in \Gamma$.
In view of (18), we can write from (17) that

$$b\beta d(a)\alpha M\Gamma b\beta d(a) = 0 \ldots \quad (19)$$

for all $a, b \in I$ and $\alpha, \beta, \gamma \in \Gamma$.
Also, we have

$$f(b)\beta(f(a) - a)\alpha M\Gamma f(b)\beta(f(a) - a) = 0 \ldots \quad (20)$$

for all $a, b \in I$ and $\alpha, \beta, \gamma \in \Gamma$.
By semiprimeness of M in (19), we find that $b\beta d(a) = 0$ for all $a, b \in I$ and $\beta \in \Gamma$.
That is, $I\Gamma d(I) = 0$.
Applying the same in (20), we see that

$$f(b)\beta(f(a) - a) = 0 \ldots \quad (21)$$

for all $a, b \in I$ and $\beta \in \Gamma$.
Now, $I\Gamma d(I) = 0$ implies,

$$f(a\alpha b) = f(a)\alpha b + a\alpha d(b) = f(a)\alpha b \text{ for all } a, b \in I \text{ and } \alpha \in \Gamma.$$

This shows that $f(I) = f(I)\Gamma I$.
Next, (21) implies that

$$f(b\alpha a)\beta(f(a) - a) = 0 \text{ and } f(b)\beta(f(a\alpha a) - a\alpha a) = 0$$

for all $a, b \in I$ and $\alpha, \beta \in \Gamma$.
Since $I\Gamma d(I) = 0$, these two identities respectively yield

$$0 = f(b)\alpha a\beta(f(a) - a) + b\alpha d(a)\beta(f(a) - a) = f(b)\alpha a\beta(f(a) - a) \ldots \quad (22)$$

and

$$0 = f(b)\beta(f(a)\alpha a + a\alpha d(a) - a\alpha a) = f(b)\beta(f(a)\alpha a - a\alpha a) \ldots \quad (23)$$

for all $a, b \in I$ and $\alpha, \beta \in \Gamma$.
Using the condition (*) in (23), and subtracting (23) from (22), we get

$$0 = f(b)\alpha[a, f(a) - a]_\beta = f(b)\alpha[a, f(a)]_\beta.$$

That is, $f(b)\alpha[a, f(a)]_\beta = 0$ for all $a, b \in I$ and $\alpha, \beta \in \Gamma$.
Substituting $b\gamma c$ for b (with $c \in I$ and $\gamma \in \Gamma$), we have

$$0 = f(b\gamma c)\alpha[a, f(a)]_\beta = f(b)\gamma c\alpha[a, f(a)]_\beta + b\gamma d(c)\alpha[a, f(a)]_\beta.$$

In view of $I\Gamma d(I) = 0$, the above relation becomes

$$f(b)\gamma c\alpha[a, f(a)]_\beta = 0 \text{ for all } a, b, c \in I \text{ and } \alpha, \beta, \gamma \in \Gamma.$$

Since I is left ideal of M, we find that

$$[b,f(b)]_\beta \Gamma M\delta c\alpha[a,f(a)]_\beta = 0 \text{for all } a,b,c \in I \text{ and } \alpha,\beta,\gamma,\delta \in \Gamma.$$

It follows that

$$I\Gamma[a,f(a)]_\beta \Gamma M\Gamma I\Gamma[a,f(a)]_\beta = 0 \text{for all } a,b \in I \text{ and } \beta \in \Gamma.$$

By semiprimenessof M, we get $I\Gamma[a,f(a)]_\beta = 0$ for all $a \in I$ and $\beta \in \Gamma$. □

Theorem 2.4 Let $I \neq 0$ be an ideal of a semiprime Γ-ring M that satisfies the condition (*) and let $f \neq 0$ be a generalized derivation of M with an associated derivation $d$of M. If $f(a\alpha b) = f(a)\alpha f(b)$ for all $a,b \in I$and $\alpha \in \Gamma$, then $d(I) = 0$ and $[a,f(a)]_\beta = 0$ for all $a \in I$ and $\beta \in \Gamma$. **In particular**, if M is a prime Γ-ring, then $d = 0$ and f is an identity mapping of M.

