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Interior Hölder Regularity and Infinite Time Extinction to a doubly Nonlinear Degenerate Parabolic Equations of Signed Solutions

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ABSTRACT

We study doubly nonlinear parabolic equation with sign changing solutions. We established the Hölder regularity and the infinite time extinction of it within a parabolic domain.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ and for $T > 0$ define the cylindrical doamin $\Omega_T := \Omega \times (0, T]$. Consider the following doubly nonlinear parabolic equation

$$\partial_t(|u|^{p-2}u) - \operatorname{div}(|Du|^{p-2}Du) = 0 \quad \text{weakly in} \quad \Omega_T \tag{1.1}$$

where $\Delta_p u := \operatorname{div}(|Du|^{p-2}Du)$ is the p-Laplacian. For the case $p = 2$ then this operator transforms to well known heat equation. In this manuscript, the weak solution u is unknown and assumed to be locally bounded, real function which depends on both the time and space variables namely x and t in the cylindrical domain.

In our context, the term structural data indicates the parameters p and N. It is also assumed that the constant $\gamma > 0$, need to be evaluated quantitatively apriori in terms of the structural data.

1.1 Interior regularity

Denote $\Gamma_T := \partial\Omega_T - \bar{\Omega} \times \{T\}$ to be the parabolic boundary of the cylindrical domain Ω_T, and for any compact subset C of Ω_T parabolic p-distance from C to Γ by

$$\operatorname{dist}_p(C, \Gamma_T) \overset{def}{=} \inf_{\substack{(x,t)\in C \\ (y,s)\in\Gamma_T}} \left\{ |x-y| + |t-s|^{\frac{1}{p}} \right\}$$

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For $\varrho > 0$ denote the cube $K_\varrho(x_0)$ with center at $x_0 \in \mathbb{R}^N$ and edge ϱ. For $\theta > 0$, consider the following backward cylinders of the form

$$(x_0, t_0) + Q_\varrho(\theta) = (x_0, t_0) + K_\varrho(0) \times (-\theta\varrho^p, 0] = K_\varrho(x_0) \times (t_0 - \theta\varrho^p, t_0].$$

For the case $\theta = 1$, we will call it as Q_ϱ.

The conclusions to be derived from the discoveries presented in this article are summarized as follows.

Theorem 1.1. *Let's consider a bounded domain with a smooth boundary, denoted as $\partial\Omega$. Given that u constitutes a local weak solution bounded by (1.1) in Ω_T, it follows that u exhibits local Hölder continuity within Ω_T. Precisely, there exist constants $\gamma > 1$ and $\beta \in (0, 1)$, predetermined based on the data, such that for any compact subset $C \subset \Omega_T$, the inequality*

$$|u(x_1, t_1) - u(x_2, t_2)| \le \gamma \|u\|_{\infty,\Omega_T} \left(\frac{|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{p}}}{dist_p(C, \Gamma_T)} \right)^\beta ,$$

holds true for any pair of points $(x_1, t_1), (x_2, t_2) \in C$.

Theorem 1.2. *Assume that at the initial time $t = 0$, the function u has a value of u_0, which is an element of the Sobolev space $W_0^{1,p}(\Omega)$. Additionally, u_0 is non-negative, not constantly zero, and remains bounded within the domain Ω. If we consider u as a local weak solution to equation (1.1), then it implies that eventually, as time goes to infinity, the function u will completely vanish, meaning that $u(\cdot, t)$ approaches zero as time progresses indefinitely.*

The following oscillation decay will be demonstrated as part of the proof of the aforementioned theorem:

$$\operatorname*{ess\,osc}_{(x_0,t_0)+Q_r} u \le \gamma \operatorname*{ess\,osc}_{(x_0,t_0)+Q_\varrho} u \left(\frac{r}{\varrho} \right)^\beta ,$$

for any pair of cylinders $(x_0, t_0) + Q_r \Subset (x_0, t_0) + Q_\varrho \Subset \Omega_T$. A typical covering argument can be used to draw the conclusion of Theorem 1.1 at the end. A weak solution is defined in Definition 2.2, and [20] examines the weak solution's global existence.

