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# A Comparative Analysis of Crank-Nicolson Scheme between Finite Volume Method and Finite Difference Method for Pricing European Option

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## ABSTRACT

The value of European option (call and put) is evaluated in this work by discretizing the Black-Scholes (BS) equation using two numerical procedures: finite volume method (FVM) and finite difference method (FDM). Both methods can be classified into three schemes: explicit, implicit, and Crank-Nicolson. The primary goal of this work is to compare the results of Crank-Nicolson FVM (CNFVM) and Crank-Nicolson FDM (CNFDM), considering the BS model as a benchmark. From the comparison, we conclude that CNFVM gives more accurate approximations than CNFDM.



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## 1 Introduction

Partial differential equations (PDE) have an incredible assortment of uses in various fields of science, for example, engineering, physics, biology, and so on. There are just a couple of their applications in finance. Among them, a notable model including PDE is the Black-Scholes model [1,2]. It was first presented by Fischer Black and Myron Scholes in 1973 [1]. The model is utilized to ascertain the price of European options (call and put) for non-dividend paying stock using the five parameters such as current stock price (S), option's strike price (K), expected volatility ($\sigma$), time to termination (T) and risk free interest rate (r). In the model, it is assumed that underlying asset's price is dependent on Geometric Brownian Motion (GBM) which is $dS = \mu S dt + \sigma S dB$, where, $\mu > 0, \sigma > 0$ and B, are constant drift, constant volatility, and standard Brownian motion respectively. The general thought behind the model is that an investor could perfectly hedge all option hazard by purchasing and selling options over time.

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In recent articles, FVM, FDM and FEM [5–9] have been widely utilized in various engineering fields, for example, fluid mechanics, heat and mass transfer, etc.

FVM is a discretization strategy suitable for the mathematical reproduction of various types of conservation laws. It can now be used in financial mathematics related problems. In 2001, Zvan [11] successfully used FVM to solve numerical problems on an option case. He portrayed a general finite volume structure for two-factor contingent claims PDE valuation models and afterward exhibited that numerous two-factor cases can be evaluated utilizing a similar general system. In 2004, Song Wang [12] directed exploration on an FVM to decide the cost of European options with dividends. According to the findings of his investigation, the process resulting from an FVM in a space with an implicit process at time is consistent and stable, hence it is assumed to be convergent as the applicable financial solution. In 2009, Kai Zhang and Song Wang [3] used FVM to determine the prices of European and American options considering the jump diffusion process. Recently, Xiaoting Gan and Junfeng Yin [15] used FVM to develop a new numerical method for evaluating the price of American options under the regime-switching model. Xiaoting Gan and Dengguo Xu [14] presented Crank-Nicolson FVM to value the American option on a zero-coupon bond. They used Cox–Ingersoll–Ross (CIR) model as their governing PDE. They also discussed the consistency of their new model.

The FDM attempts to solve PDE by replacing the differential operators with finite difference approximations [4, 16–18]. By using this methodology, a PDE can be transformed into a set of equations that can be resolved using matrix algebraic methods. Brennan and Schwartz introduced FDM to option pricing in 1978 [17].

The rest of this paper is structured as follows. The BS model and associated option pricing formulae are covered in the next section. In Sections 3 and 4, respectively, the discretization of the BS equation using CNFVM and CNFDM is provided. Numerical experiments are narrated in Section 5. Finally, Section 6 provides our work's conclusion.

# 2 The BS Model and Auxiliary Results

One of the most important concepts in modern financial theory is the Black-Scholes model. It is considered the standard model for pricing options [1, 2, 10]. If H is the price of an option, S is the underlying asset, r is the risk-free interest rate and then BS equation can be written as

$$\frac{\partial H}{\partial t} + rS\frac{\partial H}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 H}{\partial S^2} = rH. \tag{2.1}$$

When compared to all the other derivatives that can be described with S as the underlying variable, Equation (2.1) has a lot of solutions. When the problem is solved on the boundary conditions, the specific solution is obtained.

## 2.1 Initial and Boundary Conditions

### 2.1.1 Conditions for European Call Option

At maturity $t = T$, the payoff function of a European call option is given by:

$$H(S,t) = \max(S-K,0) = (S-K)^+ \quad \text{at} \quad t = T \quad \text{for} \quad 0 \le S < \infty$$

when $S = 0$, $H(S,t) = 0 \quad$ for $0 \le t \le T$.

when $S = \infty$, $H(S,t) = S - Ke^{-r(T-t)} \quad$ for $0 \le t \le T$.

