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Automorphic Forms and Holomorphic Functions on the Upper Half-plane

Md. Shafiul Alam [*a]

[a] *Department of Mathematics, University of Barishal, Barishal-8254, Bangladesh*

ABSTRACT

We define a set of holomorphic functions in terms of the Hauptmodul of a quotient Riemann surface and prove that these functions are holomorphic on the upper half-plane. It is also shown that these functions are automorphic forms of weight k with respect to a Fuchsian group.

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1 Introduction

The group $\mathrm{SL}(2,\mathbb{R})$ is defined by

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R},\ ad-bc=1 \right\}$$

and the group

$$\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I_2\},$$

where I_2 is the 2×2 identity matrix (see [14, Chapter VII]). Let $\mathbb{H}$ denote the upper half-plane $\{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\}$. The boundary of $\mathbb{H}$ is $\mathbb{R}\cup\infty$. The group $\mathrm{PSL}(2,\mathbb{R})$ acts on $\mathbb{H}$ as follows:

$$\tau \mapsto \gamma\cdot\tau = \frac{a\tau+b}{c\tau+d},$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R})$, $\tau\in\mathbb{H}$. All transformations of $\mathrm{PSL}(2,\mathbb{R})$ are conformal.

*Corresponding author. *E-mail address:* msalam@bu.ac.bd

A Fuchsian group is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$, i.e., it is a group of orientation-preserving isometries of $\mathbb{H}$. The study of Fuchsian group is a very interesting topic in many fields of Mathematics. Many mathematicians studied Fuchsian group and various subgroups of Fuchsian group, for example, see [9], [13], [18] and [17]. The Hecke group which is a subgroup of Fuchsian group is studied in [1] and [2] to investigate Ramanujan's modular equations.

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R})$ and let $\mathrm{tr}(\gamma)$ denote the trace of γ, then the element γ is said to be

- elliptic when $|\,\mathrm{tr}(\gamma)| < 2$,
- parabolic when $|\,\mathrm{tr}(\gamma)| = 2$,
- hyperbolic when $|\,\mathrm{tr}(\gamma)| > 2$.

If $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is a Fuchsian group and $\gamma \in \Gamma$ is an elliptic element, then a point $\tau \in \mathbb{H}$ is called an elliptic point of Γ if $\gamma(\tau) = \tau$. Also, for a parabolic element $\sigma \in \Gamma$, a point $x \in \mathbb{R} \cup \{\infty\}$ is called a cusp of Γ if $\sigma(x) = x$. If a Fuchsian group Γ acts on $\mathbb{H}$ properly discontinuously, then we have the quotient Riemann surface $\Gamma\backslash\mathbb{H}$. For a detailed discussion, see [3] and [11].

Let $\mathbb{H}^*$ denote the union of the upper half-plane $\mathbb{H}$ and the set of cusps of a Fuchsian group Γ. Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $\tau \in \mathbb{H}$ and $f : \mathbb{H} \to \mathbb{C}$ is a holomorphic function. Then the function f is called an automorphic form of weight k with respect to Γ if

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau).$$

If $k = 0$, then

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau)$$

and f is called an automorphic function. When the genus of the quotient Riemann surface $\Gamma\backslash\mathbb{H}^*$ is zero, an automorphic function is called a Hauptmodul. If an automorphic function has no poles, then it is constant according to the consequence of maximum modulus principle. For details, we refer the reader to [6], [7], [10], and [12].

In many areas of mathematics, especially in number theory, automorphic forms are studied extensively. In [5], automorphic forms for Schottky groups are studied. In [8], the authors have established various results related to automorphic forms of triangle groups. The famous mathematician Goro Shimura extensively investigated many arithmetic properties of automorphic forms (see [15] and [16]). Motivated by these works, we study the automorphic forms of weight k with respect to the Fuchsian group Γ with signature $(0; n_1, \ldots, n_r)$. In the previous works mentioned above, the explicit forms of the holomorphic functions on the upper half-plane are not defined. In this work, we explicitly define a set of functions which are holomorphic on the upper half-plane $\mathbb{H}$. These holomorphic functions are expressed in terms of the Hauptmodul of the quotient Riemann surface $\Gamma\backslash\mathbb{H}^*$ and are automorphic forms of weight k with respect to Γ.

