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Forest Dynamics and the Analysis of a Reaction-Diffusion Forest Model

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ABSTRACT

In this research, we study the dynamics of ecological models governed by differential equations. Ecology provides insight into the interaction between forest organisms, and interrelated processes regarding flora, fauna, and numerous other species. In particular, the results support population thresholds necessary for survival in some instances. Subsequently, we explore the response of mangroves of different regions to natural calamities and man-made disasters. A reaction-diffusion model has been developed to see the dynamics of the tidal woods and wetlands. Sundarbans, the largest mangrove of Bangladesh, are victim to a vast amount of hazardous events. This mathematical study of mangroves is unprecedented in such a manner. Furthermore, we continue the study of a forest ecosystem governed by an age-structured parabolic-ordinary system. We study some properties of an abstract parabolic equation, the dynamical system, and the limit sets. The model demonstrates a Lyapunov function which denotes some vital properties of the limit sets.

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1 Introduction

Ecology provides insight into the interaction between forest organisms, and interrelated processes regarding flora, fauna, and numerous other species [1, 2, 3, 4, 5, 6]. Over the past couple of decades, due to the present threats of climate change and global warming, there has been extensive literature and research on theoretical ecology. The study combine pioneer research works in the field of forest ecosystems and theoretical ecology [7, 8, 9, 10]. Forest ecosystems are resilient in nature, nonetheless hazards at time occur at such massive immensity it goes beyond the capacity of the forest species leading to loss of vital functionality of ecosystem [11, 12, 13]. While some ecosystems have reacted slowly to disturbances, drastic changes were observed in others.

Reaction-diffusion's systems are of growing interest to explore the nature of various habitat fragments in nature. One study demonstrated the effects of two distribution functions when two species are distributed with their corresponding resource function [14, 15, 16, 17]. If the diffusion rate is sufficiently large trajectories of

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this study tend to be uniformly promulgated, whereas nonuniform distribution leads to questionable consequences [18, 19, 20]. The discrepancy between the directed and regular diffusion strategy for the multi-species competition model was considered in [14, 21, 22].

In this study, we consider an age-structured forest partial differential equation model with two age groups and involve seed dynamics with a diffusion equation. A simpler version of this model has been introduced in which a cross-diffusion term was considered. Ecological mechanisms regarding wavefront propagation and the existence of solutions for forest areas were studied in this research [23]. In [24] an age structured reaction-diffusion-advection model has been taken into account. The advection equation governs atmospheric moisture dynamics over forested areas to emphasis examples such as the Amazon forest. Water resource and evapotranspiration plays a vital role regarding the good health of the forest. It is worth noting that, discrepancy between techniques aids to better estimates of moisture conservation in a specific area [11].

This paper main objectives are:

1. Create an age-structured reaction-diffusion model that integrates the influences of atmospheric moisture transport and water resources.

2. The first goal is to provide some analytical results based on different diffusion rates. It also provides the local stability analysis, and global asymptotic stability.

3. To focus on the stability of a particular stationary solution.

We consider the following PDE model to describe the forest dynamics [25, 26, 27, 28]:

$$\begin{cases} \dfrac{\partial p(t,x)}{\partial t} = b\eta s - k(q)p - \omega p, & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial q(t,x)}{\partial t} = \omega p - \gamma q, & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial s(t,x)}{\partial t} = D\Delta s - \eta s + \mu q, & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial s}{\partial n} = 0, & \text{in } \partial\Omega \times (0,\infty). \end{cases} \tag{1.1}$$

With initial conditions, $p(x,0) = p_0(x)$, $q(x,0) = q_0(x)$, and $s(x,0) = s_0(x)$ in a two-dimensional bounded domain Ω. In this study, the functions $p(t,x)$ denote young age trees such as saplings, shrubs, $q(t,x)$ demonstrates mature plantation and the seed density is given by $s(t,x)$. The dynamics of seed density includes diffusion rate D which plays a vital role in this forest kinematic model. The seed production rate and establishment rate is denoted by b and η. The competition between old trees and saplings are given by the quadratic function $k(q) = \alpha(q-h)^2 + z$, which is dependent on old tree density. Moreover, the growth rate of old trees is ω, with mortality rate is γ.