### 2.1.2 Conditions for European Put Option

At maturity $t = T$, the payoff function of a European put option is given by

$$H(S,t) = \max(K-S,0) = (K-S)^+ \quad \text{at} \quad t = T \quad \text{for} \quad 0 \le S < \infty$$

when $S = 0$, $H(S,t) = Ke^{-r(T-t)}$ for $0 \leq t \leq T$.

when $S = \infty$, $H(S,t) = 0$ for $0 \leq t \leq T$.

### 2.2 BS Option Pricing Formulae

The price of European call ($E_c$) and put ($E_p$) option is given by

$$E_c = S_0 N(d_1) - Ke^{-r(T-t)} N(d_2), \tag{2.2}$$

$$E_p = Ke^{-r(T-t)} N(-d_2) - S_0 N(-d_1), \tag{2.3}$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t},$$

Here, $N(x)$ is the standard normal cumulative distribution function (CDF).

## 3 Discretization of BS PDE by Crank-Nicolson FVM

In this section, we use fitted FVM [3, 12] for discretizing BS equation (2.1). First we transform the equation (2.1) into the following conservative form:

$$\frac{\partial H}{\partial \tau} = \frac{\partial}{\partial S}\left(xS^2 \frac{\partial H}{\partial S} + ySH\right) - zH, \tag{3.1}$$

where, $\tau = T - t$, $x = \frac{1}{2}\sigma^2$, $y = r - \sigma^2$, $z = 2r - \sigma^2$.

### 3.1 Spatial Discretization

First, we need to change the underlying asset $S$ region from the interval $(0, \infty)$ to the interval $I = \Big(0, S_{max}\Big)$ for computational convenience. Now, we define two partitions for $I$ as $I_i = \Big(S_i, S_{i+1}\Big)$, $i = 1, 2, 3, ..., m$ with $0 = S_1 < S_2 < S_3 < ... < S_{m+1} = S_{max}$ and $J_i = \Big[S_{i-\frac{1}{2}}, S_{i+\frac{1}{2}}\Big]$, where, $S_{i-\frac{1}{2}} = \frac{S_{i-1}+S_i}{2}$ and $S_{i+\frac{1}{2}} = \frac{S_i+S_{i+1}}{2}$ for each $i = 1, 2, ..., m+1$ with $S_1 = S_{\frac{1}{2}}$ and $S_{m+1} = S_{m+\frac{3}{2}}$.
Integrating both sides of equation (3.1) over the control volumes $J_i = \Big(S_{i-\frac{1}{2}}, S_{i+\frac{1}{2}}\Big)$, we have

$$\int_{S_{i-\frac{1}{2}}}^{S_{i+\frac{1}{2}}} \frac{\partial H}{\partial \tau} dS = \int_{S_{i-\frac{1}{2}}}^{S_{i+\frac{1}{2}}} \frac{\partial}{\partial S}\left(xS^2 \frac{\partial H}{\partial S} + ySH\right) dS - \int_{S_{i-\frac{1}{2}}}^{S_{i+\frac{1}{2}}} zH dS,$$

$$\text{or, } \frac{\partial H}{\partial \tau} \Big[S\Big]_{S_{i-\frac{1}{2}}}^{S_{i+\frac{1}{2}}} = \left[xS^2 \frac{\partial H}{\partial S} + ySH\right]_{S_{i-\frac{1}{2}}}^{S_{i+\frac{1}{2}}} - zH\Big[S\Big]_{S_{i-\frac{1}{2}}}^{S_{i+\frac{1}{2}}},$$

$$\text{or, } \frac{\partial H}{\partial \tau}\Big(S_{i+\frac{1}{2}} - S_{i-\frac{1}{2}}\Big) = \left[S\left(xS\frac{\partial H}{\partial S} + yH\right)\right]_{S_{i-\frac{1}{2}}}^{S_{i+\frac{1}{2}}} - zH\Big(S_{i+\frac{1}{2}} - S_{i-\frac{1}{2}}\Big),$$

for each $i = 2, 3, ..., m$.