1.2 Originality and importance

The equation (1.1) is a standard equation and is known as Trudinger's equation which is also famously termed as a doubly nonlinear parabolic equation due to the non-linearity of both the solution and its spatial gradient. It is of special interest to know why we are taking this type of equation for research as it has a splendid mathematical structure and generate mixed types of degeneracy and/or singularity in partial differential equations, and connection to physical models, including dynamics of glaciers [24], shallow water flows [1, 9, 12] and friction dominated flow in a gas network [22]. The Trudinger equation is also naturally connected to the non-linear eigenvalue problem $-\Delta_p u = \lambda |u|^{p-2} u$ [23], which plays an essential role in the nonlinear potential theory. The Hölder regularity of signed solutions studied by V. Bögelein, F. Duzzar and N. Liao in [2] for a more general equations with structure conditions. The Hölder regularity of this equation is studied by Trudinger [30, 31], via Moser's iteration, to possess a Harnack inequality for non-negative weak solutions, resembling to heat equation. The HÖlder regularity of nonnegative weak solutions is established by using this Harnack inequality in [10, 18, 19]. Now its turn to mention our contribution here, we eradicate the restriction of non-negativity of solutions instead we choose sign changing solutions to hold for the Hölder regularity. In our approach the Harnack inequality is not applicable as this estimate is valid only for non-negative solutions. We establish our desired goal using expansion of positivity.

Our proof's of interior Hölder regularity evolve along two key instances such as when the solution is close to zero or when it is away from zero, through comparisons between the oscillation and the supremum/infimum of the solution. For the former case, we will take the advantage of the scaling invariant property of our equation and obtain the expansion of positivity Proposition 3.1 without intrinsic scaling techniques [32] which parallels the classical parabolic theory in [21]. Whereas in the latter case, the solution behaves like the one to the parabolic p-Lalacian equation, i.e., $u_t = \Delta_p u$. Therefore, this second case rely upon the possibility to treat such a degenerate ($p > 2$) or singular ($1 < p < 2$) equation, for which we exploit the existing theory in

[4, 5, 6]. The existence of weak solution to (1.1) is shown in [26]. The Hölder regularity for doubly nonlinear equations has also been studied in [7, 13, 14, 15, 16, 33, 34]. The establishment of Hölder continuity for weak solutions to Trudinger's equation serves several visible objectives in the realm of mathematical analysis and partial differential equations. Some of these objectives include:

- Understanding regularity properties: Hölder continuity is a measure of the smoothness or regularity of a function. By proving Hölder continuity for weak solutions, mathematicians gain insights into the regularity properties of solutions to Trudinger's equation. This is crucial for understanding the behavior of these solutions in various contexts.
- Existence and uniqueness of solutions: The study of Hölder continuity contributes to establishing the existence and uniqueness of solutions to Trudinger's equation. Regularity results often play a fundamental role in proving the well-posedness of mathematical models, ensuring that solutions exist, are unique, and depend continuously on the data.
- Applicability of Sobolev spaces: Hölder continuity within a Sobolev space signifies that solutions possess a certain level of smoothness and fall within a well-defined function space. This is important for the applicability of Sobolev spaces in describing the behavior of solutions, allowing for a systematic and rigorous analysis of Trudinger's equation.
- Behavior of solutions in bounded domains: The restriction to a bounded domain in the establishment of Hölder continuity provides insights into the behavior of solutions within confined regions. This can have implications for problems arising in specific physical or mathematical contexts where the domain of interest is limited.
- Infinite-time extinction property: The observation of infinite-time extinction for nonnegative weak solutions is a specific characteristic of interest. Understanding this property contributes to the knowledge of the long-term behavior of solutions, which can be essential in applications where the evolution of certain quantities over time is a critical consideration.

2 Preliminaries

We establish certain notations and tools for technical analysis that will be utilized subsequently.

2.1 Notation

2.1.1 Concept of local weak solution

Let u be a function belonging to

$$u \in C(0,T; L^p_{\mathrm{loc}}(\Omega)) \cap L^p_{\mathrm{loc}}(0,T; W^{1,p}_{\mathrm{loc}}(\Omega)) \tag{2.1}$$

It is considered a local weak sub(super)-solution to (1.1) if, for every compact subset C of Ω and each sub-interval $[t_1,t_2] \subset (0,T]$

$$\int_C |u|^{p-2}u\;\zeta\;dx\Big|_{t_1}^{t_2} + \iint_{C\times(t_1,t_2)} [-|u|^{p-2}u\zeta_t + |Du|^{p-2}Du\cdot D\zeta]\,dx\,dt \le (\ge) 0 \tag{2.2}$$

holds for all non-negative test functions

$$\zeta \in W^{1,p}_{\mathrm{loc}}(0,T; L^p(C)) \cap L^p_{\mathrm{loc}}(0,T; W^{1,p}_0(C)).$$

ensuring the convergence of all integrals in (2.2). A function u satisfying both the conditions of being a local weak subsolution and a local weak supersolution to (2.2) is termed a local weak solution.