Proof. In view of Theorem 2.3, we see that $I\Gamma d(I) = 0$, $f(I) = f(I)\Gamma I$ and $I\Gamma[a,f(a)]_\beta = 0$ for all $a \in I$ and $\beta \in \Gamma$. Here, $I\Gamma d(I) = 0$ yields

$$0 = I\Gamma d(M\Gamma I) = I\Gamma(d(M)\Gamma I + M\Gamma d(I))$$
$$= I\Gamma d(M)\Gamma I + I\Gamma M\Gamma d(I) = I\Gamma d(M)\Gamma I, \text{ since } I\Gamma M = M\Gamma I.$$

It gives $I\Gamma d(M)\Gamma I = 0$, and hence we obtain

$$I\Gamma d(M)\Gamma M\Gamma I\Gamma d(M) = 0 \text{and } d(M)\Gamma I\Gamma M\Gamma d(M)\Gamma I = 0.$$

As M is semiprime, $I\Gamma d(M) = 0 = d(M)\Gamma I$.
Then $I \subseteq Ann(d(M))$, where $(d(M))$ denotes the ideal generated by $d(M)$ and $Ann(d(M))$ denotes the left or right annihilator of $(d(M))$.
Since every annihilator ideal of a semiprimeΓ-ring is invariant under all derivations of M (we know), we have

$$d(I) \subseteq d\left(Ann\big(d(M)\big)\right) \subseteq d(M) \cap Ann(d(M)) = 0, \text{ as } M \text{ is semiprime.}$$

Thus, we have $d(I) = 0$.
Now, let us assume that $I\Gamma[a,f(a)]_\beta = 0$ for all $a \in I$and $\beta \in \Gamma$.
Since $[a,f(a)]_\beta \Gamma M \subseteq I$ for all $a \in I$and $\beta \in \Gamma$, we have $[a,f(a)]_\beta \Gamma M\Gamma[a,f(a)]_\beta = 0$.
The semiprimeness of M forces that $[a,f(a)]_\beta = 0$ for all $a \in I$and $\beta \in \Gamma$.
Next, if M is prime, $d(I) = 0$ implies that $0 = d(M\Gamma I) = d(M)\Gamma I + M\Gamma d(I) = d(M)\Gamma I$.
Since I is an ideal of M, $M\Gamma I \subseteq I$,
and so $d(M)\Gamma M\Gamma I \subseteq d(M)\Gamma I = 0$.
Since $I \neq 0$ and M is prime, we find that $d(M) = 0$.
Now, from our assumption, we have $f(a)\alpha b = f(a)\alpha f(b)$ for all $a,b \in I$and $\alpha \in \Gamma$.
It follows that

$$f(a)\alpha(b - f(b)) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (24)$$

for all $a,b \in I$ and $\alpha \in \Gamma$.
Replacing a by $a\beta m$ (where $m \in M$and $\beta \in \Gamma$) in (24), we obtain

$$0 = f(a\beta m)\alpha(b - f(b)) = \big(f(a)\beta m + a\beta d(m)\big)\alpha(b - f(b))$$
$$= f(a)\beta m\alpha(b - f(b)), \text{ since } d(m) = 0 \text{ for all } m \in M.$$

Thus, we have $f(a)\Gamma M\Gamma(b - f(b)) = 0$.
Since M is prime, $f(a) = 0$ for all $a \in I$ or $b - f(b) = 0$ for all $b \in I$.
If $f(a) = 0$ for all $a \in I$, then we get

$$0 = f(m\alpha s\beta a) = f(m)\alpha s\beta a + m\alpha d(s\beta a) = f(m)\alpha s\beta a,$$

for all $m,s \in M$, $a \in I$ and $\alpha,\beta \in \Gamma$.
Therefore, we have $f(m)\Gamma M\Gamma I = 0$ for all $m \in M$, and since $I \neq 0$.
In view of the primeness of M forces that $f(m) = 0$ for all $m \in M$.
But, it is a contradiction to the fact that $\neq 0$.
So, we have $b - f(b) = 0$ for all $b \in I$.
Then putting $m\alpha b$ in place of b (for all $m \in M$and $\alpha \in \Gamma$), we get

$$0 = m\alpha b - f(m)\alpha b - m\alpha d(b) = m\alpha b - f(m)\alpha b = \big(m - f(m)\big)\alpha b.$$

