Numerical Approximations of a Nonlinear Volatility Model with European Options

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ABSTRACT

Black-Scholes model plays a very significant role in the world of quantitative finance. In this paper, the focus are on both nonlinear and linear Black-Scholes (BS) equations with numerical approximations. We aim to find an effective numerical approximations for Black-Scholes model. Several models from the most relevant class of nonlinear Black-Scholes equations with European option are analyzed in this study. The problem is approached by transforming the problem into a convection-diffusion equation and later it is approximated with the help of finite difference method (Crank-Nicolson). The result of finite difference schemes (Crank-Nicolson) for several volatility models are presented, including the Risk Adjusted Pricing Methodology (RAPM), Leland's model and the Barles'-Soner's Model. At the same time, it is attempted to illustrate a comparison of different volatility models. In the case of linear Black-Scholes model, we approximate the model with finite difference method (FDM) and finite element method (FEM) and compare the results. All the numerical schemes are implemented in MATLAB and corresponding graphs are also presented here.

1.   Introduction

Black-Scholes equation is the most preferable model in the field of option pricing as it can calculate the price of an option more correctly [1, 2, 4, 17, 18]. A large number of researchers and investors have shown their interests in the study of Black-Scholes model in these days as it is one of the most important tools to compute the value of an option [1, 31]. Fisher Black, Myron Scholes and Robert Merton, these three great economists [31] contributed to build-up the model in 1970 [1, 2, 4, 19]. It is possible to banish risks that does exist in markets through Black-Scholes model that can create a risk-less hedging portfolio [18, 20, 21].

Option is a kind of contract that gives the owner the right to buy or sell the asset for a specified price within a

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specified time [2, 3, 9, 11, 31]. Option is a part of financial derivative. Option is a way to minimize the loss in case there is any uncertainty. Options are hugely applied for hedging [11]. The fixed price is called the strike price or exercise price and it is denoted by E [1, 4, 11, 31]. The fixed time is called the expiration time or the maturity time and it is denoted by T [1, 4, 11, 31]. Basically call option and put option are exercised in real market [11]. Call option gives the owner the right to buy the underlying real asset [2, 4, 5, 31] whereas put option gives the owner the right to sell the underlying real asset [2, 4, 5].

The most commonly applied options are European option and American option [2, 6, 11, 31]. There are some other types of options, for example, Bermudan options, Asian options, Barrier options, etc [11]. To work on option pricing it is enough to have idea on European option, American option and Asian option.

European option can be exercised only at the expiration time T [5, 7, 8, 9, 31]. Lower premium cost is one of the advantages of European option [13, 14]. Index options are applicable in European option and investors get the chance to close the option before the expiration date [14]. On the other hand, American option can be exercised at any time until the expiration time T [2, 5, 9, 31]. American option is more flexible [14]. Stocks that are optionable follow the American Option [15]. To be able to exercise the option before the expiry time is considered as a huge advantage of American option which can be a helpful tool to maximize the profit [16]. We consider the average price of the underlying asset in Asian option [6, 10]. Asian options possess less volatility [10, 11]. Asian call or put options are calculated in two different forms, one is arithmetic average form and another one is geometric average form [11, 12].

More frequently, Black-Scholes model is exercised to evaluate European option, American option and Asian option. In this paper we consider European option for simplicity. Finite Difference Method (FDM) and Galerkin's Finite Element Method (FEM) are two very significant schemes in financial engineering [5, 17, 20, 24] and these schemes are applied for solving the linear case. The Crank-Nicolson method is applied for solving the nonlinear case.

Nonlinear model is considered as a model with transaction cost whereas linear model does not consider transaction cost [23, 52]. In linear model, we restrict ourselves by considering some assumptions which can sometimes become unreal [2, 23, 53]. On the other hand, in nonlinear model we have the opportunity to relax some of the restrictions for which the model becomes more realistic [2, 23]. For example, in nonlinear model, we consider volatility is non-constant due to transaction cost [2, 23, 42, 45, 52]. That's why considering all the circumstances we are giving priority to nonlinear BS model over the linearized version.

To approximate the nonlinear Black-Scholes model and to analyze the parameter dynamics of the model are both challenging. A few articles can be found where the researchers approximate the nonlinear BS model with a volatility corrector as a variable. Keeping this fact in mind this article focuses on to propose a few schemes both for linear and nonlinear Black-Scholes model so that researchers can use it to analyze such complicated models. To be specific, finite difference method and finite element method have been employed to approximate BS model as these two schemes are very easy and powerful schemes in financial engineering. The main objective of this paper is to study FDM and FEM for approximating linear Black-Scholes model and to analyze different volatility models with finite difference scheme.

In section 2, there is a discussion on Black-Scholes model including both nonlinear and linear version. In section 3, we describe the solving methodology of nonlinear Black-Scholes equation. Next we move on to solving the linear version with the help of FDM and FEM methods in section 4. We have done stability and error analysis of Black-Scholes model in section 5. Different types of numerical results of nonlinear model are discussed in section 6. In section 7, we present a numerical discussion on the linear BS model and do a comparison between FDM and FEM scheme. In section 8, we summarize our total observations through the conclusion.

2. The Model Description

Black-Scholes model creates an adjustment between stocks and options [22]. The Black-Scholes equation is one kind of parabolic equation [17, 31] and at the same time it depends on two independent variables, first is the time and the second is the stock price which follows a random path [1, 2]. Basically Black-Scholes model equation represents a partial differential equation (PDE) [2, 7, 9, 20, 23, 24] and it is very important to note that if we wish to solve a PDE, then we may get infinite number of solutions [9]. That's why to get the unique solution, we impose some boundary conditions in BS model [2, 7, 9].

It is possible to derive Black-Scholes equation from Brownian motion by using Ito's lemma [1, 2, 4, 9, 25]. We assume some facts and conditions while we are interested to calculate a fair value of an option by using the Black-Scholes pricing model [1, 9, 25].

Black-Scholes model is divided into two types, linear and nonlinear [2, 23]. In both linear and nonlinear Black-Scholes equation, some assumptions are same [9, 24].

2.1 Preliminaries of nonlinear Black-Scholes model

Nonlinear Black-Scholes equation deals with the transaction costs, volatility, market liquidity, uncertain risks and so on [2, 9, 23, 31]. In nonlinear Black-Scholes model, the model equation is reduced to a nonlinear parabolic type equation [2, 9, 44]. Sometimes the solution of the nonlinear Black-Scholes model shows asymptotic behavior [9, 21]. The solution of the nonlinear Black-Scholes model equation can converge even for large market frictions [2, 9, 21].