2.1.2 Function Spaces on a time-space area

In this subsection materials are arranged from [3, 8, 28]. We define several function spaces that operate in space-time domains. For $1 \le p, q \le \infty$, $L^q(t_1, t_2\,;\, L^p(\Omega))$ represents a collection of measurable real-valued functions defined on $\Omega \times (t_1, t_2)$, encompassing a finite-region in both space and time and characterized by a norm that may not be bounded:

$$\|v\|_{L^q(t_1,t_2\,;\,L^p(\Omega))} := \begin{cases} \left(\int_{t_1}^{t_2} \|v(t)\|^q_{L^p(\Omega)}\, dt\right)^{1/q} & \text{if } 1 \le q < \infty \\ \operatorname{ess\,sup}_{t_1 \le t \le t_2} \|v(t)\|_{L^p(\Omega)} & \text{if } q = \infty \end{cases}$$

where

$$\|v(t)\|_{L^p(\Omega)} := \begin{cases} \left(\int_\Omega |v(x,t)|^p\, dx\right)^{1/p} & \text{if } 1 \le p < \infty \\ \operatorname{ess\,sup}_{x \in \Omega} |v(x,t)| & \text{if } p = \infty \end{cases}$$

For simplicity, we use $L^p(\Omega \times (t_1, t_2)) = L^p(t_1, t_2\,;\, L^p(\Omega))$ when $p = q$. For $1 \le p < \infty$, the Sobolev Space $W^{1,p}(\Omega)$ consists of weakly differentiable measurable real-valued functions whose weak derivatives are p-th integrable on Ω, with the norm

$$\|w\|_{W^{1,p}(\Omega)} := \left(\int_\Omega |w|^p + |\nabla w|^p\, dx\right)^{1/p}$$

where $\nabla w = (w_{x_1}, \dots, w_{x_n})$ indicates, in a distribution sense, the gradient of w, and let $W_0^{1,p}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ with the norm $\|\cdot\|_{W^{1,p}}$. Additionally, we define $L^q(t_1, t_2\,;\, W_0^{1,p}(\Omega))$ as a function space of measurable real-valued functions on a space-time region with a bounded norm:

$$\|w\|_{L^q(t_1,t_2\,;\,W_0^{1,p}(\Omega))} := \left(\int_{t_1}^{t_2} \|w(t)\|^q_{W^{1,p}(\Omega)}\, dt\right)^{1/q}$$

Consider $\Omega \subset \mathbb{R}^n$ as a bounded domain. The truncation of a function v for a real number m can be expressed as

$$(v - m)_+ := \max\{(v - m), 0\}; \quad (v - m)_- := -\min\{(v - m), 0\}. \tag{2.3}$$

For a measurable function v in $L^1(\Omega)$ and real numbers $m < n$, we introduce the sets

$$\begin{cases} \Omega \cap \{v > n\} := \{x \in \Omega \,:\, v(x) > n\} \\ \Omega \cap \{v < m\} := \{x \in \Omega \,:\, v(x) < m\} \\ \Omega \cap \{m < v < n\} := \{x \in \Omega \,:\, m < v(x) < n\}. \end{cases}$$

2.2 Necessary tools

Let begin by recalling De Giorgi's inequality (refer to [4]).

Proposition 2.1 (Inequality of De Giorgi). *Consider $v \in W^{1,1}(B)$ and real numbers $k, m \in \mathbb{R}$ satisfying $k < m$. Then there exists a positive constant C dependent solely on p as well as n in a way that*

$$(k - m)\big|B \cap \{v > k\}\big| \le C \frac{\rho^{n+1}}{\big|B \cap \{v < m\}\big|} \int_{B \cap \{k < v < m\}} |\nabla v|\, dx. \tag{2.4}$$

Following the approach in [4], we introduce the auxiliary function

$$\begin{cases} A^+(k, u) := +(p-1) \displaystyle\int_k^u |s|^{p-2}(s - k)_+\, ds \\ A^-(k, u) := -\displaystyle\int_k^u |s|^{p-2}(s - k)_-\, ds \end{cases} \tag{2.5}$$

for $u, k \in \mathbb{R}$. In the special case of $k = 0$, we simplify as

$$A^+(u) = A^+(0, u) \quad \text{and} \quad A^-(u) = A^-(0, u).$$

It is evident that $A^\pm \geqslant 0$. We introduce bold notation $\mathbf{b}^\alpha$ to represent the signed α-exponent of b, as defined below

$$\mathbf{b}^\alpha = \begin{cases} |b|^{\alpha-1}\, b, & b \neq 0, \\ 0, & b = 0. \end{cases}$$

We present a known lemma; cf. ([1, 11, 25] for $\alpha > 1$. This lemma is utilized in the proof of the subsequent lemma:

Lemma 2.2. *For each positive value of α, there exists a specific constant β, denoted as $\beta(\alpha)$, for which the inequality below holds for any pair of real numbers a, b:*

$$\frac{1}{\beta}|\boldsymbol{b}^\alpha - \boldsymbol{a}^\alpha| \leqslant (|a| + |b|)^{\alpha-1}|b - a| \leqslant \beta|\boldsymbol{b}^\alpha - \boldsymbol{a}^\alpha|.$$

Building upon the aforementioned lemma, we establish the following result.