Thus, we have $\big(m - f(m)\big)\Gamma I = 0$ for all $m \in M$.
Since $M\Gamma I \subseteq I$, we obtain $\big(m - f(m)\big)\Gamma M\Gamma I \subseteq \big(m - f(m)\big)\Gamma I = 0$.
By the primenessof M, it shows that $f(m) = m$ for all $m \in M$. □

Theorem 2.5 Let M be a semiprime Γ-ring satisfying the condition (*), $I \neq 0$ a left ideal of M and $f \neq 0$ a generalized derivation of M with an associated derivation $d$of M. If $f(a\alpha b) = f(b)\alpha f(a)$ for all $a, b \in I$and $\alpha \in \Gamma$, then $[I, I]_\Gamma \Gamma d(I) = 0$.

Proof. For all $a, b \in I$and $\alpha \in \Gamma$,we have

$$f(a\alpha b) = f(b)\alpha f(a). \dots \quad (25)$$

Substituting $a\beta b$ for a (where $b \in I$and $\beta \in \Gamma$) in (25), we obtain

$$f(a\beta b\alpha b) = f(b)\alpha f(a\beta b). \dots \quad (26)$$

By the condition (*), we get $f(a\alpha b\beta b) = f(b)\alpha f(a\beta b)$.
This implies, $f(a\alpha b)\beta b + a\alpha b\beta d(b) = f(b)\alpha(f(a)\beta b + a\beta d(b))$,
and hence, $\big(f(a\alpha b) - f(b)\alpha f(a)\big)\beta b + a\alpha b\beta d(b) = f(b)\alpha a\beta d(b)$.
By using (25), we obtain $a\alpha b\beta d(b) = f(b)\alpha a\beta d(b)$.
That is,

$$(a\alpha b - f(b)\alpha a)\beta d(b) = 0 \dots \quad (27)$$

for all $a, b \in I$ and $\alpha, \beta \in \Gamma$.
Replacing a by $f(c)\gamma a$ (for $c \in I$and $\gamma \in \Gamma$) in (27), we have

$$(f(c)\gamma a\alpha b - f(b)\alpha f(c)\gamma a)\beta d(b) = 0. \dots \quad (28)$$

From (27), we now get

$$(f(c)\gamma a\alpha b - f(c)\gamma f(b)\alpha a)\beta d(b) = 0. \dots \quad (29)$$

Comparing (28) and (29), we obtain

$$f(b)\alpha f(c)\gamma a\beta d(b) - f(c)\gamma f(b)\alpha a\beta d(b) = 0.$$

Using the condition (*), we find $f(b)\alpha f(c)\gamma a\beta d(b) - f(c)\alpha f(b)\gamma a\beta d(b) = 0$,
which gives by (25) that

$$f(b\alpha c)\gamma a\beta d(b) - f(c\alpha b)\gamma a\beta d(b) = 0. \dots \quad (30)$$

That is, for all $a, b, c \in I$and $\alpha, \beta, \gamma \in \Gamma$, we have

$$f([b, c]_\alpha)\gamma a\beta d(b) = 0. \dots \quad (31)$$

Substituting $c\beta b$ for c in (31), we obtain

$$\begin{aligned} 0 &= f([b, c\beta b]_\alpha)\gamma a\beta d(b) \\ &= f(c\beta[b, b]_\alpha + [b, c]_\alpha\beta b)\gamma a\beta d(b) \\ &= f([b, c]_\alpha\beta b)\gamma a\beta d(b) \\ &= f([b, c]_\alpha)\beta b\gamma a\beta d(b) + [b, c]_\alpha\beta d(b)\gamma a\beta d(b) \\ &= [b, c]_\alpha\beta d(b)\gamma a\beta d(b) \end{aligned}$$