2 Cauchy Problem

We rewrite the system (1.1) as the following Cauchy problem for an abstract evolution equation,

$$\begin{cases} \dfrac{dP}{dt} + AP = F(P), \quad 0 < t < \infty \\ P(0) = P_0, \end{cases} \tag{2.1}$$

for X defined as,

$$X = \left\{ \begin{pmatrix} p \\ q \\ q \end{pmatrix} ; p, q \in L_\infty(\Omega), s \in L_2(\Omega) \right\} \tag{2.2}$$

Furthermore, the initial function space is given by,

$$\mathcal{I} = \left\{ \begin{pmatrix} p \\ q \\ q \end{pmatrix} ; 0 \leqslant p, q \in L_\infty(\Omega), 0 \leqslant s \in L_2(\Omega) \right\} \tag{2.3}$$

A is the sectorial operator of the product space X defined on the domain $\mathscr{D}(A)$ as a diagonal matrix,

$$A = \left\{ \begin{pmatrix} \omega & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & b \end{pmatrix}; p, q \in L_\infty(\Omega), s \in L_2(\Omega) \right\} \tag{2.4}$$

b is a positive definite self-adjoint operator defined as $b = -D\Delta + \eta$ in $L_2(\Omega)$ equipped with homogeneou $H_N^2(\Omega)$s Neumann boundary conditions $\dfrac{\partial s}{\partial n}$ on $\partial\Omega$ with domain $H_N^2(\Omega)$, where $H_N^2(\Omega)$ is a closed subspace of $H^2(\Omega)$ consisting of functions satisfying the boundary conditions [7]. Now, domain for Λ is,

$$\mathscr{D}(\Lambda^\theta) = \begin{cases} H^{2\theta}(\Omega), & 0 \leqslant \theta < 3/4, \\ H_N^{2\theta}(\Omega), & 3/4 < \theta \leqslant 1. \end{cases} \tag{2.5}$$

The nonlinear operator F is defined as, $\mathscr{D}(A^\nu)$ for $1/2 < \nu < 1$,

$$F(P) = \begin{pmatrix} b\eta s - k(q)p \\ \omega p \\ \mu q \end{pmatrix}, \quad P = \begin{pmatrix} p \\ q \\ s \end{pmatrix} \in \mathscr{D}(A^\nu). \tag{2.6}$$

3 Global Solutions

Let us consider the local solution of the system (2.1) on $[0, T_U]$ in the function space,

$$\left\{ \begin{array}{l} 0 \leqslant p, q \in \mathcal{C}([0, T_U]; L_\infty(\Omega)) \cap \mathcal{C}^1((0, T_U]; L_\infty(\Omega)), \\ 0 \leqslant s \in \mathcal{C}([0, T_U]; L_2(\Omega)) \cap \mathcal{C}((0, T_U]; H_N^2 \cap \mathcal{C}^1([0, T_U]; L_2(\Omega)). \end{array} \right. \tag{3.1}$$

Then the priori estimates for local solutions can be established [25],

Proposition 1. *In the function space (3.1) then for a constant $C > 0$ and exponent $r > 0$ the following inequality holds for any local solution U.*

$$\|P(t)\|_X \leqslant C[e^{-rt}\|P_0\|_X + 1], \quad 0 \leqslant t \leqslant T_U, \tag{3.2}$$

Considering the priori estimates for the local solutions of the equation (3.2) from the Proposition 1 with initial conditions $P_0 \in \mathcal{I}$, we state the next theorem.

Theorem 1. *[25] The system (2.1) has a unique global solution $P = (p, q, s)$ in the function space,*

$$\left\{ \begin{array}{l} 0 \leqslant p, q \in \mathcal{C}([0, \infty); L_\infty(\Omega)) \cap \mathcal{C}_1((0, \infty); L_\infty(\Omega)), \\ 0 \leqslant s \in \mathcal{C}([0, \infty); H_N^2(\Omega) \cap \mathcal{C}((0, \infty); L_2(\Omega)) \cap \mathcal{C}_1((0, \infty) \cap L_2(\Omega)). \end{array} \right. \tag{3.3}$$

The first and second equation of the model (1.1) satisfy the equations,

$$p(t) = e^{-\int_0^t [k(q)q(v)+\omega]dv} p_0 + b\eta \int_0^t e^{-\int_v^t [k(q)q(\tau)+\omega]d\tau} s(v)dv, \quad 0 \leqslant t < \infty, \tag{3.4}$$

$$q(t) = e^{-t\Lambda} q_0 + \omega \int_0^t e^{-(t-v)\gamma} p(v)dv, \quad 0 \leqslant t < \infty. \tag{3.5}$$

For the seed density equation of the model (1.1) the solution satisfies the integral equation,

$$s(t) = e^{-t\Lambda} s_0 + \mu \int_0^t e^{-(t-v)\Lambda} q(v)dv, \quad 0 \leqslant t < \infty, \tag{3.6}$$

Here, $e^{-t\Lambda}$ denotes the linear semigroup by Λ.