Lemma 2.3. *There exists a constant $\beta = \beta(p)$ such that the following inequality holds for all $w, k \in \mathbb{R}$ and $\alpha > 0$:*

$$\frac{1}{\beta}(|w| + |k|)^{p-2}(w - k)_\pm^2 \leqslant A^\pm(k, w) \leqslant \beta(|w| + |k|)^{p-2}(w - k)_\pm^2$$

We introduce a type of time mollification for the solution u to enhance its time regularity:

$$[u]_h(x,t) \overset{def}{=} \frac{1}{h}\int_0^t e^{\frac{s-t}{h}} u(x,s)\, ds \quad \text{for any} \quad u \in L^1(\Omega_T) \tag{2.6}$$

Lemma 2.4. *(Properties of mollification) [17]*

(i) *If $u \in L^p(\Omega_T)$, then $\|[u]_h(x,t)\|_{L^p(\Omega_T)} \leqslant \|u\|_{L^p(\Omega_T)}$ and*

$$\frac{\partial [u]_h}{\partial t} = \frac{u - [u]_h}{h} \in L^p(\Omega_T).$$

Moreover, $[u]_h \to u$ in $L^p(\Omega_T)$ as $h \to 0$.

(ii) *If, additionally, $\nabla([u]_h) = [\nabla u]_h$ componentwise,*

$$\|\nabla([u]_h)\|_{L^p(\Omega_T)} \leqslant \|\nabla u\|_{L^p(\Omega_T)}$$

and $\nabla[u]_h \to \nabla u$ in $L^p(\Omega_T)$ as $h \to 0$.

(iii) *Furthermore, if $u_k \to u$ in $L^p(\Omega_T)$, then*

$$[u_k]_h \to [u]_h \text{ and } \frac{\partial [u_k]_h}{\partial t} \to \frac{\partial [u]_h}{\partial t}$$

in $L^p(\Omega_T)$, and $\nabla[u]_h \to \nabla u$ in $L^p(\Omega_T)$ as $h \to 0$.

(iv) *If $\nabla u_k \to \nabla u$ in $L^p(\Omega_T)$, then also $\nabla[u_k]_h \to \nabla[u]_h$ in $L^p(\Omega_T)$.*

(v) *Similar results hold for weak convergence in $L^p(\Omega_T)$.*

(vi) *Lastly, if $\varphi \in C(\bar{\Omega}_T)$, then $[\varphi]_h(x,t) + e^{-\frac{t}{h}}\varphi(x,0) \to \varphi(x,t)$ uniformly in Ω_T as $h \to 0$.*

Moving forward, we will employ the following energy estimate (as found in [27]). We briefly outline the estimate before proceeding with the main proof.

Proposition 2.5. *Assume that u serves as a subsolution in a local sense for equation (1.1). In this context, there exists positive constant $\gamma(p)$ in a way that for any cylinders $Q_{R,S} = K_R(x_0) \times (t_0 - S, t_0) \Subset \Omega_T$, the subsequent inequality is satisfied for every non-negative piecewise smooth cutoff function ζ that vanishes along $\partial K(x_0) \times (t_0 - S, t_0)$, as well as for any $k \in \mathbb{R}$:*

$$\begin{aligned} \operatorname*{ess\,sup}_{t_0 - S < t < t_0} \int_{K_R(x_0)\times\{t\}} \zeta^p A^{\pm}(k,u)\,dx + \iint_{Q_{R,S}} \zeta^p |D(u-k)_{\pm}|^p\,dx\,dt \\ \leqslant \gamma \iint_{Q_{R,S}} [|D\zeta|^p (u-k)_{\pm} + A^{\pm}(k,u)|\partial_t \zeta^p|]\,dx\,dt \\ + \int_{K_R(x_0)\times\{t_0 - S\}} \zeta^p A^{\pm}(k,u)\,dx \end{aligned} \tag{2.7}$$

3 Positivity expansion

Consider $K \subset \mathbb{R}^n$ and a cylinder $\mathcal{Q} \overset{def}{=} K \times (t_1, t_2) \subset \Omega_T$. Throughout this section, we will utilize the following notations:

$$\mu^+ \geq \operatorname*{ess\,sup}_{\mathcal{Q}} u, \quad \mu^- \leq \operatorname*{ess\,inf}_{\mathcal{Q}} u, \quad \omega = \mu^+ - \mu^-.$$

We also assume that $(x_0, t_0) \in \mathcal{Q}$ for defining the forward cylinder

$$K_{8\varrho}(x_0) \times (t_0, t_0 + (8\varrho)^p) \subset \mathcal{Q}. \tag{3.1}$$

In this context, we present the proposition regarding the extension of positivity. The complete proof can be found in [25].