(since $b\gamma a \in I$, and so by (31), $f([b, c]_\alpha)\beta b\gamma a\beta d(b) = 0$).
Thus, for all $a, b, c \in I$and $\alpha, \beta, \gamma \in \Gamma$, we have

$$[b, c]_\alpha\beta d(b)\gamma a\beta d(b) = 0. \dots \quad (32)$$

Since, for all $m \in M$, $m\mu[b, c]_\alpha \in I$, we replace a by $m\mu[b, c]_\alpha$ in (32) and obtain

$$[b, c]_\alpha\beta d(b)\gamma m\mu[b, c]_\alpha\beta d(b) = 0 \dots \quad (33)$$

for all $m \in M$.
As M is semiprime, we get

$$[b, c]_\alpha\beta d(b) = 0 \dots \quad (34)$$

for all $b, c \in I$ and $\alpha, \beta \in \Gamma$.
Replacing c by $m\gamma a$ (for any $m \in M$) in (34), and using (34), we have

$$0 = [b, m\gamma a]_\alpha\beta d(b) = m\gamma[b, a]_\alpha\beta d(b) + [b, m]_\alpha\gamma a\beta d(b) = [b, m]_\alpha\gamma a\beta d(b).$$

Thus, for all $m \in M$, $a, b \in I$ and $\alpha, \beta, \gamma \in \Gamma$, we get

$$[b, m]_\alpha\gamma a\beta d(b) = 0. \dots \quad (35)$$

Since I is a left ideal of M, for all $b \in I$ and $\alpha \in \Gamma$, we obtain

$$[b, M]_\alpha\Gamma M\Gamma I\Gamma d(b) = 0. \dots \quad (36)$$

As M is semiprime, it must contain a family $\mathcal{P} = \{ P_\alpha : \alpha \in \Lambda\}$ of prime ideals such that $\bigcap_{\alpha\in\Lambda} P_\alpha = 0$.
If P is a typical member of $\mathcal{P}$and $b \in I$, (36) shows that $[b, M]_\alpha \subseteq P$ or $I\Gamma d(b) \subseteq P$.
For fixed P, let $A = \{b \in I : [b, M]_\alpha \subseteq P\}$ and $B = \{b \in I : I\Gamma d(b) \subseteq P\}$.
Then it is clear that A and B are additive subgroups of I such that $A \cup B = I$.
Since a group cannot be the set theoretic union of two of its proper subgroups, therefore $A = I$or $B = I$.
If $A = I$, then $[I, M]_\Gamma \subseteq P$. If $B = I$, then $I\Gamma d(I) \subseteq P$.
Together of these two implies that $[I, I]_\Gamma\Gamma d(I) \subseteq P$ for any $P \in \mathcal{P}$.
Therefore, it follows that $[I, I]_\Gamma\Gamma d(I) \subseteq \bigcap_{\alpha\in\Lambda} P_\alpha = 0$; that is, $[I, I]_\Gamma\Gamma d(I) = 0$. □

Theorem 2.6 Let M be a semiprime Γ-ring that satisfies the condition (*), let $I \neq 0$ be an ideal of M, and let $f \neq 0$ be a generalized derivation of M with an associated derivation d of M. If $f(a\alpha b) = f(b)\alpha f(a)$ for all $a, b \in I$and $\alpha \in \Gamma$, then $d(I) = 0$ or M contains a non-zero central ideal. **In particular**, if M is a prime Γ-ring, then M is commutative and f is a left multiplier mapping of M.