When all terms except the first one in the right side are subjected to the Gaussian one point quadrature rule, we obtain

$$\frac{\partial H_i}{\partial \tau} f_i = \Big[S\rho\big(H\big)\Big]_{S_{i-\frac{1}{2}}}^{S_{i+\frac{1}{2}}} - zH_i fi = S_{i+\frac{1}{2}}\rho\big(H\big)\Big|_{S_{i+\frac{1}{2}}} - S_{i-\frac{1}{2}}\rho\big(H\big)\Big|_{S_{i-\frac{1}{2}}} - zH_i fi, \tag{3.2}$$

for $i = 2, 3, ..., m$, where, $f_i = S_{i+\frac{1}{2}} - S_{i-\frac{1}{2}}$ is the length of the interval, $H_i$ represents the nodal approximation to $H\big(S_i, \tau\big)$ which is needed to be determined and $\rho(H)$ is the flux related to $H$ described by

$$\rho\big(H\big) = xS\frac{\partial H}{\partial S} + yH.$$

Now, we have to determine a rough estimate of the continuous flux $\rho\big(H\big)$. The two cases that are being discussed here, are for $i \geq 1$ and $i = 0$, respectively.

**Case 1:** Approximation of $\rho$ at $S_{i-\frac{1}{2}}$ and $S_{i+\frac{1}{2}}$ for $i \geq 1$.

Let's think about the following two point boundary value problem (BVP):

at the point $S_{i+\frac{1}{2}}$

$$\frac{\partial}{\partial S}\left(xS\frac{\partial H}{\partial S} + yH\right) = 0, \qquad S \in I_i, \tag{3.3}$$

$$H\big(S_i\big) = H_i, \quad H\big(S_{i+1}\big) = H_{i+1}. \tag{3.4}$$

at the point $S_{i-\frac{1}{2}}$

$$\frac{\partial}{\partial S}\left(xS\frac{\partial H}{\partial S} + yH\right) = 0, \qquad S \in I_i, \tag{3.5}$$

$$H\big(S_i\big) = H_i, \quad H\big(S_{i-1}\big) = H_{i-1}. \tag{3.6}$$

Solving the above BVPs, we get

at the point $S_{i+\frac{1}{2}}$

$$\rho_i\big(H\big) = y\frac{H_{i+1}S_{i+1}^p - H_i S_i^p}{S_{i+1}^p - S_i^p}, \tag{3.7}$$

at the point $S_{i-\frac{1}{2}}$

$$\rho_i\big(H\big) = y\frac{H_{i-1}S_{i-1}^p - H_i S_i^p}{S_{i-1}^p - S_i^p}, \tag{3.8}$$

where, $p = \frac{y}{x}$.

**Case 2:** The analysis in **Case 1**, does not apply to the approximation of the flux on $I_1 = \big(0, S_2\big)$, because the equations (8) and (10) are degenerate for $S \to 0$. In this situation, we take into account the following two point BVP:

$$\frac{\partial}{\partial S}\left(xS\frac{\partial H}{\partial S} + yH\right) = D, \qquad S \in I_1, \tag{3.9}$$

$$H\big(0\big) = H_1, \quad H\big(S_2\big) = H_2, \tag{3.10}$$

where, $D$ is an unknown constant to be determined.
Solving [3, 12, 13] this problem, we get the following solution:

$$\rho_1(H) = \frac{1}{2}\Big[(x+y)H_1 - (x-y)H_1\Big], \tag{3.11}$$

$$H = H_1 + \frac{S}{S_2}\Big(H_2 - H_1\Big).$$

Substituting (3.7), (3.8)and (3.11) into equation (3.2), we get

$$\frac{\partial H_i}{\partial \tau} = \alpha_i H_{i-1} + \beta_i H_i + \gamma_i H_{i+1}, \tag{3.12}$$

for $i = 2, 3, ......, m$, where,

$$\alpha_2 = \frac{S_2(x-y)}{4f_2},$$

$$\beta_2 = -\frac{S_2(x+y)}{4f_2} - \frac{yS_{\frac{3}{2}}S_2^p}{f_2\Big(S_3^p - S_2^p\Big)} - z,$$

$$\gamma_2 = \frac{yS_{\frac{3}{2}}S_3^p}{f_2\Big(S_3^p - S_2^p,\Big)},$$

and

$$\alpha_i = \frac{yS_{i-\frac{1}{2}}S_{i-1}^p}{S_{i-1}^p - S_i^p},$$

$$\beta_i = -\frac{yS_{i-\frac{1}{2}}S_i^p}{S_{i-1}^p - S_i^p} - \frac{yS_{i+\frac{1}{2}}S_i^p}{S_{i+1}^p - S_i^p} z f_i,$$

$$\gamma_i = \frac{yS_{i+\frac{1}{2}}S_{i+1}^p}{S_{i+1}^p - S_i^p},$$

for $i = 3, 4, ......, m$.