Proposition 2. *Let, $P(t) = (p(t), q(t), s(t))$ be the global solution to the system (2.1) with $P_0 \in \mathcal{I}$, then the following estimates hold for a continuously increasing function, $f(.)$.*

$$\|p(t)\|_{L_\infty} \leqslant f(\|P_0\|_X), \quad 0 \leqslant t < \infty, \tag{3.7}$$

$$\|q(t)\|_{L_\infty} \leqslant f(\|P_0\|_X), \quad 0 \leqslant t < \infty, \tag{3.8}$$

$$\|s(t)\|_{L_2} \leqslant f(\|P_0\|_X), \quad 0 \leqslant t < \infty. \tag{3.9}$$

Proof. It is known that,

$$\|P(t)\|_{L_2} \leqslant f(\|P_0\|_{L_2}), \quad 0 \leqslant t < \infty.$$

It follows from equation (3.6),

$$\begin{aligned}
\|s(t)\|_{H^{2\nu}} &\leqslant C\left\{\|\Lambda^\nu e^{-t\Lambda} s_0\| + \int_0^t \|\Lambda^\nu e^{-(-t-v)\Lambda} p(v)\|_{L_2} dv\right\} \\
&\leqslant C\left(1+t^{-\nu}\right) e^{-bt}\|s_0\|_{L_2} + f\|P_0\|_{L^2}\int_0^t \left(1+(t-v)^{-\nu}\right) e^{-\omega(t-v)} dv \\
&\leqslant C\left(1+t^{-\nu}\right) f\left(\|P_0\|_{L_2}\right), \quad 0 < t < \infty.
\end{aligned}$$

Since $\|s(t)\|_{L_\infty} < C\|s(t)\|_{H^{2\nu}}$, we obtain the desired result,

$$\|s(t)\|_{L_\infty} \leqslant (1+t^{-\nu}) f\|P_0\|_{L_2}, \quad 0 < t < \infty. \tag{3.10}$$

Now, we use equation (3.4) to obtain,

$$\|p(t)\|_{L_\infty} \leqslant \|p_0\|_{L_\infty} + \int_0^t e^{-\omega(t-v)}(1+v^{-\nu}) dv f\|P_0\|_{L_2} \leqslant f\|P_0\|_X, \quad 0 \leqslant t < \infty. \tag{3.11}$$

Similarly, we can obtain inequalities (3.7) and (3.8). □

Proposition 3. *For the derivative* $P'(t) = (p'(t), q'(t), s'(t))$,

$$\|p'(t)\|_{L_\infty} \leqslant (1+t^{-\nu}) f_1(\|P_0\|_X), \quad 0 < t < \infty, \tag{3.12}$$
$$\|q'(t)\|_{L_\infty} \leqslant f_1(\|P_0\|_X), \quad 0 < t < \infty, \tag{3.13}$$
$$\|s'(t)\|_{L_\infty} + \|s(t)\|_{H^2} \leqslant (1+t^{-1}) f_1(\|P_0\|_X), \quad 0 < t < \infty. \tag{3.14}$$

where $f_1(.)$ *is an appropriate continuously increasing function.*

Proposition 4. *For the second order derivative* $P''(t) = (p''(t), q''(t), s''(t))$,

$$\|p''(t)\|_{L_\infty} \leqslant (1+t^{-1-\nu}) f_2(\|P_0\|_X), \quad 0 < t < \infty, \tag{3.15}$$
$$\|q''(t)\|_{L_\infty} \leqslant (1+t^{-\nu}) f_2(\|P_0\|_X), \quad 0 < t < \infty, \tag{3.16}$$
$$\|s''(t)\|_{L_\infty} + \|s'(t)\|_{H^2} \leqslant (1+t^{-2}) f_2(\|P_0\|_X), \quad 0 < t < \infty. \tag{3.17}$$

where $f_2(.)$ *is an appropriate continuously increasing function.*

4 Dynamical System

Let $P(t) = P(t, P_0)$ be the global solution of (2.1) for any $P_0 \in \mathcal{I}$ in the space (3.3). Then, from (3.2), the following is obtained,

$$\|P(t, P_0)\|_X \leqslant C[e^- rt\|P_0\|_X + 1], \quad 0 \leqslant t \leqslant \infty, \quad P_0 \in \mathcal{I}. \tag{4.1}$$

Since, Now, let $G(t)P_0 = P(t, P_0)$ for $0 \leqslant t < \infty$, where $G(t)$ is a nonlinear semigroup on the initial condition space $\mathcal{I}$. Since the semigroup is continuous on $\mathcal{I}$ and equation (4.1) satisfies the system it can be concluded that system (2.1) is a dynamical system $(G(t), \mathcal{I}, X)$ [7, 8].