In nonlinear BS model, we assume that drift μ is constant and volatility σ is non-constant [2, 23, 42, 45] which is defined as $\tilde{\sigma}^2 = \tilde{\sigma}^2(t, S, V_S, V_{SS})$ [23, 43, 52, 53]. We want to hedge the position such that there are no risks [9, 52]. There are some assumptions on nonlinear Black-Scholes model [2, 9, 23, 43, 44]. Because of transaction costs, the drift μ and volatility σ depend on some other parameters, such as stock price S, time t, derivatives of the option price V etc [42, 45, 52]. That means the nonlinearity arises as an output of transaction costs [2, 9, 44, 45, 52]. The nonlinear Black-Scholes equation is [23, 42, 43, 45, 52, 53]

$$V_t + r S V_S + \frac{1}{2}\tilde{\sigma}^2(t, S, V_S, V_{SS})S^2 V_{SS} - rV = 0 \quad ; 0 \le S < \infty, \ 0 \le t \le T. \tag{2.1}$$

The terminal and boundary conditions can be defined in the following way [23, 52, 53].

European Call Option

Terminal condition is $\qquad V(S,T) = \max(S - E, 0) \ ; 0 \le S < \infty$,

Boundary condition is
$$V(0,t) = 0 \ ; 0 \le t \le T$$
$$V(S,t) = S - Ee^{-r(T-t)} \ ; S \to \infty.$$

European Put Option

Terminal condition is $\qquad V(S,T) = \max(E - S, 0) \ ; 0 \le S < \infty$,

Boundary condition is
$$V(0,t) = Ee^{-r(T-t)} \ ; 0 \le t \le T$$
$$V(S,t) = 0 \ ; S \to \infty.$$

As the analytical solution of nonlinear BS model (transaction cost models) is unknown to us [23], we have tried to approximate the numerical solution of different volatility models. In this case, we solve nonlinear BS

equation with FDM (Crank-Nicolson) [23].

We have studied some transaction cost models with different volatilities. We have considered three types of volatility models [2, 23, 52].

- **Leland's Model**

In Leland's model, the volatility of the nonlinear Black-Scholes equation is defined as [2, 9, 23, 43, 52]

$$\tilde{\sigma}^2 = \sigma^2 \left(1 + Le \ sign(V_{SS})\right) \tag{2.2}$$

where the symbols have the following meaning [2, 9, 23, 42, 43],

$\kappa =$ Round trip transaction cost for per unit dollar of transaction,
$\sigma =$ Original volatility,
$\partial t =$ Transaction frequency,
$Le = \sqrt{\dfrac{2}{\pi}} \dfrac{\kappa}{\sigma\sqrt{\partial t}} =$ Leland's number.

- **Barles' and Soner's Model**

The volatility of Barles' and Soner's model is defined as [2, 9, 23, 43, 46, 52]

$$\tilde{\sigma}^2 = \sigma^2 \left(1 + \Psi\left(e^{r(T-t)} a^2 S^2 V_{SS}\right)\right). \tag{2.3}$$

Here, $a = \dfrac{\kappa}{\sqrt{\varepsilon}}$ and $\Psi(x)$ is the solution for the following ordinary differential equation [9, 43, 46]

$$\Psi'(x) = \dfrac{\Psi(x) + 1}{2\sqrt{x\Psi(x) - x}}, \ \ x \neq 0.$$

We have, initial condition $\Psi(0) = 0, \lim\limits_{x\to\infty} \dfrac{\Psi(x)}{x} = 1, \lim\limits_{x\to-\infty} \Psi(x) = -1$.

For identity $\Psi(x) = x$, then the volatility of (2.3) becomes [9, 23, 43, 46]

$$\tilde{\sigma}^2 = \sigma^2 \left(1 + e^{r(T-t)} a^2 S^2 V_{SS}\right). \tag{2.4}$$

- **Risk Adjusted Pricing Methodology (RAPM) Model**

The volatility of this model is [2, 9, 23, 43, 52, 53]

$$\tilde{\sigma}^2 = \sigma^2 \left(1 + 3\left(\dfrac{C^2 M}{2\pi} S V_{SS}\right)^{\frac{1}{3}}\right). \tag{2.5}$$

Here, $M \geq 0$ and $C \geq 0$ denotes the measure of transaction cost and risk premium respectively [2, 9, 23, 43].

2.2 Introduction to Linear Black-Scholes Model

We have also considered the linearized version of BS model. If σ is constant, then nonlinear BS model will become linear BS model. When we say Black-Scholes model equation, we mainly mean to say linear Black-

Scholes equation. The famous and well-known linear Black-Scholes (BS) partial differential equation is [2, 5, 7, 9, 20, 23, 26, 31]

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 ,\tag{2.6}$$

where the symbols have their usual meaning. In linear BS equation, we assume that μ, r, σ are constants [2]. The solution of equation (2.1) and (2.6) are defined on the domain $0 \le S < \infty$ and $0 \le t \le T$ [7, 5, 25, 26]. We are considering European option.

If we want to find the solution of linear Black-Scholes equation, we need to provide boundary and terminal conditions [2, 9, 24, 25, 31].

European call option

We have the following conditions [1, 2, 5, 7, 26, 27, 31]

$$V(S,T) = \max(S - E, 0) = (S - E)^+ \quad \text{for } 0 < S < \infty$$
$$V(0,t) = 0 \quad \text{for } t \in [0,T]\tag{2.7}$$
$$V(S,t) = S - Ee^{-r(T-t)} \quad \text{for } S \to \infty.$$

European put option

The conditions are as follows [1, 2, 5, 7, 27]

$$V(S,T) = \max(E - S, 0) = (E - S)^+ \quad \text{for } 0 < S < \infty$$
$$V(0,t) = Ee^{-r(T-t)} \quad \text{for } t \in [0,T]\tag{2.8}$$
$$V(S,t) = 0 \quad \text{for } S \to \infty.$$

The analytical solution of linear BS equation can be obtained with the help of Fourier transformation from heat equation [7, 28, 29]. The solution that we get is unique [1]. The theoretical solution of BS model are as follows [2, 7, 30, 31]

$$V(S,t) = S\,N(d_1) - E\,e^{-r(T-t)}N(d_2) \quad ;[\text{For Call Option}]$$

$$V(S,t) = E\,e^{-r(T-t)}N(-d_2) - S\,N(-d_1) \quad ;[\text{For Put Option}].$$

Here $N(d)$ is the standard normal distribution of the function d .