## 3.2 Time Discretization

Let $\tau_j = j\Delta\tau$, for $j = 0, 1, 2, ..., n$ and $\Delta\tau = T/n$. Now, let us consider

$$\begin{aligned}\frac{H_i^{j+1} - H_i^j}{\Delta\tau} =& \theta\Big(\alpha_i H_{i-1}^{j+1} + \beta_i H_i^{j+1} + \gamma_i H_{i+1}^{j+1}\Big) \\ &+ (1-\theta)\Big(\alpha_i H_{i-1}^j + \beta_i H_i^j + \gamma_i H_{i+1}^j\Big),\end{aligned} \tag{3.13}$$

where, $H_i^j$ denotes the nodal approximation to $H(S_i, \tau_j)$ and $\theta$ stands for temporal weighting and holds, that $0 \leq \theta \leq 1$.

Simplifying equation (3.13), we get

$$\begin{aligned}\big[1 - \theta\Delta\tau\beta_i\big]H_i^{j+1} - \theta\Delta\tau\big[\alpha_i H_{i-1}^{j+1} + \gamma_i H_{i+1}^{j+1}\big] =& \big[1 + (1-\theta)\Delta\tau\beta_i\big]H_i^j + \\ & (1-\theta)\Delta\tau\big[\alpha_i H_{i-1}^j + \gamma_i H_{i+1}^j\big],\end{aligned} \tag{3.14}$$

for $i = 1, 2, 3, ..., m$ and $j = 0, 1, 2, ..., n$.
For CNFVM, we put $\theta = \frac{1}{2}$ in equation (3.14) and then we obtain the following equation

$$-\frac{\Delta\tau\alpha_i}{2}H_{i-1}^{j+1} + \Big(1 - \frac{\Delta\tau\beta_i}{2}\Big)H_i^{j+1} - \frac{\Delta\tau\gamma_i}{2}H_{i+1}^{j+1} = \frac{\Delta\tau\alpha_i}{2}H_{i-1}^{j} + \Big(1 + \frac{\Delta\tau\beta_i}{2}\Big)H_i^{j} + \frac{\Delta\tau\gamma_i}{2}H_{i+1}^{j}. \tag{3.15}$$

The matrix form of equation (3.15) can be written as

$$A\underline{H}^{(j+1)} = B\underline{H}^{(j)} + \underline{W}, \tag{3.16}$$

where,

$$\underline{H}^j = \Big[H_1^j, H_2^j, ..., H_m^j\Big]^T,$$

$$\underline{W} = \Big[\frac{\Delta\tau\alpha_2}{2}(H_1^{j+1} + H_1^j), 0, ....., 0, \frac{\Delta\tau\gamma_m}{2}(H_{m+1}^{j+1} + H_{m+1}^j)\Big]^T,$$

$$A = \begin{bmatrix} 1 - \frac{\Delta\tau\beta_2}{2} & -\frac{\Delta\tau\gamma_2}{2} & 0 & \cdots & \cdots & 0 \\ -\frac{\Delta\tau\alpha_3}{2} & 1 - \frac{\Delta\tau\beta_3}{2} & -\frac{\Delta\tau\gamma_3}{2} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & -\frac{\Delta\tau\gamma_{m-1}}{2} \\ 0 & \cdots & \cdots & 0 & -\frac{\Delta\tau\alpha_m}{2} & 1 - \frac{\Delta\tau\beta_m}{2} \end{bmatrix}, \quad \text{and}$$

$$B = \begin{bmatrix} 1 + \frac{\Delta\tau\beta_2}{2} & \frac{\Delta\tau\gamma_2}{2} & 0 & \cdots & \cdots & 0 \\ \frac{\Delta\tau\alpha_3}{2} & 1 + \frac{\Delta\tau\beta_3}{2} & \frac{\Delta\tau\gamma_3}{2} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \frac{\Delta\tau\gamma_{m-1}}{2} \\ 0 & \cdots & \cdots & 0 & \frac{\Delta\tau\alpha_m}{2} & 1 + \frac{\Delta\tau\beta_m}{2} \end{bmatrix}.$$

# 4 Numerical Implementation and Result Discussions

In this section, we use the following four data sets to evaluate the values of European call option. For all data sets, we choose, space grid points $M = 1000$ and time steps $N = 1000$ (we can also take $m \neq n$).

**Data Set 1**: $\{K = 50, \quad T = 0.25, \quad r = 0.12, \quad \sigma = 0.3, \quad t = 0, \quad 45 \leq S_0 \leq 55\}$ [2].