4.1 Lyapunov Function

Here, we develop a Lyapunov function $V(P)$ for a dynamical system $(G(t), \mathcal{I}, X)$ and let $G(t)P_0 = P(t, P_0)$. Let,

$$\begin{aligned}
&\chi(t) = \omega p - \gamma q \\
&\Rightarrow \frac{\partial \chi(t)}{\partial t} = \omega \frac{\partial p}{\partial t} - \gamma \frac{\partial q}{\partial t} \\
&\Rightarrow \frac{\partial \chi(t)}{\partial t} = \omega[b\eta s - k(q)p - \omega p] - \gamma[\omega p - \gamma q]
\end{aligned}$$

$$\Rightarrow \frac{\partial \chi(t)}{\partial t} = \omega b\eta s - [k(q)+\omega+\gamma]\chi - \gamma[k(q)+\omega]q. \tag{4.2}$$

We multiply (4.2) by $\chi(t) = \frac{\partial q}{\partial t}$ and then integrate over Ω

$$\int_\Omega \chi(t)\frac{\partial \chi(t)}{\partial t}dx = \int_\Omega \left[\omega b\eta s\frac{\partial q}{\partial t} - [k(q)+\omega+\gamma]\chi^2 - \gamma[k(q)+\omega]q\frac{\partial q}{\partial t}\right]dx$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\int_\Omega \chi^2 dx + \gamma\frac{d}{dt}\int_\Omega \Gamma(q)dx - \omega b\eta\int_\Omega \frac{\partial q}{\partial t}sdx = -\int_\Omega [k(q)+\omega+\gamma]\left(\frac{\partial q}{\partial t}\right)^2 dx, \tag{4.3}$$

where $\Gamma(q) = \int_0^q [k(q)q+\omega q]dq$.
Now, we multiply the equation of s in (2.1) by $\frac{\partial s}{\partial t}$ and integrate the product over Ω and obtain,

$$-\int_\Omega \left(\frac{\partial s}{\partial t}\right)^2 dx = \frac{d}{2}\frac{d}{dt}\int_\Omega (|\nabla s|)^2 dx + \frac{b}{2}\frac{d}{dt}\int_\Omega s^2 dx - \mu\int_\Omega q\frac{\partial s}{\partial t}dx. \tag{4.4}$$

Equating equation (4.3) and (4.4)

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \left[\frac{\mu}{2}\chi^2 + \frac{d\omega b\eta}{2}|\nabla s|^2 + \gamma\mu\Gamma(q) + \frac{\omega b^2\eta}{2}s^2 - (\omega\mu b\eta)qs\right]dx\\ &= -\int_\Omega \left\{\mu[k(q)+\omega\gamma]\left(\frac{\partial q}{\partial t}\right)^2 + \omega b\eta\left(\frac{\partial s}{\partial t}\right)^2\right\}dx \le 0.\end{aligned} \tag{4.5}$$

Here, we note that,

$$\frac{\mu}{2}(\omega p - \gamma q)^2 + \mu\Gamma(q) + \frac{\omega b^2\eta}{2}s^2 - (\omega\mu bk)qs \ge C, \quad \text{for all } q, s \ge 0,$$

with some constant $C > 0$. So, the functional given by

$$V(P) = \int_\Omega \left[\frac{\mu}{2}(\omega p - \gamma q)^2 + \frac{f\omega b\eta}{2}|\nabla s|^2 + \gamma\mu\Gamma(q) + \frac{\omega b^2\Gamma}{2}s^2 - (\omega\mu b\eta)qs\right]dx, \tag{4.6}$$

becomes a Lyapunov function for the dynamical system $(G(t), \mathcal{I}, X)$, where $P \in \mathscr{D}(A^{1/2})$ [25]. The Lyapunov function provides the following two propositions.

Proposition 5. *For any solution $G(t)P_0 = P(t)$, the following inequality holds,*

$$\int_1^\infty \left\|\frac{dP}{dt}(t)\right\|_{L_2}^2 dt < \infty.$$

Proof. Integrating both sides of eqref on the interval $[1, T]$,

$$\begin{aligned}&\int_1^T \int_\Omega \left\{\mu[k(q)+\omega+\gamma]\left(\frac{\partial q}{\partial t}\right)^2 + \omega b\eta\left(\frac{\partial s}{\partial t}\right)^2\right\}dxdt \leqslant\\ &\int_\Omega \left[\frac{\mu}{2}\chi(1)^2 + \frac{D\omega b\eta}{2}|\delta s(1)|^2 + \gamma\mu\Gamma(q(1)) + \frac{\omega b^2\eta}{2}s(1)^2 + \omega\mu b\eta q(T)s(T)\right]dx.\end{aligned} \tag{4.7}$$

Utilizing equation (3.2) we have,

$$\int_1^T \int_\Omega \left\{\mu[k(q)+\omega+\gamma]\left(\frac{\partial q}{\partial t}\right)^2 + \omega b\eta\left(\frac{\partial s}{\partial t}\right)^2\right\}dxdt \leqslant \infty. \tag{4.8}$$