$$d_1 = \frac{\ln\left(\frac{S}{E}\right) + (T-t)\left(r + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-t}}.$$

$$d_2 = \frac{\ln\left(\frac{S}{E}\right) + (T-t)\left(r - \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}.$$

3. A Simple Scheme for Nonlinear BS Equation

The working procedure to find the numerical solution of nonlinear BS equation is similar as linear BS equation [2]. We transfer the nonlinear Black-Scholes equation of the European call option into a convection-diffusion equation through the following transformation of variables [23, 53]

$$x = \ln\left(\frac{S}{E}\right), \tau = \frac{1}{2}\sigma^2(T-t), V(S,t) = E\,e^x\,u(x,\tau) \ . \tag{3.1}$$

Then volatility for Leland's model becomes [23]

$$\tilde{\sigma}^2 = \sigma^2\left(1 + Le\ sign(u_{xx} + u_x)\right).$$

Volatility of Barles' and Soner's model is [23]

$$\tilde{\sigma}^2 = \sigma^2\left(1 + \Psi\left(e^{\frac{2r\tau}{\sigma^2}}\,a^2\,E\,e^x\left(u_{xx} + u_x\right)\right)\right).$$

Volatility of RAPM model is [23, 53]

$$\tilde{\sigma}^2 = \sigma^2\left(1 + 3\left(\frac{C^2 M}{2\pi}\left(u_{xx} + u_x\right)\right)^{\frac{1}{3}}\right).$$

The convection-diffusion equation is [23, 53]

$$u_\tau - D u_x - \frac{\tilde{\sigma}^2}{\sigma^2}\left(u_{xx} + u_x\right) = 0 \ , \tag{3.2}$$

where $D = \dfrac{2r}{\sigma^2}$.

We solve convection-diffusion equation for the transformed mesh $-\infty < x < \infty$, $0 \le \tau \le \tilde{T}$ [23, 53]. The initial and boundary conditions are defined in the following manner [23, 53].

Initial condition is $u(x,0) = \max(1 - e^{-x}, 0)$; $x \in \mathrm{P}$.
Boundary condition is

$$u(x,\tau) = 0 \ ; x \to -\infty$$
$$u(x,\tau) = 1 - e^{-D\tau - x} \ ; x \to \infty.$$

We assume the followings for volatility models [23], for example,

$$u_{xx}(x_i, \tau_n) = \frac{U_{i+2}^n - 2U_i^n + U_{i-2}^n}{4\Delta x^2} \ ,$$

$$z_i^n = 3\left(\frac{C^2 M}{2\pi}\left(u_{xx}(x_i, \tau_n) + \frac{U_{i+1}^n - U_{i-1}^n}{2\Delta x}\right)\right)^{\frac{1}{3}} \ ; [\text{RAPM model}].$$

We can convert the convection-diffusion equation in the following system of equations [23]

$$P^n U^{n+1} = Q^n U^n + W^n \ , \tag{3.3}$$

where

$$U^n = \begin{pmatrix} U_{-N+1}^n 0 \\ \vdots \\ U_0^n \\ \vdots \\ U_{N-1}^n \end{pmatrix}, W^n = \begin{pmatrix} q_{-1}U_{-N}^n - p_{-1}U_{-N}^{n+1} \\ 0 \\ \vdots \\ 0 \\ q_1 U_N^n - p_1 U_N^{n+1} \end{pmatrix}, P^n = \begin{pmatrix} p_0 & p_1 & 0 & \cdots & 0 \\ p_{-1} & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & p_1 \\ 0 & \cdots & 0 & p_{-1} & p_0 \end{pmatrix}, Q^n = \begin{pmatrix} q_0 & q_1 & 0 & \cdots & 0 \\ q_{-1} & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & q_1 \\ 0 & \cdots & 0 & q_{-1} & q_0 \end{pmatrix}$$

and

$$r = \frac{\Delta \tau}{\Delta x^2}, \mu = \frac{\Delta \tau}{\Delta x}, \lambda = -(1+D),$$

$$p_0 = 1 + r(1+z_i^n), q_0 = 1 - r(1+z_i^n), p_{-1} = z_i^n\left(-\frac{r}{2}+\frac{\mu}{4}\right)-\frac{r}{2}-\frac{\lambda\mu}{4}, q_{-1} = z_i^n\left(\frac{r}{2}-\frac{\mu}{4}\right)+\frac{r}{2}+\frac{\lambda\mu}{4},$$

$$p_1 = z_i^n\left(-\frac{r}{2}-\frac{\mu}{4}\right)-\frac{r}{2}+\frac{\lambda\mu}{4}, q_1 = z_i^n\left(\frac{r}{2}+\frac{\mu}{4}\right)+\frac{r}{2}-\frac{\lambda\mu}{4}.$$

Equation (3.3) is executed iteratively for the uniform grid points (x_i, τ_n). We apply Crank-Nicolson method as this method is unconditionally stable.

Domain Discretization

We have the domain as: $-\infty < x < \infty, 0 \leq \tau \leq \tilde{T} = \frac{\sigma^2 T}{2}$. Grid points are taken as: $x_i = i\Delta x, \tau_n = n\Delta \tau$

where Δx is the step size for space and $\Delta \tau$ is the step size for time. $N = \frac{R}{\Delta x}, i = -N : N$.

4. Methods Applied to Solve Linear BS Model

As heat equation can provide us well-posed solution [17, 26], the Black-Scholes equation is reduced to a heat equation [5, 7, 17, 25, 26, 31]. The heat equation is a key to financial engineering [17]. It is an extreme material of Black-Scholes (BS) equation [17]. The boundary conditions of the heat equation can also be applied for BS equation.

To reduce BS equation to a heat equation, we apply the following change of variables [1, 5, 7, 26]

$$\left.\begin{array}{c} t = T - \dfrac{\tau}{\dfrac{1}{2}\sigma^2} \\ S = Ee^x \\ V(S,t) = Ev(x,\tau) \\ \dfrac{2r}{\sigma^2} = k \\ v(x,\tau) = e^{\alpha x + \beta \tau} \\ \alpha = -\dfrac{(k-1)}{2} \\ \beta = -\dfrac{(k+1)^2}{4} \end{array}\right\}. \tag{4.1}$$

The one dimensional heat equation is [5, 7, 17, 26]

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}. \tag{4.2}$$

The heat equation is defined for the following domain [5, 7, 26]

$$\left\{(x,\tau): -\infty < x < \infty, 0 \leq \tau \leq \frac{\sigma^2 T}{2}\right\}.$$

Equation (4.2) is the one-dimensional heat equation and we can compare it to the temperature distribution $u(x,\tau)$ along a rod or slab of length l for time τ [17, 36, 39, 40]. The heat-flow equation (4.2) is also a parabolic partial differential equation of second order [37, 39, 40].