Now substituting the above parameters into equation (3.16) and (A.6) and using the conditions for European call option, we get

Table 1: Option values of BS (exact), CNFVM and CNFDM for different stock prices ($S_0$).

| **Stock price** | **BS** | **CNFVM** | **CNFDM** |
|---|---|---|---|
| 45.00 | 1.37922193 | 1.37904034 | 1.37903217 |
| 46.00 | 1.74105788 | 1.74084639 | 1.74083780 |
| 47.00 | 2.15837795 | 2.15814318 | 2.15813433 |
| 48.00 | 2.63165952 | 2.63140932 | 2.63140038 |
| 49.00 | 3.16026469 | 3.16000738 | 3.15999853 |
| 50.00 | 3.74254380 | 3.74228749 | 3.74227889 |
| 51.00 | 4.37598412 | 4.37573617 | 4.37572794 |
| 52.00 | 5.05738676 | 5.05715330 | 5.05714557 |
| 53.00 | 5.78305500 | 5.78284073 | 5.78283356 |
| 54.00 | 6.54897964 | 6.54878771 | 6.54878115 |
| 55.00 | 7.35100983 | 7.35084193 | 7.35083602 |

Table 1 represents the the approximate values of European call option acquired by using CNFVM and CNFDM respectively for different stock prices. Comparing the approximate values with the exact (BS) values, we observe that the values of both CNFVM and CNFDM are very close to BS values.

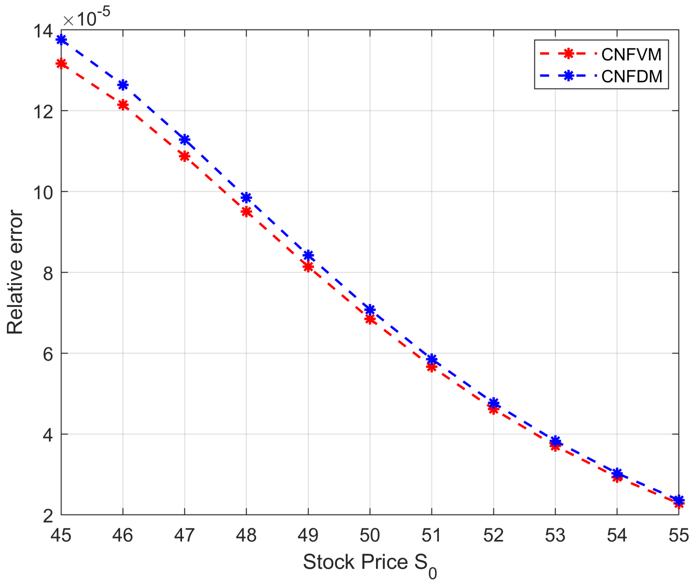


Figure 1: Comparison between the relative errors of CNFVM and CNFDM for different stock prices ($S_0$).

Figure 1 shows up the relative errors found by both CNFVM and CNFDM for various stock prices. We have clearly seen that CNFVM produces less relative errors than CNFDM.

**Data Set 2**: $\{S_0 = 50, \quad K = 50, \quad T = 0.25, \quad r = 0.12, \quad t = 0, \quad 15\% \leq \sigma \leq 60\%\}$ [2].

Now substituting the above parameters into equation (3.16) and (A.6) and using the conditions for European call option, we get

Table 2: Option values of BS (exact), CNFVM and CNFDM for different volatilities ($\sigma$).

| **Volatility** (%) | **BS** | **CNFVM** | **CNFDM** |
|---|---|---|---|
| 15 | 2.32871664 | 2.32824987 | 2.32815620 |
| 20 | 2.79093858 | 2.79056799 | 2.79053033 |
| 25 | 3.26407399 | 3.26377008 | 3.26375267 |
| 30 | 3.74254380 | 3.74228749 | 3.74227889 |
| 35 | 4.22385349 | 4.22363256 | 4.22362828 |
| 40 | 4.70670169 | 4.70650798 | 4.70650602 |
| 45 | 5.19032644 | 5.19015422 | 5.19015353 |
| 50 | 5.67423841 | 5.67408099 | 5.67407984 |
| 55 | 6.15809815 | 6.15791899 | 6.15790135 |
| 60 | 6.64165420 | 6.64123160 | 6.64109339 |

Table 2 illustrates the European call option prices obtained by using CNFVM, CNFVM and also BS (exact) for different values of volatility. We observe that CNFVM generates better results than CNFDM.