Differentiating the equation of p from (1.1) with respect to t, we get,

$$\frac{\partial^2 p}{\partial t^2} = b\eta\frac{\partial s}{\partial t} - k'(q)\frac{\partial q}{\partial t} - [k(q)+\omega]\frac{\partial p}{\partial t}$$

$$\Rightarrow \frac{\partial^2 p}{\partial t^2} = b\eta \frac{\partial s}{\partial t} - 2ap(q-b)\frac{\partial q}{\partial t} - [k(q)+\omega]\frac{\partial p}{\partial t}. \tag{4.9}$$

We multiply equation (4.9) by $\dfrac{\partial p}{\partial t}$ and integrate the product over ω. Then,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\frac{\partial p}{\partial t}\right)^2 dx &= \int_\Omega \left(b\eta\frac{\partial s}{\partial t} - k'(q)p\frac{\partial q}{\partial t}\right)\frac{\partial p}{\partial t}dx - \int_\Omega (k(q)+\omega)\left(\frac{\partial p}{\partial t}\right)^2 dx \\ &\leqslant C(\|P_0\|_X{+}1)\int_\Omega \left[\left(\frac{\partial q}{\partial t}\right)^2 + \left(\frac{\partial s}{\partial t}\right)^2\right] dx - \frac{\omega}{2}\int_\Omega \left(\frac{\partial p}{\partial t}\right)^2 dx. \end{aligned} \tag{4.10}$$

Integrating the above inequality over $[1, T]$, we get

$$\frac{\omega}{2}\int_1^T \int_\Omega \left(\frac{\partial p}{\partial t}\right)^2 dxdt \leqslant \frac{1}{2}\int_\Omega \left(\frac{\partial p}{\partial t}(1)\right)^2 dx + C(\|P_0\|_X{+}1)\int_\Omega \left[\left(\frac{\partial q}{\partial t}\right)^2 + \left(\frac{\partial s}{\partial t}\right)^2\right] dxdt.$$

Using equation (4.8), we conclude that $\dfrac{\omega}{2}\displaystyle\int_1^\infty \int_\Omega \left(\frac{\partial p}{\partial t}\right)^2 dxdt < \infty$, completing our proof. □

Theorem 2. *For any solution $G(t)P_0 = P(t)$, as $t \to \infty$ the derivative $\frac{dP}{dt}(t) \to 0$ in the L_2 topology.*

Proof. Let us consider that as $t \to \infty$ the derivative $\dfrac{dP}{dt}(t)$ does not tend to 0 in the $L_2(\Omega)$. Then, there exists, $\epsilon > 0$ and a time sequence $t_n \to \infty$ such that,

$$\left\|\frac{dP}{dt}(t_n)\right\|^2_{L_2} \geqslant \epsilon, \quad n = 1, 2, 3, \ldots$$

Using propostion 3 and 4, we have,

$$\left|\frac{d}{dt}\left\|\frac{dP}{dt}(t)\right\|^2_{L_2}\right| = 2\left|\left(\frac{d^2P}{dt^2}(t), \frac{dP}{dt}(t)\right)_{L_2}\right| \leqslant M, \quad 1 \leqslant < \infty.$$

Where M is some constant. Furthermore, by mean-value theorem,

$$\left\|\frac{dP}{dt}(t_n)\right\|^2_{L_2} \geqslant \begin{cases} M\left(1 - t_n + \frac{\epsilon}{M}\right), & t_n - \frac{\epsilon}{M} \leqslant t < t_n, \\ -M\left(1 - t_n - \frac{\epsilon}{M}\right), & t_n \leqslant t < t_n + \frac{\epsilon}{M}, \end{cases} \tag{4.11}$$

Which is a contradiction to the fact that $\left\|\frac{dP}{dt}(t_n)\right\|^2_{L_2}$ is integrable in $(1, \infty)$. □

4.2 Limit Sets

In the study of dynamical systems, a limit set is the state a dynamical system reaches after an infinite amount of time has passed, by either going forward or backwards in time. In this study, limit sets will be denoted by, $\mathcal{W}$ and defined by,

$$\mathcal{W}(P_0) = \bigcap_{t\geqslant 0} \overline{\{G(t)P_0; t < \tau < \infty\}}.$$

Now, if there exists a time sequence $\{t_n\} \to \infty$ such that $G(t_n)P_0 \to \overline{P}$ if and only if $\overline{P} \in \mathcal{W}(P_0)$. Since, there exists a solution that despite starting from continuous initial conditions converges to a discontinuous stationary solution the limit set $\mathcal{W}(P_0) = \emptyset$.We now define the L_2 limit set, $L_2 - \mathcal{W}(P_0)$. A sequence $\{(p_n, q_n, s_n)\} \in X$ converges to $\{(p_0, q_0, s_0)\} \in X$ as $n \to \infty$, if

$$\begin{cases} p_n \to p_0 & \text{strongly in } L_2(\Omega), \\ q_n \to q_0 & \text{strongly in } L_2(\Omega), \\ s_n \to s_0 & \text{strongly in } L_2(\Omega). \end{cases}$$