Once we solve the heat equation, then we return to the solution of BS equation with the help of backward substitution [1, 5, 7, 23, 25, 26]. We mainly solve heat equation by different numerical schemes. We consider [-1, 1] spatial domain [7].

4.1 A Finite Difference Scheme of the Model

Finite difference method (FDM) is widely applied in quantitative finance [17, 20]. This scheme is a well-known section of numerical analysis [17, 37, 39]. In this method, we approximate the derivatives by using finite differences [7, 17, 36, 37, 38, 39]. That is, the differential equation is replaced by algebraic equations and then the algebraic equations are solved at each mesh point [7, 36, 37, 38, 39]. In short, FDM solves a discrete set of equations [7, 36, 37, 38, 39].

FDM is a relatively simple way to find the approximate solution $u(x, \tau)$ of the heat equation [7, 17, 37, 38, 39]. We mainly consider explicit, implicit, Crank-Nicolson (CN), these 3 types of FDM to solve Black-Scholes equation [17].

Explicit method is also known as FTCS (Forward-Time Central-Space) method [17, 36, 37, 38, 40]. That is, we apply a forward difference in time τ and central difference in space x [36, 37, 38, 39, 40]. We need condition $s = \dfrac{\Delta\tau}{\Delta x^2} \le \dfrac{1}{2}$ for applying explicit scheme [17, 34, 36, 37, 38, 39, 40]. That's why we say that explicit (FTCS) method is conditionally stable [36, 37, 38, 39]. If this condition is not specified, the solution becomes unstable and begins to oscillate [40].

Implicit method is called as BTCS (Backward-Time Central-Space) method [17, 36, 40]. This method is sometimes called "Fully implicit scheme (or, backward in time)". In this case, we use backward difference in time τ and central difference in space x [36, 37, 38, 39, 40]. We do not have any condition on implicit scheme [38, 39]. That's why implicit (BTCS) scheme is unconditionally stable [36, 37, 38, 39, 40].

Crank-Nicolson method is an improvement of the implicit method (BTCS) [36]. This is usually known as CTCS (Central-Time Central-Space) method [7, 17, 36, 40]. Basically CN scheme is the average of forward Euler (explicit) and backward Euler (implicit) schemes [39, 40]. That is, we take central difference at time τ and central difference at space x [36, 37, 38, 39, 40]. CN scheme is unconditionally stable [36, 37, 38, 39].

Explicit, implicit and CN scheme have errors of order $O(\Delta\tau, \Delta x^2), O(\Delta\tau, \Delta x^2), O(\Delta\tau^2, \Delta x^2)$ respectively. Crank-Nicolson scheme is always preferable in FDM as it can give more accurate result comparing to the other schemes (FTCS & BTCS) [17, 36, 39]. That's why Crank-Nicolson method is recommended if we want to find the solution of Black-Scholes equation through FDM [20].

In general, we can express finite difference method as "weight formula" or "θ method". We can write heat equation as [37, 41]

$$\frac{u_j^{m+1} - u_j^m}{\Delta\tau} = (1 - \theta)\frac{u_{j-1}^m - 2u_j^m + u_{j+1}^m}{\Delta x^2} + \theta\frac{u_{j-1}^{m+1} - 2u_j^{m+1} + u_{j+1}^{m+1}}{\Delta x^2}. \tag{4.3}$$

If $\theta = 0$, we get explicit Euler scheme.
If $\theta = 1$, we get implicit Euler scheme.
If $\theta = 0.5$, we reach to Crank-Nicolson scheme.
If $\lambda = \dfrac{\Delta\tau}{\Delta x^2}$, we can rewrite the above equation as

$$u_j^{m+1} - u_j^m = \lambda\left[(1-\theta)\left(u_{j-1}^m - 2u_j^m + u_{j+1}^m\right) + \theta\left(u_{j-1}^{m+1} - 2u_j^{m+1} + u_{j+1}^{m+1}\right)\right]. \tag{4.4}$$

We can write the above equation in the following form

$$AU^{m+1} = BU^m + C, \tag{4.5}$$

where

$$A = \begin{bmatrix} (1+2\lambda\theta) & -\lambda\theta & 0 & 0 & 0 \\ -\lambda\theta & (1+2\lambda\theta) & -\lambda\theta & 0 & 0 \\ 0 & -\lambda\theta & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & -\lambda\theta \\ 0 & 0 & 0 & -\lambda\theta & (1+2\lambda\theta) \end{bmatrix},$$

$$B = \begin{bmatrix} (1-2\lambda(1-\theta)) & \lambda(1-\theta) & 0 & 0 & 0 \\ \lambda(1-\theta) & (1-2\lambda(1-\theta)) & \lambda(1-\theta) & 0 & 0 \\ 0 & \lambda(1-\theta) & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \lambda(1-\theta) \\ 0 & 0 & 0 & \lambda(1-\theta) & (1-2\lambda(1-\theta)) \end{bmatrix},$$

$$C = \begin{pmatrix} \lambda(1-\theta)u_1^m + \lambda\theta u_1^{m+1} \\ 0 \\ \vdots \\ 0 \\ \lambda(1-\theta)u_{end}^m + \lambda\theta u_{end}^{m+1} \end{pmatrix}.$$

We are going to apply FDM (CN) with the help of "weight formula" to the heat equation.

For European call option, the initial and boundary conditions are defined as follows [1, 5, 7, 25, 26].

Initial condition is

$$u(x,0) = \max\left(e^{\frac{(k+1)}{2}x} - e^{\frac{(k-1)}{2}x}, 0\right) \text{ for} -\infty < x < \infty.$$

Boundary condition is

$$u(x,\tau) = 0 \quad \text{for } x \to -\infty$$

$$u(x,\tau) = e^{\left\{\frac{k-1}{2}x + \frac{(k+1)^2}{4}\tau\right\}}\left(e^x - e^{-k\tau}\right) \text{ for } x \to \infty.$$

For European put option, the initial and boundary conditions are defined as follows [1, 5, 7, 25, 26].