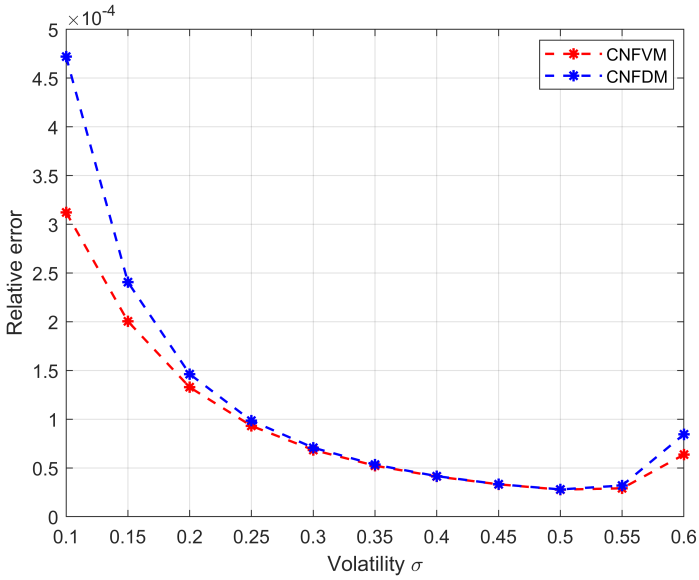


Figure 2: Comparison between the relative errors of CNFVM and CNFDM for different volatilities ($\sigma$).

From the Figure 2, we can conclude that CNFVM gives more accurate results than CNFDM for different volatilities..

**Data Set 3**: $\{S_0 = 50, \quad K = 50, \quad T = 0.25, \quad \sigma = 0.3, \quad t = 0, \quad 10\% \leq r \leq 20\%\}$ [2].

Now substituting the above parameters into equation (3.16) and (A.6) and using the conditions for European call option, we get

Table 3: Option values of BS (exact), CNFVM and CNFDM for different rates of interest ($r$).

| Rate of interest (%) | BS | CNFVM | CNFDM |
|---|---|---|---|
| 10 | 3.61044507 | 3.61018635 | 3.61018105 |
| 11 | 3.67617508 | 3.67591751 | 3.67591065 |
| 12 | 3.74254380 | 3.74228749 | 3.74227889 |
| 13 | 3.80954608 | 3.80929115 | 3.80928064 |
| 14 | 3.87717657 | 3.87692312 | 3.87691053 |
| 15 | 3.94542973 | 3.94517788 | 3.94516305 |
| 16 | 4.01429985 | 4.01404969 | 4.01403247 |
| 17 | 4.08378101 | 4.08353267 | 4.08351289 |
| 18 | 4.15386715 | 4.15362071 | 4.15359823 |
| 19 | 4.22455201 | 4.22430757 | 4.22428224 |
| 20 | 4.29582916 | 4.29558683 | 4.29555852 |

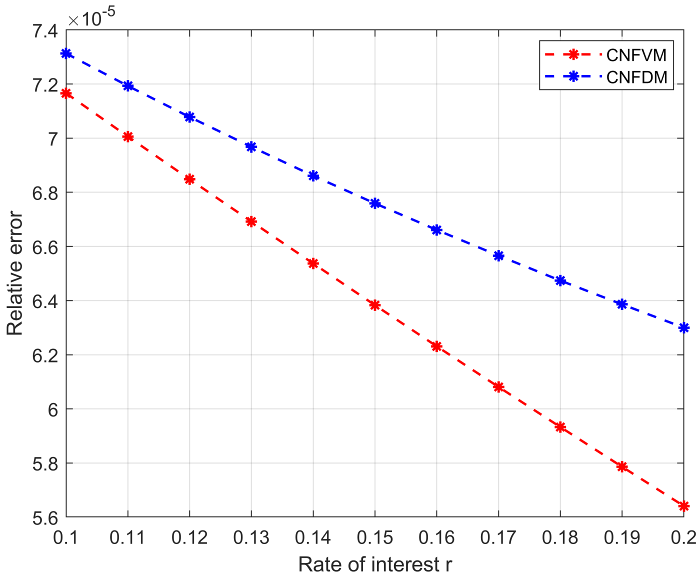


Figure 3: Comparison between the relative errors of CNFVM and CNFDM for different rates of interest ($r$).