Let F be the fundamental domain for the Fuchsian group Γ. Let X and $\hat{X}$ denote the quotient Riemann surfaces $\Gamma\backslash\mathbb{H}$ and $\Gamma\backslash\mathbb{H}^*$, respectively. If F is compact, then it has finitely many vertices which are elliptic points and cusps of $\hat{X} = \Gamma\backslash\mathbb{H}^*$. Let $P_1, \ldots, P_r$ be the vertices whose orders are $n_1, n_2, \ldots, n_r$, respectively. If the number of elliptic elements and cusps of Γ are m and l, respectively, then $m + l = r$. If g is the genus of $\hat{X}$, then we say that Γ has signature $(g; n_1, \ldots, n_r)$. For a more detailed discussion, the reader may consult Section 2.1 of [4], Chapter 4 of [11], and Section 2 of [18]. Let us denote by A_k the space of automorphic forms of weight

k with respect to Γ. The basis for A_k on a Shimura curve X with genus 0 is determined in Theorem 4 of [19]. The following theorem is written according to Theorem 2.23 of [15] to determine the dimension of A_k.

Theorem 1 ([15, Theorem 2.23]). *For a Fuchsian group Γ with signature $(g; n_1, \ldots, n_r)$, let g be the genus of the compact quotient Riemann surface $\hat{X} = \Gamma\backslash\mathbb{H}^*$. Then, the dimension, $\dim A_k$, of A_k for an even integer k is given by*

$$\dim A_k = \begin{cases} 0 & \text{if } k < 0, \\ 1 & \text{if } k = 0, \\ g & \text{if } k = 2, \\ (g-1)(k-1) + \sum_{i=1}^{r} \left\lfloor \frac{k}{2}(1 - \frac{1}{n_i}) \right\rfloor & \text{if } k \geq 4. \end{cases}$$

In the following section, we present our main results and their proofs.

2 Main Results

Let $(0; n_1, \ldots, n_r)$ be the signature of a Fuchsian group Γ, i.e., the genus of the quotient Riemann surface $\hat{X} = \Gamma\backslash\mathbb{H}^*$ is 0 and let $d = \dim A_k$. Then, for an even integer $k \geq 4$, we have from Theorem 1

$$d = 1 - k + \sum_{i=1}^{r} \left\lfloor \frac{k}{2}(1 - \frac{1}{n_i}) \right\rfloor.$$

In the following theorem, we define the functions h_j for $j = 0, \ldots, d-1$ so that the functions are holomorphic on $\mathbb{H}$. Also, these functions are automorphic forms of weight k with respect to Γ.

Theorem 2. *Consider the Fuchsian group Γ with signature $(0; n_1, \ldots, n_r)$ and the compact quotient Riemann surface $\hat{X} = \Gamma\backslash\mathbb{H}^*$. Let $\tau_1, \ldots, \tau_r$ be the inequivalent vertices (elliptic points or cusps of $\hat{X}$) of the fundamental domain of Γ of orders $n_1, \ldots, n_r$, respectively, and let $w(\tau)$ be a Hauptmodul of $\hat{X}$. For an even integer $k \geq 4$, let*

$$a_i = \left\lfloor \frac{k}{2}\left(1 - \frac{1}{n_i}\right) \right\rfloor$$

and

$$d = \dim A_k = 1 - k + \sum_{i=1}^{r} a_i.$$

If $w(\tau_i) = w_i$ for $i = 1, \ldots, r$ and the functions $h_j(\tau)$ are defined by

$$h_j(\tau) = \frac{\left(w'(\tau)\right)^{k/2} \left(w(\tau)\right)^j}{\prod_{i=1, w_i \neq \infty}^{r} \left(w(\tau) - w_i\right)^{a_i}} \tag{2.1}$$

for $j = 0, \ldots, d-1$ and $\tau \in \mathbb{H}$, then the functions $h_j(\tau)$ are holomorphic on $\mathbb{H}$.

Proof. We need to consider the following three cases:

1. the Hauptmodul $w(\tau)$ does not have any pole at the points τ_i for $i = 1, \ldots, r$;
2. the Hauptmodul $w(\tau)$ has a pole at one of the points τ_i for $i = 1, \ldots, r$;
3. the Hauptmodul $w(\tau)$ has a pole at another point, say $\tau = \tau_0$, except the points $\tau_1, \ldots, \tau_r$.