Proposition 3.1. *Given that u is locally limited and acts as a sub(super)solution on a local scale for equation (1.1) within the domain Ω_T, and for a specific point $(x_0, t_0) \in \Omega_T$, as well as for constants M, α, and ϱ, where $M > 0$, and α belongs to the interval $(0,1)$, while $\varrho > 0$ the ensuing conditions are met: (3.1) and*

$$|\{\pm(\mu^{\pm} - u(.,t_0)) \geq M\} \cap K_\varrho(x_0)| \geq \alpha |K_\varrho|.$$

Subsequently, constants ξ, δ, and η all falling within the range of $(0,1)$, can be identified based solely on the provided information and the value of α. This leads to either

$$|\mu^{\pm}| > \xi M$$

or

$$\pm(\mu^{\pm} - u) \geq \eta M \quad \textit{almost everywhere in} \quad K_{2\varrho}(x_0) \times (t_0 + \delta(\frac{1}{2}\varrho)^p, t_0 + \delta\varrho^p),$$

where

$$\xi = \begin{cases} 2\eta, & \textit{if } p > 2, \\ 8, & \textit{if } 1 < p \leq 2. \end{cases}$$

The proof of Proposition 3.1 follows directly from three lemmas presented in subsequent sections. Here, we provide the statements of these lemmas, which collectively form the foundation for proving the expansion of positivity. For detailed proofs, please refer to [27].

3.1 Extension of Positivity in Measure

Lemma 3.2. *Take any positive M and $\alpha \in (0,1)$ into account. Consequently, there are δ and ε within the range of $(0,1)$, and their values are exclusively determined by the provided information and the value of α. In cases where u functions as a locally restricted sub(super)-solution to equation (1.1) within Ω_T, adhering to the condition*

$$|\{\pm(\mu^{\pm} - u(\cdot, t_0)) \geq M\} \cap K_\varrho(x_0)| \geq \alpha |K_\varrho|,$$

we have either

$$|\mu^{\pm}| > 8M$$

or

$$|\{\pm(\mu^{\pm} - u(\cdot, t)) \geq \varepsilon M\} \cap K_\varrho(x_0)| \geq \frac{\alpha}{2}|K_\varrho| \quad \textit{for all } t \in (t_0, t_0 + \delta\varrho^p). \tag{3.2}$$

3.2 Lemma of shrinking

Lemma 3.3. *Given the assumptions in Lemma 3.2, the second option (3.2) is true. Let $Q = K_\varepsilon(x_0) \times (t_0, t_0 + \delta\varepsilon^p]$ denote the corresponding cylindrical domain, and let $\widetilde{Q} = K_{4\varepsilon}(x_0) \times (t_0, t_0 + \delta\varrho^p] \subset \Omega_T$. A positive constant γ, which is exclusively dependent on the given data and α, exists. This constant is such that for any positive integer j_*, when $1 < p < 2$, the following inequality is legitimate:*

$$\left| \left\{ \pm(\mu^{\pm} - u) \leq \frac{\varepsilon M}{2^{j_*}} \right\} \cap \widetilde{Q} \right| \leq \frac{\gamma}{j_*^{\frac{p-1}{p}}} |\widetilde{Q}|,$$

Similarly, if $p > 2$, the same result holds when $|\mu^{\pm}| < \varepsilon M 2^{-j_*}$.

3.3 Lemma of the DeGiorgi type

Within this section, we introduce a Lemma resembling DeGiorgi's lemma, but it pertains to cylinders in the format of $Q_\varrho(\theta)$. In the scope of its application, the value of the parameter θ will be a constant universally determined by the provided data. Remarkably, this constant θ remains unaffected by changes in the solution and remains consistent.