To define the $s^* - \mathcal{W}-$ limit set of $G(t)P_0, \quad P_0 \in \mathcal{I}$, we define the weak topology for $L_\infty(\Omega)$. A sequence $\{(p_n, q_n, s_n)\} \in X$ converges to $\{(p_0, q_0, s_0)\} \in X$ as $n \to \infty$, if

$$\begin{cases} p_n \to p_0 & \text{strongly in } L_\infty(\Omega), \\ q_n \to q_0 & \text{strongly in } L_\infty(\Omega), \\ s_n \to s_0 & \text{strongly in } L_2(\Omega). \end{cases}$$

The limit set, $s^* - \mathcal{W}$ is given by,

$$s^* - \mathcal{W}(P_0) = \bigcap_{t \geqslant 0} \overline{\{G(t)P_0; t < \tau < \infty\}}.$$

Where the above set is the closure in the weak topology of X.

Theorem 3. *For each $P_0 \in \mathcal{I}, \mathcal{W}(P_0) \subset L_2 - \mathcal{W} \subset s^* - \mathcal{W}(P_0)$ is nonempty.*

Proof. By definition, the first relation $\mathcal{W}(P_0) \subset L_2 - \mathcal{W}(P_0)$ is true. Now, let, $\overline{P} = (\overline{p}, \overline{q}, \overline{s}) \in L_2 - \mathcal{W}(P_0)$. Then, there exists a sequence $\{t_n\} \to \infty$ such that, $P(t_n)P_0 = (p(t_n), q(t_n), s(t_n)) \to \overline{P}$ in the L_2 topoligy of X. Let $\psi \in L_1(\Omega)$. For any $g \in L_2(\Omega)$,

$$\left| \int_\Omega \psi\{p(t_n) - \overline{p}\} dx \right| \leqslant \|\psi - g\|_{L_1} \|p(t_n) - \overline{p}\|_{L_\infty} + \left| \int_\Omega g\{p(t_n) - \overline{p}\} dx \right|.$$

Since, $L_2(\Omega)$ is dense in $L_1(\Omega 0)$ and equation (3.7) holds, as $t \to \infty$,

$$\left| \int_\Omega \psi\{p(t_n) - \overline{p}\} dx \right| \to 0.$$

Thus, in the weak topology of $L_\infty(\Omega)$, $p(t_n) \to \overline{p}$ holds. Similarly it can be shown that, $q(t_n) \to \overline{q}$ is true. Thus, $\overline{P} \in s^* - \mathcal{W}(P_0)$. □

Theorem 4. *For $P_0 \in \mathcal{I}$, let there exists a time sequence $\{t_n\} \to \infty$ such that $G(t_n)P_0 \to (p(t_n), q(t_n), s(t_n))$ converges to the function $\overline{P} = (\overline{p}, \overline{q}, \overline{s}) \in X$ almost everywhere in Ω . Then, $\overline{P} \in L_2 - \mathcal{W}(P_0)$.*

Proof. Since equation (3.7), (3.8), and (3.10) are true and the almost everywhere convergence implies the convergence of L_2 for each sequence $(p(t_n), q(t_n)$, and $s(t_n)$. Therefore, $\overline{P} \in L_2 - \mathcal{W}(P_0)$ holds. □

Proposition 6. *For each $P_0 \in \mathcal{I}$, $L_2 - \mathcal{W}(P_0)$ is an invariant set of $G(t)$, i.e.,*

$$G(t)(L_2 - \mathcal{W}(P_0)) \subset L_2 - \mathcal{W}(P_0), \quad 0 < t < \infty.$$

Proof. Firstly, it is important to show that $G(t)$ is continuous from $\mathcal{I}$ into itself in the L_2 topology. Let, $(p_1(t), q_1(t), s_1(t))$ and $(p_2(t), q_2(t), s_2(t))$ be the solutions to the cauchy problem (2.1) with initial conditions $P_{01} = (p_{01}, q_{01}, s_{01})$ and $P_{02} = (p_{02}, q_{02}, s_{02})$ in $\mathcal{I}$ respectively. Let, $T > 0$. Then, from (3.4),

$$p_i(t) = e^{-\int_0^t [k(q_i)+\omega] dv} p_{01} + b\eta \int_0^t e^{-\int_v^t [k(q_i)+\omega] d\tau} s_i(\tau) d\tau, \quad 0 \leqslant t < \infty.$$