If a function has a zero of order ≥ 0 and has a pole of order ≤ 0 at a point, then there is no principal part in the expansion of the function at that point, i.e., the function is holomorphic. Thus, we have to show that the functions h_j have

- a zero of order ≥ 0 at $\tau = \tau_i$ for Case 1,
- a pole of order ≤ 0 at $\tau = \tau_i$ for Case 2,
- a pole of order ≤ 0 at $\tau = \tau_0$ for Case 3.

Case 1: If $w(\tau)$ does not have any pole at τ_i, then $w(\tau_i) = w_i \neq \infty$ for $i = 1, \ldots, r$. Since τ_i is a vertex of order n_i, in a neighbourhood of $\tau = \tau_i$, we have

$$w(\tau) - w(\tau_i) = b_i(\tau - \tau_i)^{n_i} + O\big((\tau - \tau_i)^{n_i+1}\big) \tag{2.2}$$

or,

$$w(\tau) - w_i = (\tau - \tau_i)^{n_i} w^*(\tau), \tag{2.3}$$

where $b_i \in \mathbb{C} \setminus \{0\}$, $w^*(\tau)$ is analytic in a neighbourhood of $\tau = \tau_i$ and $w^*(\tau_i) \neq 0$ for $i = 1, \ldots, r$. Therefore, in a neighbourhood of $\tau = \tau_i$, one can define a single-valued analytic n_i-th root of $(w - w_i)$ and this can be done at all points which are equivalent to τ_i under the action of the Fuchsian group Γ. Since $w(\tau) - w_i \neq 0$ for $\tau \neq \tau_i$ and $(w - w_i)$ is analytic on the other part of $\mathbb{H}$, its n_i-th root is analytic at each point of the remainder of $\mathbb{H}$. As $(w(\tau) - w_i)^{n_i}$ is locally analytic and single-valued at each $\tau \in \mathbb{H}$, so it follows from monodromy theorem that a single-valued and analytic n_i-th root of $(w - w_i)$ can be defined on the whole $\mathbb{H}$.

From (2.3), we observe that $\big(w(\tau) - w_i\big)$ has a zero of order n_i at $\tau = \tau_i$ and

$$\prod_{i=1, w_i \neq \infty}^{r} \big(w(\tau) - w_i\big)^{a_i}$$

has a zero of order $n_i a_i = n_i \left\lfloor \frac{k}{2}\Big(1 - \frac{1}{n_i}\Big) \right\rfloor$ at $\tau = \tau_i$. Also, we have from (2.2)

$$w'(\tau) = b_i n_i (\tau - \tau_i)^{n_i - 1} + O\big((\tau - \tau_i)^{n_i}\big). \tag{2.4}$$

Consequently, at $\tau = \tau_i$, $\big(w'(\tau)\big)^{k/2}$ has a zero of order $\frac{k}{2}(n_i - 1)$. Since

$$\frac{k}{2}(n_i - 1) - n_i \left\lfloor \frac{k}{2}\Big(1 - \frac{1}{n_i}\Big) \right\rfloor \geq 0,$$

we conclude from (2.1) that the functions h_j have a zero of order ≥ 0 at $\tau = \tau_i$. Hence, the functions h_j are holomorphic on $\mathbb{H}$.

Case 2: Assume that $w(\tau)$ has a pole at one of the points τ_i for $i = 1, \ldots, r$. Without loss of generality, suppose $w(\tau)$ has a pole at τ_1, i.e., $w(\tau_1) = w_1 = \infty$. Since τ_1 is a vertex of order n_1, it follows that

$$w(\tau) = \frac{b_1}{(\tau - \tau_1)^{n_1}} + O\big((\tau - \tau_1)^{1-n_1}\big), \quad b_1 \in \mathbb{C} \setminus \{0\}$$

and

$$w'(\tau) = -\frac{b_1 n_1}{(\tau - \tau_1)^{n_1+1}} + O\big((\tau - \tau_1)^{-n_1}\big).$$

In this case, from (2.1) we have

$$h_j(\tau) = \frac{\big(w'(\tau)\big)^{k/2}\big(w(\tau)\big)^j}{\displaystyle\prod_{i=2, w_i \neq \infty}^{r} \big(w(\tau) - w_i\big)^{a_i}}. \tag{2.5}$$