Lemma 3.4. *Examine a locally bounded function u, which serves as a local sub(super)-solution to equation (1.1) within Ω_T. Consider the set $(x_0, t_0) + Q_\varrho(\theta) = K_\varrho(x_0) \times (t_0 - \theta\varrho^p, t_0] \Subset \Omega_T$. A constant $\nu \in (0,1)$, relying solely on the given data and θ, is present. If the condition holds that*

$$\left|\left\{\pm(\mu^{\pm} - u) \leq M\right\} \cap (x_0, t_0) + Q_\varrho(\theta)\right| \leq \nu |Q_\varrho(\theta)|,$$

then either $|\mu^{\pm}| > 8M$, or

$$\pm(\mu^{\pm} - u) \geq \frac{1}{2}M \quad \textit{a.e. in} \quad (x_0, t_0) + Q_{\frac{1}{2}\varrho}(\theta).$$

4 Proof of the Theorem 1.1: Degenerate Case

4.1 Proof.

Here we fix $(x_0, t_0) \in \Omega_T$ as well as $A \geqslant 1$ to be determined later and $\varrho > 0$ be so small that

$$Q_0 \stackrel{def}{=} K_\varrho(x_0) \times (t_0 - A\varrho^p, t_0] \Subset \Omega_T.$$

Without sacrificing generality, we may presume that $(x_0, t_0) = (0, 0)$. We proceed to define the following symbols:

$$\mu^+ = \operatorname*{ess\,sup}_{Q_0} u, \;\; \mu^- = \operatorname*{ess\,inf}_{Q_0} u \quad\;\; \omega = \mu^+ - \mu^-.$$

Similar to the singular case, the proof of degenerate case unfolds along two main cases, such as

$$\begin{cases} \text{when } u \text{ is near zero: } \mu^- \leq \xi\omega \text{ , and } \mu^+ \geq -\xi\omega; \\ \text{when } u \text{ is away from zero: } \mu^- > \xi\omega \text{ or } \mu^+ < -\xi\omega. \end{cases} \tag{4.1}$$

4.2 Deduction of Oscillation Around Zero-1st Alternative

In this context, we assume that the first option described in 4.1 is active, and that u represents a super solution in close proximity to its infimum. Suppose that for some $\bar{t} \in (-(A-1)\varrho^p, 0]$,

$$|\{u \leqslant \mu^- + \frac{1}{4}\omega\} \cap (0, \bar{t}) + Q_\varrho| \leqslant \nu |Q_\varrho|, \tag{4.2}$$

where ν is the absolute constant as in Lemma 3.4. Choose $M = \frac{1}{4}\omega$, then applying Lemma 3.4, gives

$$u \geqslant \mu^- + \frac{1}{8}\omega \text{ a.e. in } (0, \bar{t}) + Q_{\frac{1}{2}\varrho},$$

Applying Proposition 3.1 with $2^p A$ instead of A gives $\xi = \xi(A, data), \eta = \eta(A, data) \in (0,1)$, such that either $|\mu^-| > \xi\omega$ or

$$u > \mu^- + \eta\omega \quad \text{a.e. in} \quad \tilde{Q_1} \overset{def}{=} K_{\frac{1}{2}\varrho} \times \left(-\left(\frac{1}{2}\varrho\right)^p, 0\right]. \tag{4.3}$$

Then we immediately get the inequality

$$\operatorname*{ess\,osc}_{\tilde{Q_1}} u \leqslant (1-\eta)\omega.$$

It remains to deal with the case $\mu^- < -\xi\omega$. Since $\mu^+ \geq -\xi\omega$, we also have $\mu^- > -2\omega$.Then we proceed further with the assumptions

$$\begin{cases} -2\omega < \mu^- < -\xi\omega. \\ u(\cdot, \bar{t} - (\frac{1}{2}\varrho)^p) \geqslant \mu^- + \frac{1}{8}\omega \quad \text{a.e. in} \quad K_{\frac{1}{2}\varrho}. \end{cases} \tag{4.4}$$

The next step is to establish pointwise propagation within the second alternative, extending it all the way up to the upper limit of the cylinder Q_0. This process is accomplished through the application of the following lemma.

Lemma 4.1. *If the hypotheses (4.4) holds true, then a constant $\eta_1 = \eta_1(\xi, A, \ data) \in (0,1)$ can be found in a way that*

$$u \geqslant \mu^- + \eta_1\omega \quad \text{a.e. in} \quad K_{\frac{1}{4}\varrho} \times \left(\bar{t} - \left(\frac{1}{2}\varrho\right)^p, 0\right]. \tag{4.5}$$

As a consequence, the reduction of oscillation is

$$\operatorname*{ess\,osc}_{\widehat{Q_1}} u \leqslant (1-\eta_1)\omega, \quad \text{where} \quad \widehat{Q_1} = K_{\frac{1}{4}\varrho} \times \left(-\left(\frac{1}{2}\varrho\right)^p, 0\right].$$

Proof. For our convenience we set $\bar{t} - \left(\frac{1}{2}\varrho\right)^p = 0$. Introduce the parameters