Initial condition is

$$u(x,0) = \max\left(e^{\frac{(k-1)}{2}x} - e^{\frac{(k+1)}{2}x}, 0\right) \text{ for} -\infty < x < \infty.$$

Boundary condition is

$$u(x,\tau) = e^{\left\{\frac{k-1}{2}x + \frac{(k-1)^2}{4}\tau\right\}} \quad \text{for } x \to -\infty$$

$$u(x,\tau) = 0 \quad \text{for } x \to \infty.$$

4.2 A Piece-wise Polynomial Scheme of the Model

In Finite element method (FEM), the domain is discretized with the help of finite element. We have used residual formula in FEM [5, 32]. The basic equations that are required for finite element analysis can easily be obtained by weighted residual method [5, 32]. We define the weak formulation or integral formulation [5, 32] of the heat equation. We also define stiffness matrix and mass matrix [5, 32]. In this case, we consider linear element and linear shape functions L_1, L_2 where Lagrangian polynomials are applied [5, 32, 34]. To approximate the solution, we assume a test function in Galerkin's weighted residual method. It is important that the test function w should be a member of $H_0^1(\Omega)$ [47, 48, 50] where Ω is the polygonal domain [51] and $\partial\Omega$ is the boundary [47, 50, 51]. As $w \in H_0^1(\Omega)$, we get a unique solution u of the heat equation [47].

Local coordinate variable ξ is used whose value varies from -1 to $+1$ [5, 32]. That is, space variable x and shape functions $L_1(x), L_2(x)$ are expressed as functions of ξ [5, 32, 33].

$$L_1(\xi) = \frac{1}{2}(1-\xi), \ L_2(\xi) = \frac{1}{2}(1+\xi).$$

$$x = x_1 L_1(\xi) + x_2 L_2(\xi).$$

In FEM, we have to do integrations that are taken over elements [33]. Like, for linear elements [5, 32, 34, 35, 56],

$$\int_e L_i(x)L_j(x)dx \ ; i, j = 1, 2. \tag{4.6}$$

Let h be the length of the element. Integrals over elements can easily be computed with the help of ξ coordinate [5]. After completing the integration (4.6), we reach to a mass matrix N [5].

$$N = \frac{h}{3}\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}.$$

Similarly, we often need to use element integrals involving shape functions [5, 32, 34, 35]. Like, for linear elements [5, 32, 34, 35, 56],

$$\int_e \frac{dL_i}{dx}\frac{dL_j}{dx}dx \ ; i, j = 1, 2. \tag{4.7}$$

We compute this integral using ξ. Finally, from this integration, we obtain stiffness matrix M [5, 32, 35, 56].

$$M = \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The shape function L must be differentiable for the existence of the integration (4.7). That is, the derivatives $\frac{dL_i}{dx}, \frac{dL_j}{dx}$ must be well-behaved [48].

The integral (4.6) & (4.7) exist and they both are bounded as the function belongs to S. For our problem, we want to find the approximate solution u such that $u \in H_0^1$ [48, 49]. More specifically, $u(.,\tau) \in H_0^1$ where

$(\cdot,\cdot)$ indicates the inner product in L^2 (or L^2 norm) [50, 56, 51] and $u_\tau \in L^2$ [50].

Galerkin's weighted residual method is applied to obtain the trial solution w [5, 32] where $w \in H_0^1$ [48, 49]. H_0^1 is such a space that consists of all admissible functions for boundary value problem and therefore all the trial functions exist in H_0^1 [48].

$$w = L_1 w_1 + L_2 w_2 \, .$$

We assume that the test function or the trial solution w is a smooth function. At the same time, w is a differentiable and continuous function [47]. We also assume some conditions for H_0^1. For example H_0^1 will form a linear space i.e. $w = L_1 w_1 + L_2 w_2 \in H_0^1$ and H_0^1 is finite dimensional [48].

Here we see that L_1, L_2 are multiplied with the unknown parameters w_1, w_2. So, L_1, L_2 are weighting functions or weight functions [5]. Residual function g is defined as follows [32]

$$g = \frac{\partial^2 w}{\partial x^2} - \frac{\partial w}{\partial \tau} = 0 \, . \tag{4.8}$$

The weak formulation or integral formulation is obtained by multiplying the residual function g and weight function L, integrating over the domain $x_1 \leq x \leq x_n$ of the problem, equating to zero [33]. That is,

$$\int_{x_1}^{x_n} g \, L_j \, dx = 0 \quad ; j = 1, 2, \ldots, n \, . \tag{4.9}$$

The weak form $\int g \, L \, dx$ is an inner product which belongs to $L^2(\Omega)$ [47]. The integral $\int g \, L \, dx = \int \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial w}{\partial \tau} \right) L \, dx$ will exist and be finite (i.e. $\int g \, L \, dx < \infty$) [48] if the test function w is twice differentiable w.r.t. x and one time differentiable w.r.t. τ.

Equation (4.9) is called weighted residual equation. From (4.9), we get,

$$\int_{x_1}^{x_n} g \, L^T \, dx = 0$$

$$i.e. \left[L^T \frac{\partial w}{\partial x} \right]_{x_1}^{x_n} - \left[\int_{x_1}^{x_n} \frac{\partial L^T}{\partial x} \frac{\partial L}{\partial x} dx \right] w - \left[\int_{x_1}^{x_n} L^T L \, dx \right] \dot{w} = 0. \tag{4.10}$$

We compute the above equation for each element. The 1st term of equation (4.10) is $\left[L^T \frac{\partial w}{\partial x} \right]_{x_1}^{x_n}$. If we take element with two nodes, then for each element, we get [33],

$$F = \left[L^T \frac{\partial w}{\partial x} \right]_{x_j}^{x_{j+1}} \quad ; \, j = \overline{1, n-1}$$

$$i.e. \ F = \begin{bmatrix} -\dfrac{\partial}{\partial x} w_j \\[2mm] \dfrac{\partial}{\partial x} w_{j+1} \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial}{\partial x} w(x_j, \tau) \\[2mm] \dfrac{\partial}{\partial x} w(x_{j+1}, \tau) \end{bmatrix}.$$

From (4.10), we get [5],

$$N \dot{w} + M w = F. \tag{4.11}$$

Equation (4.11) is a first-order differential equation which is solved to get the solution. To solve (4.11), we use backward FDM [5]. That is, we take backward difference (i.e. backward Euler method) [33, 56] in time τ to approximate the time derivative $\dot{w} = \dfrac{\partial w}{\partial \tau}$. We use $\dot{w} = \dfrac{w_i - w_{i-1}}{\Delta \tau} = \dfrac{w_i^j - w_{i-1}^j}{\Delta \tau}$. We put the value of $\dot{w}$ in (4.11) and we get [5],

$$N \left(\frac{w_i^j - w_{i-1}^j}{\Delta \tau} \right) + M w_i^j = F$$

$$\text{i.e. } w_i^j = \frac{1}{\Delta \tau} N \left(\frac{1}{\Delta \tau} N + M \right)^{-1} w_{i-1}^j + F \left(\frac{1}{\Delta \tau} N + M \right)^{-1}. \tag{4.12}$$

The above system of equations (4.12) is computed for each $i = 2, 3, \cdots, m$. For $i = 1$, we get the approximation (w_1^j) for the initial time τ_1. For this problem, the approximate solution $u = u(x, \tau)$ is defined for the domain for $x \in \Omega$ and $\tau \in (\tau_a, \tau_b)$ [50, 51].