Now, suppose that the functions $h_j(\tau)$ defined in (2.5) have a pole of order N at $\tau = \tau_1$. Since $w(\tau)$ has a pole of order n_1 at $\tau = \tau_1$, $\big(w'(\tau)\big)^{k/2}$ has a pole of order $\frac{k}{2}(n_1 + 1)$ and

$$\prod_{i=2, w_i \neq \infty}^{r} \big(w(\tau) - w_i\big)^{a_i}$$

has a pole of order $n_1 \sum_{i=2}^{r} a_i$ at $\tau = \tau_1$. As j varies from 0 to $d-1$, so the maximum value of j is $d-1 = \sum_{i=1}^{r} a_i - k$. Hence, $\big(w(\tau)\big)^j$ has a pole of order at most $n_1\Big(\sum_{i=1}^{r} a_i - k\Big)$ at $\tau = \tau_1$ and we have

$$\begin{aligned}
N &\leq \frac{k}{2}(n_1 + 1) + n_1\Big(\sum_{i=1}^{r} a_i - k\Big) - n_1 \sum_{i=2}^{r} a_i \\
&= \frac{k}{2}(n_1 + 1) + n_1\Big(\sum_{i=1}^{r} \Big\lfloor \frac{k}{2}\Big(1 - \frac{1}{n_i}\Big)\Big\rfloor - k\Big) - n_1 \sum_{i=2}^{r} \Big\lfloor \frac{k}{2}\Big(1 - \frac{1}{n_i}\Big)\Big\rfloor \\
&= -\frac{k}{2}\Big(n_1 - 1\Big) + n_1 \Big\lfloor \frac{k}{2}\Big(1 - \frac{1}{n_1}\Big)\Big\rfloor \\
&\leq 0.
\end{aligned}$$

Since $N \leq 0$, it follows that there are no principal parts in the expansions of the functions h_j. Therefore, the functions h_j are holomorphic on $\mathbb{H}$.

Case 3: Suppose that $w(\tau)$ has the value ∞ at the point $\tau = \tau_0$ and $w(\tau_i) \neq \infty$ for $i = 1, \ldots, r$. Therefore, $w(\tau)$ has a simple pole at τ_0 and we have

$$w(\tau) = \frac{b_0}{(\tau - \tau_0)} + O(1), \quad b_0 \in \mathbb{C} \setminus \{0\} \tag{2.6}$$

and

$$w'(\tau) = -\frac{b_0}{(\tau - \tau_0)^2} + O(1). \tag{2.7}$$

Let N_0 be the order of the pole of h_j defined in (2.1) at $\tau = \tau_0$. From (2.6) and (2.7), we observe that $\big(w'(\tau)\big)^{k/2}$

has a pole of order k,

$$\prod_{i=1, w_i\neq\infty}^{r} \big(w(\tau) - w_i\big)^{a_i}$$

has a pole of order $\sum_{i=1}^{r} a_i$ and w^j has a pole of order at most $d-1 = \sum_{i=1}^{r} a_i - k$. Therefore, from (2.1), it follows that

$$N_0 \leq k + \sum_{i=1}^{r} a_i - k - \sum_{i=1}^{r} a_i = 0,$$

which implies that the functions h_j are holomorphic on $\mathbb{H}$ in this case also. □

Lemma 1. *The functions h_j for $j = 0, \ldots, d-1$ defined in (2.1) are automorphic forms of weight k with respect to the Fuchsian group Γ.*

Proof. For $j = 0, \ldots, d-1$ and $a_i = \left\lfloor \frac{k}{2}\left(1 - \frac{1}{n_i}\right)\right\rfloor$, we have to show that

$$h_j\Big(\frac{a\tau + b}{c\tau + d}\Big) = (c\tau + d)^k h_j(\tau),$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\tau \in \mathbb{H}$. Since $w(\tau)$ is a Hauptmodul of $\hat{X}$, i.e., $w(\tau)$ is an automorphic function, thus we have

$$w\Big(\frac{a\tau + b}{c\tau + d}\Big) = w(\tau)$$

and

$$w'\Big(\frac{a\tau + b}{c\tau + d}\Big) = (c\tau + d)^{1/2} w'(\tau).$$

Now,

$$\begin{aligned} h_j\Big(\frac{a\tau + b}{c\tau + d}\Big) &= \frac{\Big(w'\Big(\frac{a\tau + b}{c\tau + d}\Big)\Big)^{k/2}\Big(w\Big(\frac{a\tau + b}{c\tau + d}\Big)\Big)^{j}}{\prod_{i=1, w_i\neq\infty}^{r}\Big(w\Big(\frac{a\tau + b}{c\tau + d}\Big) - w_i\Big)^{a_i}} \\ &= \frac{(c\tau + d)^k \big(w'(\tau)\big)^{k/2}\big(w(\tau)\big)^{j}}{\prod_{i=1, w_i\neq\infty}^{r}\big(w(\tau) - w_i\big)^{a_i}} \\ &= (c\tau + d)^k h_j(\tau). \end{aligned}$$