Subsequently,

$$\begin{aligned} p_2(t) - p_1(t) =& e^{-\int_0^t [k(q_1)+\omega] dv} (e^{-\int_0^t [k(q_2)-k(q_1) dv} - 1) p_{01} + \\ & e^{-\int_0^t [k(q_2)+\omega] dv} (p_{02} - p_{01}) + b\eta \int_0^t e^{-\int_0^t [k(q_2)+\omega] dv} (s_2(\tau) - s_1(\tau)) d\tau + \\ & b\eta \int_0^t e^{-\int_\tau^t [k(q_1)+\omega] dv} (e^{-\int_\tau^t [k(q_2)-k(q_1)] dv} - 1) s_1(\tau) d\tau. \end{aligned}$$

Considering (3.7), (3.8), and (3.9) it can be shown,

$$\|p_2(t) - p_1(t)\|_{L_2} \leqslant \|_{L_2} + Cf(\|P_{01}\|_X + \|P_{02}\|_X) * [\|e^{-\int_0^t [k(q_2)-k(q_1)] dv} - 1\|_{L_2}$$

$$+\int_0^t \|s_2(\tau)-s_1(\tau)\|_{L_2} d\tau + \int_0^t \|e^{-\int_0^t [k(q_2)-k(q_1)]dv} - 1\|_{L_2}(1+\tau^{-\nu})d\tau],$$
$$0 \leqslant t \leqslant T. \tag{4.12}$$

For any R, there exists a constant $C_R > 0$ such that $|e^{\xi}-1| \leqslant C_R|\xi|$ holds for all $|\xi| \leqslant R$. Using this estimate, it can be verified,

$$\|e^{-\int_0^t [k(q_2)-k(q_1)]dv} - 1\|_{L_2} \leqslant C_f(\|P_{01}\|_X + \|P_{02}\|_X) \int_0^t \|q_2(\tau)-q_1(\tau)\|_{L_2} d\tau.$$

Similarly,

$$\begin{aligned}
&\int_0^t \|e^{-\int_0^t [k(q_2)-k(q_1)]dv} - 1\|_{L_2} \tau^{-(1+\epsilon)/2} d\tau \\
&\qquad \leqslant C_f(\|P_{01}\|_X + \|P_{02}\|_X) \int_0^t \int_\tau^t \|q_2(u)-q_1(u)\|_{L_2} \tau^{-(1+\epsilon)/2} du d\tau \\
&\qquad \leqslant C_f(\|P_{01}\|_X + \|P_{02}\|_X) \int_0^t \int_\tau^t \|q_2(u)-q_1(u)\|_{L_2} du.
\end{aligned}$$

Hence, we obtain,

$$\begin{aligned}
\|p_2(t)-p_1(t)\|_{L_2} &\leqslant \|p_{02}-p_{01}\|_{L_2} + C_f\left(\|P_{01}\|_X + \|P_{02}\|_X\right) * \\
&\int_0^t \left\{\|q_2(\tau)-q_1(\tau)\|_{L_2} + \|s_2(\tau)-s_1(\tau)\|_{L_2}\right\} d\tau, \quad 0 \leqslant t \leqslant T.
\end{aligned} \tag{4.13}$$

Similarly, from (3.5) it can be shown that,

$$\|q_2(t)-q_1(t)\|_{L_2} \leqslant \|q_{02}-q_{01}\|_{L_2} + C\int_0^t \|p_2(\tau)-p_1(\tau)\|_{L_2} d\tau, \quad 0 \leqslant t \leqslant T. \tag{4.14}$$

Finally, from (3.6), we have,

$$\|s_2(t)-s_1(t)\|_{L_2} \leqslant \|s_{02}-s_{01}\|_{L_2} + \mu\int_0^t e^{-(t-\tau)}\{q_2(\tau)-q_1(\tau)\} d\tau$$
$$\|s_2(t)-s_1(t)\|_{L_2} \leqslant \|s_{02}-s_{01}\|_{L_2} + \mu\int_0^t \|q_2(\tau)-q_1(\tau)\|_{L_2} d\tau, \quad 0 \leqslant t \leqslant T. \tag{4.15}$$

Adding equations (4.13), (4.14), and (4.15) and applying Gronwall's inequality, we can conclude,

$$\begin{aligned}
&\|p_2(t)-p_1(t)\|_{L_2} + \|q_2(t)-q_1(t)\|_{L_2} + \|s_2(t)-s_1(t)\|_{L_2} \\
&\qquad \leqslant \|P_{02}-P_{01}\|_{L_2} e^{(C_f\|P_{01}\|_X + \|P_{02}\|_X)t}, \quad 0 \leqslant t \leqslant T.
\end{aligned} \tag{4.16}$$

□

Theorem 5. *For any $P_0 \in \mathcal{I}$, $L_2 - \mathcal{W}(P_0)$ consists of equilibrium of the dynamical system.*