The error of order for finite element method is $O(\Delta \tau, \Delta x^2)$ that is less than that of finite difference method (CN).

The conditions [5] for European call and put option are as follows.

European Call option

Initial condition is $u(x,0) = \max\left(e^{\frac{k+1}{2}x} - e^{\frac{k-1}{2}x}, 0 \right) \ \forall x$.

Boundary condition

$$u(x_1, \tau) = 0 \ \forall \tau$$

$$\frac{\partial}{\partial x} u(x_n, \tau) = \frac{1}{2} \left[(k+1) e^{x_n} - (k-1) e^{-k\tau} \right] e^{\left\{ \frac{k-1}{2} x_n + \frac{(k+1)^2}{4} \tau \right\}} \ \forall \tau.$$

European Put option

Initial condition is $u(x,0) = \max\left(e^{\frac{k-1}{2}x} - e^{\frac{k+1}{2}x}, 0 \right) \ \forall \tau$.

Boundary condition

$$\frac{\partial}{\partial x} u(x_1, \tau) = \left(\frac{k-1}{2} \right) e^{\frac{(k-1)^2}{4} \tau} e^{\frac{k-1}{2} x_1} \ \forall \tau$$

$$u(x_n, \tau) = 0 \ \forall \tau.$$

5. Stability and Error Analysis of Black-Scholes Model

For error analysis of FEM in case of heat equation (4.2), we assume that Ω is the domain in P^d and $\partial \Omega$ is the smooth boundary for heat equation [54,55]. Next, we multiply the heat equation (4.2) by a smooth

function φ (where $\varphi \in H_0^1$), integrate over Ω, apply Green's theorem and thus we obtain the following weak formulation [54,55]

$$\left(u_t, \varphi\right) + \left(\nabla u, \nabla \varphi\right) = \left(f, \varphi\right). \tag{5.1}$$

After that we consider a linear space S_h of functions of x which is finite dimensional. We impose the following condition on the weak formulation (5.1) and we get [54, 55]

$$\left(u_{h,t}, \chi\right) + \left(\nabla u_h, \nabla \chi\right) = \left(f, \chi\right) \forall \chi \in S_h, u_h(t) = u_h(\cdot, t). \tag{5.2}$$

We consider the L_2 error between (5.2) and (4.2). If u_h and v are the solutions of (5.2) and (4.2) respectively, then [54]

$$\| u_h(t) - u(t) \| \le \| v_h - v \| + C h^r \left(\| v \|_r + \int\limits_0^t \| u_t \|_r \, ds \right) \tag{5.3}$$

where $v_h = P_h v$. Here P_h is the orthogonal projection of v onto S_h and $\|.\|$ is the norm in $L_2 = L_2(\Omega)$. The proof of (5.3) is very straightforward and one may consult with [54] for detail information.

We have applied FDM in convection-diffusion equation (3.2) and the stability, convergence of FDM can be analyzed through various methods, such as Von Neumann method, discrete perturbation method, matrix method etc [38, 39, 56]. FDM will converge if the approximate solution approaches to exact solution when

the size of the grid becomes very small, i.e. $\lim\limits_{\Delta x, \Delta t \to 0} \left(\max\limits_{j,n} \left| u(x_j, t_n) - u_j^n \right| \right) = 0$, where $u(x_j, t_n)$ and u_j^n

stands for exact solution and approximate solution respectively [56, 39]. For equation (3.2), the above statement is true which establishes the convergence of nonlinear Black-Scholes model with FDM. On the other hand, we get $\dfrac{|u|\Delta\tau}{\Delta x} \le 1$ for (3.2) which guarantees the stability of (3.2) with FDM [57].

If we apply finite element method in (3.2), it will be also be stable [58, 59, 54]. Let the domain $\Omega \in P^d$ and $W_h \in H^1(\Omega)$ where W_h is a finite dimensional space [58]. We assume $V_h = W_h \cap H_0^1(\Omega)$ and α is a bilinear form which is defined as $W_h \times W_h \to P$ [58]. Then we have

$$\alpha(v_h, v_h) \ge C(h) \| v_h \|^2 \quad \forall v_h \in V_h \tag{5.4}$$

where C is assumed as a positive constant[58,59]. Equation (5.4) establishes the stability and uniqueness of solution of (3.2) [58, 59]. The detail description of (5.4) can be found in [58, 59]. So we can guarantee that if we attempt to apply FEM in nonlinear BS model, it will definitely be stable.

6. Numerical Illustration for the Nonlinear Model

We take the inputs as follows to solve nonlinear Black-Scholes model for European call option.

$$\sigma = 0.2, r = 0.1, E = 75, T = 1, R = 1, \Delta\tau = 0.001, \Delta x = 0.1, a = 0.02, Le = 0.6, M = 2, C = 0.01.$$