Thus, the functions h_j are automorphic forms of weight k with respect to Γ. □

Example 1. *Let Γ_1 be the triangle group $(0; 4, 6, 6)$ which is a subgroup of the Fuchsian group Γ. Thus,*

$$g = 0, \quad n_1 = 4, \quad n_2 = 6, \quad n_3 = 6.$$

If τ_1, τ_2, τ_3 are the elliptic points of orders 4, 6, 6, respectively, and $w(\tau)$ is the Hauptmodul of the quotient Riemann surface $\hat{X} = \Gamma\backslash\mathbb{H}^*$, then according to Theorem 2

$$w(\tau_1) = w_1, \quad w(\tau_2) = w_2, \quad w(\tau_3) = w_3.$$

We consider the cases $k=6$ and $k=8$. For $k=6$, we have

$$a_1 = \left\lfloor \frac{6}{2}(1-\frac{1}{4}) \right\rfloor = 2, \quad a_2 = a_3 = \left\lfloor \frac{6}{2}(1-\frac{1}{6}) \right\rfloor = 2.$$

and

$$d = \dim A_6 = 1 - 6 + \sum_{i=1}^{3} a_i = 1.$$

Therefore, we have the holomorphic function $h_0(\tau)$ on $\mathbb{H}$ defined by

$$h_0(\tau) = \frac{\big(w'(\tau)\big)^3}{(w(\tau)-w_1)^2(w(\tau)-w_2)^2(w(\tau)-w_3)^2}.$$

If we normalize $w(\tau)$ such that

$$w(\tau_1) = 0, \quad w(\tau_2) = 1, \quad w(\tau_3) = \infty,$$

then

$$h_0 = \frac{\big(w'(\tau)\big)^3}{(w(\tau))^2(w(\tau)-1)^2}.$$

For the case $k=8$, we have

$$a_1 = \left\lfloor \frac{8}{2}(1-\frac{1}{4}) \right\rfloor = 3, \quad a_2 = a_3 = \left\lfloor \frac{8}{2}(1-\frac{1}{6}) \right\rfloor = 3$$

and

$$d = \dim A_8 = 1 - 8 + \sum_{i=1}^{3} a_i = 2.$$

In this case, we have the holomorphic functions $h_0(\tau)$ and $h_1(\tau)$ on $\mathbb{H}$ defined by

$$h_0(\tau) = \frac{\big(w'(\tau)\big)^4}{(w(\tau)-w_1)^3(w(\tau)-w_2)^3(w(\tau)-w_3)^3}$$

and

$$h_1(\tau) = \frac{\big(w'(\tau)\big)^4 w(\tau)}{(w(\tau)-w_1)^3(w(\tau)-w_2)^3(w(\tau)-w_3)^3},$$

respectively. For the following normalization

$$w(\tau_1) = 0, \quad w(\tau_2) = 1, \quad w(\tau_3) = \infty,$$

we have

$$h_0 = \frac{\big(w'(\tau)\big)^4}{(w(\tau))^3(w(\tau)-1)^3}$$

and

$$h_1 = \frac{\big(w'(\tau)\big)^4 w(\tau)}{(w(\tau))^3(w(\tau)-1)^3}.$$

3 Conclusion

In this study, a set of functions has been defined explicitly in terms of the Hauptmodul w of the quotient Riemann surface $\hat{X} = \Gamma\backslash\mathbb{H}^*$ for the Fuchsian group Γ with signature $(0; n_1, \dots, n_r)$. The holomorphicity of these functions on the upper half-plane $\mathbb{H}$ has been investigated. Also, it has been shown that these holomorphic functions are automorphic forms of weight k with respect to the Fuchsian group Γ. Finally, an example has been given as an application of Theorem 2 for the triangle group $\Gamma_1 = (0; 4, 6, 6)$ which is a subgroup of the Fuchsian group Γ.

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