Proof. Let, $\overline{P} = (\overline{p}, \overline{q}, \overline{s}) \in L_2 - \mathcal{W}(P_0)$. Then there exists a time sequence $\{t_n\} \to \infty$ such that $G(t_n)P_0 \to \overline{P}$ in the L_2 topology. Since $s(t_n)$ is a bounded sequence in $H^2(\Omega)$, we can take a subsequence $\{s'(t_n)\}$ of $\{s(t_n)\}$ such that $\{s'(t_n)\} \to \overline{s}'$ strongly in $H^1(\Omega)$, thus $s'(t_n) = \overline{s}'$. Also, since (3.7) and (3.8) holds it implies, in any L_p topology with $2 \leqslant p < \infty$, the following are also true.

$$p(t_n) \to \overline{p}, \text{ and } \quad q(t_n) \to \overline{q}.$$

These facts lead to the conclusion that the Lyapunov function $V(P(t_n'))$ given by equation (4.6) is convergent to $V(\overline{P})$ as $t_n' \to \infty$.

$$V(\overline{P}) = \lim_{n'\to\infty} V(P(t_n')) = \inf_{0 \leqslant t < \infty} V(G(t)P_0) \equiv V_\infty.$$

This implies that, $V(\overline{P}) \equiv V_\infty$ for all vectors, $\overline{P} \in L_2 - \mathcal{W}(P_0)$. Since, $G(t)\overline{P} \in L_2 - \mathcal{W}(P_0)$ for every $t > 0$, using Proposition 6 we have,

$$V(G(t)\overline{P}) \equiv V_\infty, \quad 0 < t < \infty, \overline{P} \in L_2 - \mathcal{W}(P_0).$$

Subsequently, let $G(t)\overline{P} = \overline{P}(t) = (\overline{p}(t), \overline{q}(t), \overline{s}(t))$. Then, using equation (4.5), we obtain,

$$\frac{d}{dt}V(\overline{P}(t)) = -\int_\Omega \left[\mu\{k(\overline{q}) + \omega + \gamma\}\left(\frac{\partial \overline{q}}{\partial t}\right) + \omega b\eta\left(\frac{\partial \overline{s}}{\partial t}\right)\right] dx \equiv 0, \quad 0 < t < \infty.$$

Hence, $\dfrac{\partial \overline{q}}{\partial t} \equiv \dfrac{\partial \overline{s}}{\partial t} \equiv 0$ for $0 < t < \infty$. Furthermore the second equation of the system (1.1) it follows, $\omega\overline{p}(t) \equiv \gamma\overline{q}(t)$. Thus, $\dfrac{\partial \overline{p}}{\partial t} \equiv 0$ for $0 < t < \infty$. Thus, the proof is complete as $G(t)\overline{P} \equiv \overline{P}$ and $\overline{P}$ must be an equilibrium of the system. $\square$

5 Conclusion

In this study, an abstract reaction-diffusion model has been studied for age-structured class of trees. A dynamical system was introduced for the reaction-diffusion model. Due to lack of smoothing of solutions of the young trees, p and old trees, q the underlying spaces were chosen cautiously. We study three kinds of $\mathcal{W}-$ limit sets for each point P_0 of the dynamical system. We construct a Lyapunov function for the system and utilize it to show that $L_2 - \mathcal{W}(P_0)$ consists of stationary solutions. In theoretical ecology, there are numerous mathematical models governing the growth and decay of individual trees, interaction between multiple species in a forest even other vital elements of nature have been considered. Theoretical results regarding reaction-diffusion models along with numerical illustrations have been developed in a number of studies [29, 30, 31, 32, 33].

We have considered a mathematical model to represent forest ecology while concentrating on mangroves. Some scenarios have resulted in the extinction of the forest ecosystem due to excessive predation, while in other cases the forest persisted. We did a rigorous theoretical analysis of a reaction-diffusion model. The dynamical behavior of this model is interesting as at the time the trajectories converge to discontinuous stationary solutions. However, the use of Lyapunov function is utilized to study limit sets of the model. The outcomes depict important characteristics of forest ecology and resemble real-life issues, thus playing a vital role in mathematical ecology and dynamical systems.

Here, we discuss some future opportunities for further development and research as open problems.

- We can formulate the model using time-dependent parameters, which will increase the accuracy of prediction
- Harvesting rate can be considered a function that is not proportional to the intrinsic growth rate.
- For the PDE model, we consider the Neumann boundary conditions, Dirichlet or mixed boundary conditions can be studied.
- This study can be extended for multiple species with harvesting. In the case of n-species competitions, the monotone dynamical systems theory is applicable, and all outcomes can be studied using principal eigenvalue analysis.
- In this study, we have considered logistic growth law, which can be studied for other growth laws such as the Gompertz model, Gilpin-Ayala model, Smith's model, etc.

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Conflict of interest

The authors declare no conflict of interest.

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