Proof. In view of Theorem 2.5, we conclude $[I, I]_\Gamma \Gamma d(I) = 0$. This gives
$$0 = [M\Gamma I, I]_\Gamma \Gamma d(I) = M\Gamma[I, I]_\Gamma \Gamma d(I) + [M, I]_\Gamma \Gamma I \Gamma d(I) = [M, I]_\Gamma \Gamma I \Gamma d(I).$$
Thus, we have
$$0 = [M, M\Gamma I]_\Gamma \Gamma I \Gamma d(I) = M\Gamma[M, I]_\Gamma \Gamma I \Gamma d(I) + [M, M]_\Gamma \Gamma I \Gamma I \Gamma d(I) = [M, M]_\Gamma \Gamma I \Gamma I \Gamma d(I).$$
Since I is an ideal of M, it follows that
$$[M, M]_\Gamma \Gamma I \Gamma d(I) \Gamma M \Gamma [M, M]_\Gamma \Gamma I \Gamma d(I) = 0.$$
As M is semiprime, $[M, M]_\Gamma \Gamma I \Gamma d(I) = 0$.
This yields, $[M, M]_\Gamma \Gamma I \Gamma [d(I), I]_\Gamma = 0$.
In particular, we have $[d(I), I]_\Gamma \Gamma I \Gamma [d(I), I]_\Gamma = 0$.
With the fact that I is an ideal of M, we get $[d(I), I]_\Gamma \Gamma M \Gamma I \Gamma [d(I), I]_\Gamma = 0$,
and so, we obtain $(I\Gamma[d(I), I]_\Gamma)\Gamma M \Gamma (I\Gamma[d(I), I]_\Gamma) = 0$.
Since M is semiprime, $I\Gamma[d(I), I]_\Gamma = 0$.
Again, since I is an ideal of $M$and $[d(I), I]_\Gamma \Gamma M = \big(d(I)\Gamma I - I\Gamma d(I)\big)\Gamma M \subseteq I$, we find
$$[d(I), I]_\Gamma \Gamma M \Gamma [d(I), I]_\Gamma \subseteq I\Gamma[d(I), I]_\Gamma = 0.$$
By the semiprimenessof M, it follows that $[d(I), I]_\Gamma = 0$.
In view of Theorem 2.1, M contains a non-zero central ideal if $d(I) \neq 0$.
In particular, if M is prime, then by Theorem 2.2, $[d(I), I]_\Gamma = 0$ implies that $d = 0$ or M is commutative.
If $d \neq 0$, then M is commutative, and so, f acts as a homomorphism on I.
In view of Theorem 2.4, we get the contradiction that $d = 0$.
Hence $d = 0$.
So, for all $a, b \in M$and $\alpha \in \Gamma$, $f(a\alpha b) = f(a)\alpha b$;
that is, f is a left multiplier mapping of M.
Then, by assumption, we get
$$f(a)\alpha b = f(b)\alpha f(a) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (37)$$
for all $a, b \in I$ and $\alpha \in \Gamma$.
Now, replacing a by $a\beta m$ (for $m \in M$and $\beta \in \Gamma$) in (37), we find that $f(a\beta m)\alpha b = f(b)\alpha f(a\beta m)$,
and consequently, we obtain $f(a)\beta m\alpha b = f(b)\alpha f(a)\beta m$, since $d = 0$.
By using (37), it follows that $f(a)\beta m\alpha b = f(a)\alpha b\beta m$,
which then gives $f(a)\alpha m\beta b = f(a)\alpha b\beta m$, by (*).
As a result, we have $f(a)\alpha[m, b]_\beta = 0$.
Putting $a\gamma t$ in place of a (for $t \in M$and $\gamma \in \Gamma$) here, we find that
$$f(a)\gamma t\alpha[m, b]_\beta = 0 \text{for all } a, b \in I, m, t \in M \text{ and } \alpha, \beta, \gamma \in \Gamma \text{ (since } d = 0).$$
By the primenessof M, we get $f(a) = 0$ (for all $a \in I$) or $[m, b]_\beta = 0$ (for all $b \in I$, $m \in M$ and $\beta \in \Gamma$).
If $[m, b]_\beta = 0$ for all $b \in I$, $m \in M$ and $\beta \in \Gamma$, then $I \subseteq Z(M)$.
Then, by Lemma 2.1, M is commutative.
If $f(a) = 0$ for all $a \in I$, then $f = 0$, a contradiction.
This completes the proof. □

Theorem 2.7 Let M be a semiprime Γ-ring that satisfies the condition (*) with centre $Z(M) \neq 0$, let $I \neq 0$ be a left ideal of M, and let $f \neq 0$ be a generalized derivation of M with an associated derivation d of M. If $f(a\alpha b) \pm f(a)\alpha f(b) \in Z(M)$ holds for all $a, b \in I$and $\alpha \in \Gamma$, then $I\Gamma d\big(Z(M)\big) \subseteq Z(M)$.