$k_n, \tilde{k}_n, \varrho_n, \tilde{\varrho}_n, K_n, \tilde{K_n}, , Q_n, \bar{Q}_n$

$$\begin{cases} k_n = \mu^- + \frac{2\eta_1\omega}{2} + \frac{2\eta_1\omega}{2^{n+1}}, & \tilde{k}_n = \frac{k_n + k_{n+1}}{2}, \\ \varrho_n = \frac{\frac{1}{2}\varrho}{2} + \frac{\frac{1}{2}\varrho}{2^{n+1}}, & \tilde{\varrho}_n = \frac{\frac{1}{2}\varrho_n + \frac{1}{2}\varrho_{n+1}}{2} \\ K_n = K_{\frac{1}{2}\varrho_n}, & \tilde{K}_n = K_{\frac{1}{2}\tilde{\varrho}_n}(\theta) \\ Q_n = Q_{\frac{1}{2}\varrho_n}(\theta) & \bar{Q}_n = Q_{\frac{1}{2}\tilde{\varrho}_n}(\theta) \end{cases} \tag{4.6}$$

where $0 < \eta_1 < \frac{1}{8}\xi$ and $\theta > 0$ to be determined later. Here the forward cylinders are of the type $Q_{\varrho_n}(\theta) = K_n \times (0, \theta\varrho_n^p]$ and $\tilde{Q}_n = \tilde{K}_n \times (0, \theta\tilde{\varrho}_n^p]$. Initiate a cutoff function ζ that satisfies the following conditions: It ranges between 0 and 1, is not dependent on time t, disappears at the boundary of K_n, and takes on a value of 1 within the region $\tilde{K}_n$. Additionally, this function should meet the requirement:

$$|D\zeta| \le \gamma \frac{2^n}{\varrho}.$$

The boundary component at the initial time $t = 0$ on the right side of the energy inequality disappears within the domain K_n, as

$$u(\cdot, 0) \geqslant \mu^- + \frac{1}{8}\omega \geqslant \mu^- + 2\eta_1\omega \geqslant \mu^- + \eta_1\omega + \eta_1\frac{\omega}{2^n} = k_n$$

a.e. on $K_{\frac{1}{2}\varrho}$ which requires $2\eta_1 < \frac{1}{8}$. In this context, we can express the energy estimates as follows:

$$\begin{aligned}
&\omega^{p-2} \operatorname*{ess\,sup}_{0<t<\theta\varrho^p} \int_{\tilde{K}_n} (u - \tilde{k}_n)_-^2 \, dx + \iint_{\tilde{Q}_n} |D(u - \tilde{k}_n)_-|^p \, dx\, dt \\
&\le \gamma \frac{2^{pn}}{\varrho^p} (\eta_1\omega)^p |A_n|,
\end{aligned}$$

where

$$A_n = \{u < k_n\} \cap Q_n.$$

Now treating ξ as a cutoff function that becomes zero along the parabolic boundary of $\tilde{Q}_n$ and equals one in Q_{n+1}, and then applying the Sobolev embedding as described in [4, Chapter I, Proposition 3.1] with $q = p\frac{N+2}{N}$ and $m = 2$ provides us

$$\frac{M^{p-2}}{2^{p(n+3)}} \operatorname*{ess\,sup}_{-\theta\tilde{\varrho}_n^p<t<0} \int_{\tilde{K}_n} (u - \tilde{k_n})_-^2 \, dx + \iint_{\bar{Q}_n} |D(u - \tilde{k_n})_-|^p \, dx\, dt \le \gamma \frac{2^{pn}}{\varrho^p} M^p |A_n|, \tag{4.7}$$

Now, defining ϕ as a cutoff function that ranges between 0 and 1, vanishing along the parabolic boundary of $\bar{Q}_n$ while equaling 1 within Q_{n+1}, we can apply the Hölder inequality and the Sobolev embedding [4, Chapter I, Proposition 3.1] to derive the following result:

$$\begin{aligned}
\left(\frac{\eta_1\omega}{2^{n+2}}\right)^p |A_{n+1}| &\le \iint_{\tilde{Q}_n} |(u - \tilde{k_n})_-\zeta|^p \, dx\, dt \\
&\le \left[\iint_{\tilde{Q}_n} [(u - \tilde{k}_n)_-\zeta]^{p\frac{N+2}{N}} \, dx\, dt\right]^{\frac{N}{N+2}} |A_n|^{\frac{2}{N+2}} \\
&\le \gamma \left[\iint_{\tilde{Q}_n} |D[(u - \tilde{k}_n)_-\zeta]|^p \, dx\, dt\right]^{\frac{N}{N+2}} \times \left[\operatorname*{ess\,sup}_{0<t<\theta\varrho^p} \int_{\tilde{K}_n} (u - \tilde{k}_n)_-^2 \, dx\right]^{\frac{p}{N+2}} |A_n|^{\frac{2}{N+2}} \\
&\le \gamma\omega^{\frac{p(2-p)}{N+2}} \left(\frac{2^{pn}(\eta_1\omega)^p}{\varrho^p}\right)^{\frac{N+p}{N+2}} |A_n|^{1+\frac{p}{(N+2)}}
\end{aligned}$$