Table 3 represents the European call option values obtained by using BS(exact), CNFVM and CNFDM for various rates of interest. The relative errors produced by CNFVM and CNFDM are graphically presented in the Figure 3. We have observed from the Figure 3 that CNFVM generates smaller absolute errors than CNFDM.

**Data Set 4**: $\{S_0 = 50, \quad K = 50, \quad r = 0.12, \quad \sigma = 0.3, \quad t = 0, \quad 0.08 \leq K \leq 0.83\}$ [2].

Now substituting the above parameters into equation (3.16) and (A.6) and using the conditions for European call option, we get

Table 4: Option values of BS (exact), CNFVM and CNFDM for different times to maturity ($T$).

| **Time to maturity** | **BS** | **CNFVM** | **CNFDM** |
|---|---|---|---|
| 0.08 | 1.97852921 | 1.97807404 | 1.97806897 |
| 0.17 | 2.94446527 | 2.94414744 | 2.94414034 |
| 0.25 | 3.74254380 | 3.74228749 | 3.74227889 |
| 0.33 | 4.45281585 | 4.45259662 | 4.45258679 |
| 0.42 | 5.10624069 | 5.10604703 | 5.10603614 |
| 0.50 | 5.71867443 | 5.71849979 | 5.71848798 |
| 0.58 | 6.29952510 | 6.29936500 | 6.29935219 |
| 0.67 | 6.85491654 | 6.85476462 | 6.85474925 |
| 0.75 | 7.38909920 | 7.38893457 | 7.38890796 |
| 0.83 | 7.90516630 | 7.90491991 | 7.90485067 |

European call option prices, for several times to maturity, attained by BS(exact), CNFVM, CNFDM, are demonstrated in the Table 4. Here, the results produced by CNFVM and CNFDM are very close to BS (exact) values. But CNFVM gives more accurate results.

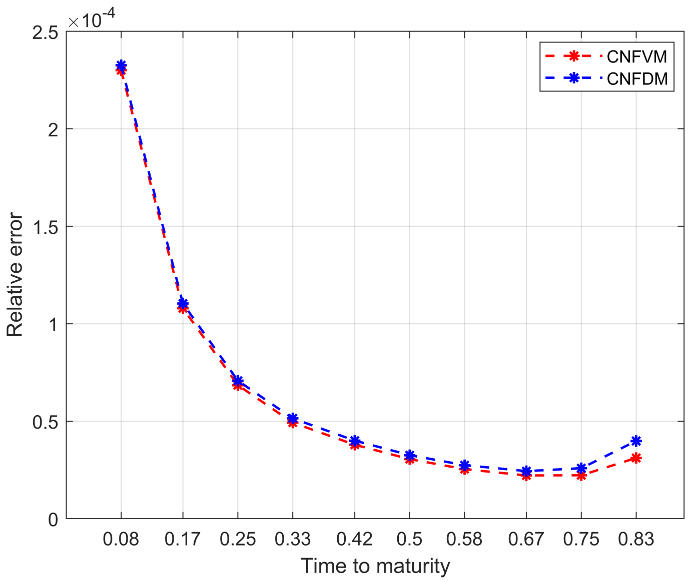


Figure 4: Comparison between the relative errors of CNFVM and CNFDM for different times to maturity ($T$).

The relative errors created by CNFVM and CNFDM are shown graphically in the Figure 4. From the figure, we have seen that the results are more accurate when obtained by CNFVM.

## 5 Conclusion

From the outcomes of our paper, we can infer that among CNFVM and CNFDM, CNFVM is the best scheme and gives us more precise outcome than CNFDM. Moreover, we can improve the approximations of these methods by increasing the values of m and n. In this work, all the results are shown graphically with the help of MATLAB.

| **Nomenclature** | |
|---|---|
| $c$ | Call premium |
| $p$ | Put premium |
| $S$ | Current stock price of underlying asset |
| $K$ | Strike price |
| $t$ | Option expiration |
| $\sigma$ | Volatility |
| $T$ | Time to maturity |
| $\rho$ | Flux |
| $r$ | Risk-free interest rate |
| $N$ | Cumulative standard normal distribution |

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# APPENDIX A: Discretization of BS PDE by Crank-Nicolson FDM

Assuming a reversal of time, equation (2.1) becomes

$$\frac{\partial H}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 H}{\partial S^2} + rS\frac{\partial H}{\partial S} - rH, \tag{A.1}$$

where, $\tau = T - t$, $T$ is maturity time, $t$ is current time i.e. $\tau$ denotes the remaining life time and $\tau \in \big(0, T\big)$. It is a second order PDE in S-space and first order in time. The intervals $\big[0, T\big]$ and $\big[0, S_{max}\big]$ are first divided into $N$ equal sub-intervals of length $\Delta\tau = \frac{T}{N}$ and $M$ equal sub-intervals of length $\Delta S = \frac{S_{max}}{M}$, respectively. A grid with a total of $(M+1)(N+1)$ points is defined by the time and stock price points.