European Call Option of RAPM model

6.1(a). RAPM Model

European Call Option of Leland's model

6.1(b). Leland's Model

European Call Option of Barles'-Soner's model (Identity)
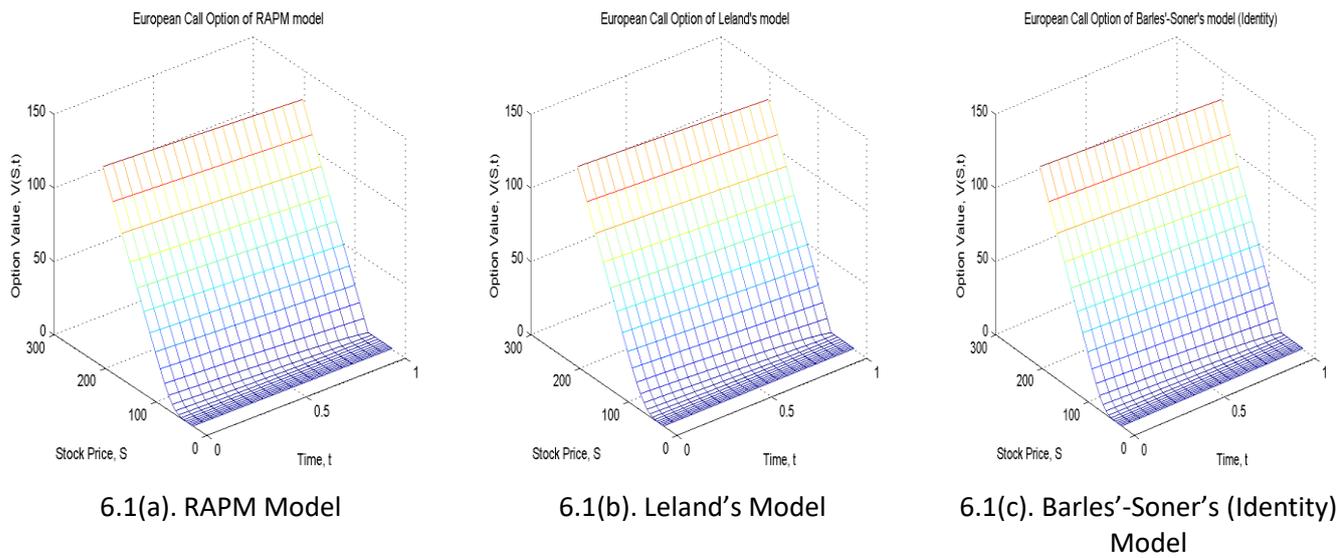

6.1(c). Barles'-Soner's (Identity) Model

Fig. 6.1. Different Types of Volatility Models

In figure 6.1, we have tried to plot different types of volatility models (nonlinear BS model) for European call option and we observe that the graphical shapes for all the models are almost same.

RAPM Model vs. Linear Model

6.2(a). RAPM Model against Linear Model

Leland's Model vs. Linear Model

6.2(b). Leland's Model against Linear Model

Barles'-Soner's Model (Identity) vs. Linear Model

6.2(c). Barles'-Soner's (Identity) Model against Linear Model
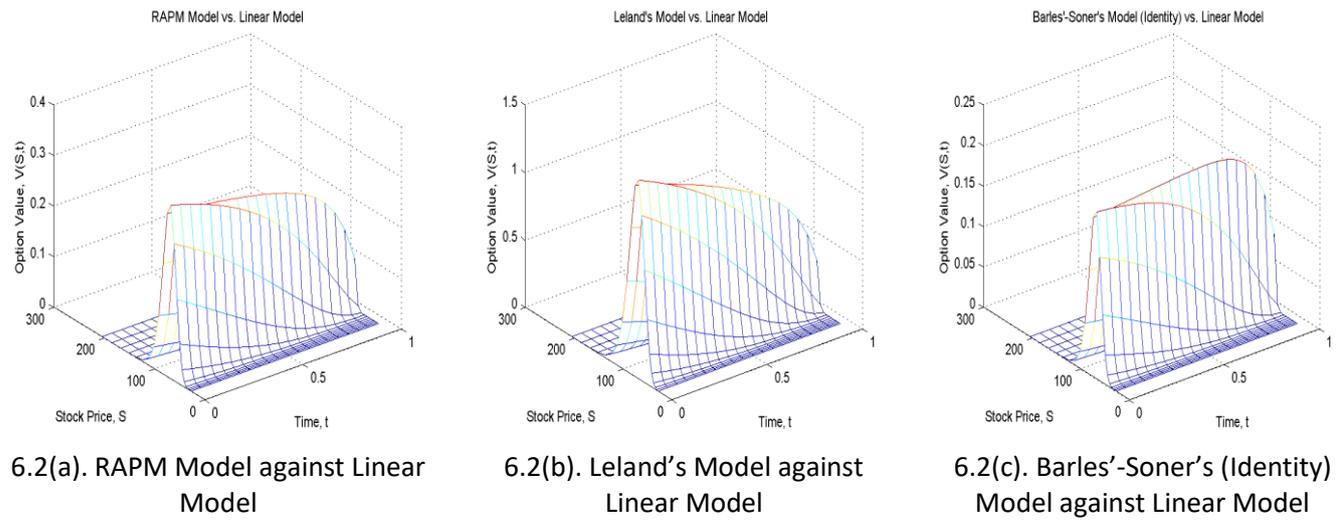
Fig. 6.2. Volatility Models against Linear BS Model

In figure 6.2, we compare solutions between volatility models (transaction cost models) and linear BS model. From figure 6.2, we observe that the difference is significant when the time to expiration is 1 year (i.e. $t = 0$) for all the volatility models [23, 52] where the stock price approximately lies between $S = 50$ to $S = 130$. That is, nonlinear price (model with transaction cost) is significantly bigger than the linear price (model without transaction cost) at the very beginning of the time (i.e. $t = 0$ or time to expiration 1 year) [23, 52]. But we note from the figure that the difference decreases when the time to expiration decreases (i.e. t increases) [52].
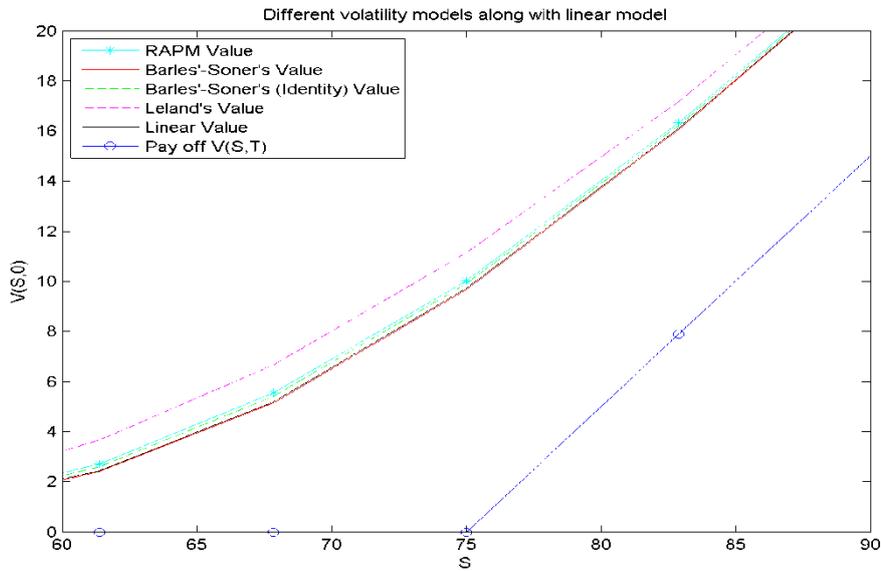
Fig. 6.3. Solution of Transaction Models along with Linear Model at Time $t = 0$

From figure 6.3, we note that the highest price is presented by Leland's model, after that RAPM model, next Barles'-Soner's (Identity) model, followed by Barles'-Soner's model and finally linear model at the time to expiration 1 year (i.e. $t = 0$).