Proof. Let us first consider that
$$f(a\alpha b) + f(a)\alpha f(b) \in Z(M) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (38)$$
holds for all $a, b \in I$ and $\alpha \in \Gamma$.
As $Z(M) \neq 0$, we choose $0 \neq c \in Z(M)$. Then $b\beta c \in I$ for any $b \in I$and $\beta \in \Gamma$.
Substituting $b\beta c$ for b in (38), we obtain
$$\begin{aligned} &f(a\alpha b\beta c) + f(a)\alpha f(b\beta c) \\ &= f(a\alpha b)\beta c + a\alpha b\beta d(c) + f(a)\alpha f(b)\beta c + f(a)\alpha b\beta d(c) \\ &= \big(f(a\alpha b) + f(a)\alpha f(b)\big)\beta c + a\alpha b\beta d(c) + f(a)\alpha b\beta d(c) \in Z(M). \end{aligned}$$

By (38), we then have

$$a\alpha b\beta d(c) + f(a)\alpha b\beta d(c) \in Z(M). \dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (39)$$

Putting $a\gamma c$ for a in (39), we get $a\gamma c\alpha b\beta d(c) + f(a\gamma c)\alpha b\beta d(c) \in Z(M)$.
This implies that

$$\begin{aligned} &a\gamma b\alpha d(c)\beta c + \big(f(a)\gamma c + a\,\gamma d(c)\big)\alpha b\beta d(c) \\ &= a\gamma b\alpha d(c)\beta c + f(a)\gamma c\alpha b\beta d(c) + a\gamma d(c)\alpha b\beta d(c) \\ &= a\gamma b\alpha d(c)\beta c + f(a)\gamma b\alpha d(c)\beta c + a\gamma d(c)\alpha b\beta d(c) \\ &= \big(a\gamma b\alpha d(c) + f(a)\gamma b\alpha d(c)\big)\beta c + a\gamma d(c)\alpha b\beta d(c) \in Z(M). \end{aligned}$$

Hence, by (39), we obtain $a\gamma d(c)\alpha b\beta d(c) \in Z(M)$.
Thus, for all $m \in M$, $a, b \in I$ and $\alpha, \beta, \gamma, \delta \in \Gamma$, we get $[a\gamma d(c)\alpha b\beta d(c), m]_\delta = 0$.
Since $c \in Z(M)$, we obtain $[a\gamma b\alpha d(c)\beta d(c), m]_\delta = 0$.
As $I \neq 0$ is a left ideal of M, we may replace a by $s\mu a$ (for $s \in M$ and $\mu \in \Gamma$) here, and we then get

$$\begin{aligned} 0 &= [s\mu a\gamma b\alpha d(c)\beta d(c), m]_\delta \\ &= s\mu[a\gamma b\alpha d(c)\beta d(c), m]_\delta + [s, m]_\delta \mu a\gamma b\alpha d(c)\beta d(c) \\ &= [s, m]_\delta \mu a\gamma b\alpha d(c)\beta d(c). \end{aligned}$$

Since I is a left ideal of M, $M\nu I \subseteq I$ for all $\nu \in \Gamma$, and so

$$[s, m]_\delta \mu M\nu a\gamma b\alpha d(c)\beta d(c) = 0. \dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (40)$$

In particular, for any $a, b \in I$and $\alpha, \beta, \gamma, \mu \in \Gamma$, we have

$$[a, b]_\gamma \mu d(c)\alpha M\beta [a, b]_\gamma \mu d(c) = 0.$$

By the semiprimenessof M, it follows that $[a, b]_\gamma \mu d(c) = 0$.
That is, we obtain $[I, I]_\Gamma \Gamma d(c) = 0$.
Now, for every $c \in Z(M)$, we have

$$[I, I\Gamma d(c)]_\Gamma = [I, I]_\Gamma \Gamma d(c) + I\Gamma [I, d(c)]_\Gamma = 0.$$

Thus, we get $I\Gamma d\big(Z(M)\big) \subseteq Z(I) \subseteq Z(M)$, since we know that the centre of a non-zero ideal of a semiprime Γ-ring is contained in $Z(M)$. □

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