Now, we use the following finite difference approximations for the time derivative $\frac{\partial H}{\partial t}$ and the stock price derivatives $\frac{\partial H}{\partial S}$, $\frac{\partial^2 H}{\partial S^2}$ as

$$\frac{\partial H}{\partial \tau} = \frac{H_i^{n+1} - H_i^n}{\Delta\tau}, \tag{A.2}$$

$$\frac{\partial H}{\partial S} = \frac{1}{2}\left[\frac{H_{i+1}^n - H_{i-1}^n}{2\Delta S} + \frac{H_{i+1}^{n+1} - H_{i-1}^{n+1}}{2\Delta S}\right], \tag{A.3}$$

$$\frac{\partial^2 H}{\partial S^2} = \frac{1}{2}\left[\frac{H_{i+1}^n - 2H_i^n + H_{i-1}^n}{\big(\Delta S\big)^2} + \frac{H_{i+1}^{n+1} - 2H_i^{n+1} + H_{i-1}^{n+1}}{\big(\Delta S\big)^2}\right], \tag{A.4}$$

Substituting equations (A.2), (A.3) and (A.4) into equation (A.1) and letting $S = i\Delta S$, we obtain,

$$\begin{aligned}\frac{H_i^{n+1} - H_i^n}{\Delta\tau} =& \frac{1}{4}\sigma^2 i^2 \big(\Delta S\big)^2 \left(\frac{H_{i+1}^{n+1} - 2H_i^{n+1} + H_{i-1}^{n+1}}{\big(\Delta S\big)^2}\right) + \frac{ri\Delta S}{2}\left(\frac{H_{i+1}^{n+1} - H_{i-1}^{n+1}}{2\Delta S}\right) \\ &- \frac{r}{2}H_i^{n+1} + \frac{1}{4}\sigma^2 i^2 \big(\Delta S\big)^2 \left(\frac{H_{i+1}^n - 2H_i^n + H_{i-1}^n}{\big(\Delta S\big)^2}\right) \\ &+ \frac{ri\Delta S}{2}\left(\frac{H_{i+1}^n - H_{i-1}^n}{2\Delta S}\right) - \frac{r}{2}H_i^n,\end{aligned}$$

Rearranging we get Crank-Nicolson scheme

$$-u_i H_{i-1}^{n+1} + (1+v_i)H_i^{n+1} - w_i H_{i+1}^{n+1} = u_i H_{i-1}^n + (1-v_i)H_i^n + w_i H_{i+1}^n, \tag{A.5}$$

for $i = 1, 2, 3, ..., M-1$ and $n = 1, 2, 3, ..., N-1$.

where,

$$u_i = \frac{1}{4}\Delta\tau\Big(\sigma^2 i^2 - ri\Big),$$

$$v_i = \frac{1}{2}\Delta\tau\Big(\sigma^2 i^2 + r\Big),$$

$$w_i = \frac{1}{4}\Delta\tau\Big(\sigma^2 i^2 + ri\Big),$$

Equation (A.5) can be written in the matrix form as,

$$X\underline{H}^{(n+1)} = Y\underline{H}^{(n)} + \underline{Z}, \tag{A.6}$$

where,

$$\underline{H}^n = \Big[H_1^n, H_2^n, ..., H_{M-1}^n\Big]^T, \quad \underline{Z} = \Big[u_1(H_0^{n+1} + H_0^n), 0, ....., 0, w_{M-1}(H_M^{n+1} + H_M^n)\Big]^T,$$

$$X = \begin{bmatrix} 1+v_1 & -w_1 & 0 & \dots & \dots & 0 \\ -u_2 & 1+v_2 & -w_2 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & -w_{M-2} \\ 0 & \dots & \dots & 0 & -u_{M-1} & 1+v_{M-1} \end{bmatrix}, \text{and } Y = \begin{bmatrix} 1-v_1 & w_1 & 0 & \dots & \dots & 0 \\ u_2 & 1-v_2 & w_2 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & w_{M-2} \\ 0 & \dots & \dots & 0 & u_{M-1} & 1-v_{M-1} \end{bmatrix}.$$