7. Numerical Discussion on Linear Model

From We take the inputs as $T = 1$ year, $r = 0.1, \sigma = 0.2, E = 10, x_a = -1 \text{ to } x_b = 1, 0 \le \tau \le \dfrac{\sigma^2 T}{2}$, number of spatial points, $n = 100$, number of temporal points, $m = 100$.
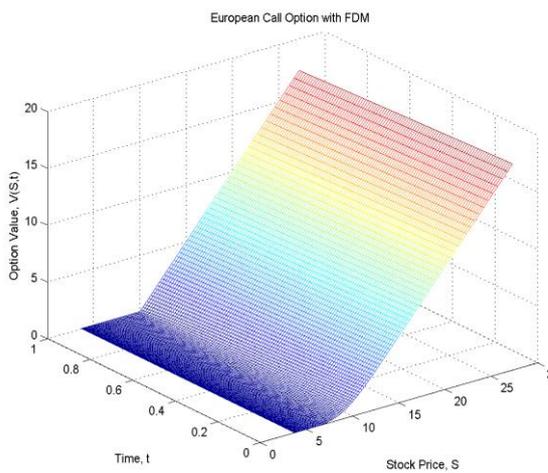
7.1(a). Linear Black-Scholes Model with FDM
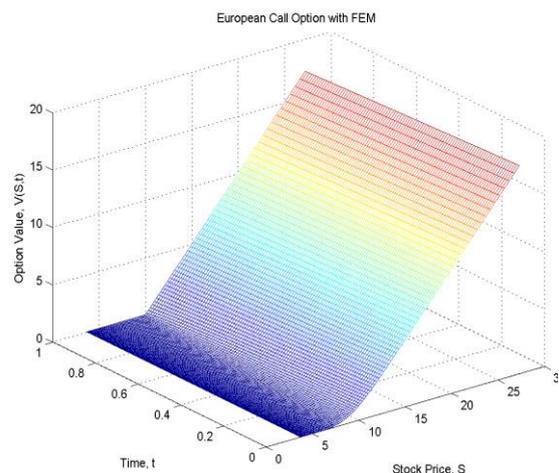
7.1(b). Linear Black-Scholes Model with FEM

Fig. 7.1. Linear Black-Scholes Model

In figure 7.1, we have attempted to present the graphical mapping of linear BS model through the well-established models finite difference scheme (CN) and Galerkin's finite element method (FEM) for European call option.
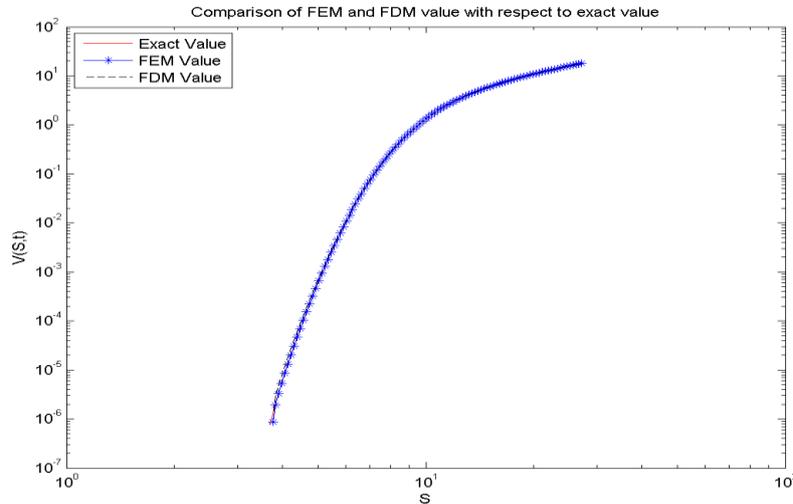

Fig. 7.2. 2-D Plot of Linear Black-Scholes Model with FDM and FEM

Figure 7.2 is shown for the issued time $t = 0$. From this figure we observe that both FEM and FDM approximate values that are very close to the exact values. That's why we can say that both FDM and FEM both are very strong schemes on option pricing.

Table 7.1 is shown for the issued time $t = 0$, Error=| Exact Value – Approximate Value|. From the table, we note that the error for FEM scheme is significantly lower than FDM method. We can conclude that, for this particular problem, FEM is approximating better than FDM scheme.

Table 7.1: Numerical Results of Linear Black-Scholes Model with FDM and FEM

Stock Price, S	Exact Value	FEM Value	Error	FDM Value	Error
4.98092	0.00054	0.00062	0.00009	0.00071	0.00017
5.18629	0.00110	0.00124	0.00014	0.00138	0.00028
5.29213	0.00155	0.00173	0.00018	0.00190	0.00035
5.40013	0.00217	0.00239	0.00022	0.00260	0.00044
5.62279	0.00412	0.00445	0.00033	0.00477	0.00065
5.73753	0.00561	0.00600	0.00039	0.00639	0.00078
5.85462	0.00757	0.00804	0.00047	0.00849	0.00092
5.97410	0.01013	0.01068	0.00055	0.01121	0.00108
6.09602	0.01344	0.01408	0.00064	0.01469	0.00125
6.22042	0.01768	0.01842	0.00073	0.01912	0.00143
6.34736	0.02308	0.02391	0.00083	0.02470	0.00162
6.47690	0.02988	0.03081	0.00093	0.03169	0.00181
6.60907	0.03836	0.03940	0.00103	0.04036	0.00200
6.74395	0.04887	0.05000	0.00113	0.05105	0.00218
6.88158	0.06176	0.06298	0.00122	0.06410	0.00234
7.02201	0.07745	0.07875	0.00130	0.07994	0.00249

8. Conclusion

In this paper, we have mainly attempted to solve the widely applied option model (Black-Scholes equation) numerically. We have discussed the solving procedure both for linear and nonlinear BS model in detail. The solution procedure was used only in the European option settings. We have advanced numerical approximation of nonlinear Black-Scholes model with the help of Crank-Nicolson finite difference method where different types of volatility models are compared to each other. Though finite element method and finite difference method (CN) are well-established and very good numerical methods, in case of linear Black-Scholes model, we have discovered that finite element method (FEM) approximates better than finite difference method (FDM). We would also like to pursue our work on both linear and nonlinear settings for European and American options by applying Rigel Compact and differential quadrature methods.

Conflict of Interests

Submitted paper is not previously published in any other journal.

Acknowledgements

Funding

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