Expressing the relationship using the variable $Y_n = \frac{|A_n|}{|Q_n|}$, we can rephrase this as:

$$Y_{n+1} \le \gamma b^n (\eta_1^{p-2}\theta)^{\frac{p}{N+2}} Y_n^{1+\frac{p}{N+2}},$$

the constant γ rely only on the data and with $b \equiv 2^{p(1+\frac{N+p}{N+2})}$. Therefore, as demonstrated in [4, Chapter I, Lemma 4.1], there exists a positive constant $\nu_0 = \nu_0(data) \in (0,1)$ such that the condition for Y_n to converge to 0 as n approaches infinity is:

$$Y_0 \leq \nu_0 \frac{\eta_1^{2-p}}{\theta}.$$

We fix $\theta = 2^p A$ and consider η_1 so small that

$$\nu_0 \frac{\eta_1^{2-p}}{\theta} \geq 1$$

which in turn implies that $\eta_1^{p-2} < \frac{\nu_0}{2^p A}$. Aggregating the bounds on η_1, we have to require that

$$\eta_1 < \min\left\{\frac{1}{16}, \frac{1}{8}\xi, \left(\frac{\nu_0}{2^p A}\right)^{\frac{1}{p-2}}\right\}$$

which established the lemma. □

4.3 Reduction of Oscillation Neighboring Zero-Continue

Here we consider u to function as a subsolution in the vicinity of its supreme by considering the first case of 4.1. We will assume that the opposite of (4.2) holds then

$$|\{u \leqslant \mu^- + \frac{1}{4}\omega\} \cap (0,\bar{t}) + Q_\varrho| > \nu|Q_\varrho|, \quad \text{for all} \quad \bar{t} \in (-(A-1)\varrho^p, 0]. \tag{4.8}$$

So, for these particular $\bar{t}$ values, there is a presence of some $s \in [\bar{t} - \varrho^p, \bar{t} - \frac{1}{2}\nu\varrho^p]$ with

$$|\{u(\cdot,s) \leqslant \mu^- + \frac{1}{4}\omega\} \cap K_\varrho| > \frac{1}{2}\nu|K_\varrho|$$

If it does not satisfied then for any s in the given interval, we have

$$|\{u \leqslant \mu^- + \frac{1}{4}\omega\} \cap (0,\bar{t}) + Q_\varrho| = \int_{\bar{t}-\varrho^p}^{\bar{t}-\frac{1}{2}\nu\varrho^p} |\{u(\cdot,s) \leqslant \mu^- + \frac{1}{4}\omega\} \cap K_\varrho| ds + \int_{\bar{t}-\frac{1}{2}\nu\varrho^p}^{\bar{t}} |\{u(\cdot,s) \leqslant \mu^- + \frac{1}{4}\omega\} \cap K_\varrho| ds < \frac{1}{2}\nu|K_\varrho|(\varrho^p - \frac{1}{2}\nu\varrho^p$$

which is a contradiction to (4.8). Since $\mu^+ - \frac{1}{4}\omega > \mu^- + \frac{1}{4}\omega$ holds always, then

$$|\{u(\cdot,s) \leqslant \mu^+ - \frac{1}{4}\omega\} \cap K_\varrho| > \frac{1}{2}\nu|K_\varrho|$$

By Proposition 3.1, there exists $\xi, \eta_2 \in (0,1)$, such that either $|\mu^+| > \xi\omega$ or

$$u \leqslant \mu^+ - \eta_2\omega \quad \text{a.e. in} \quad \tilde{Q}_1,$$

where $\tilde{Q}_1$ is introduced in (4.3). This implies nothing but the reduction of oscillation

$$essosc_{\tilde{Q}_1} u \leqslant (1-\eta_2)\omega.$$

As the case $\mu^+ < -\xi\omega$ does not hold due to the alternative $(4.1)_1$, we handle the case $\mu^+ > \xi\omega$. For the following three sub sections, our assumptions are

$$\xi\omega \leqslant \mu^+ \leqslant 2\omega, \tag{4.9}$$

and

$$\begin{cases} \text{for all} \quad \bar{t} \in (-(A-1)\varrho^p, 0] \quad \text{there exists} \quad s \in [\bar{t} - \varrho^p, \bar{t} - \frac{1}{2}\nu\varrho^p] \\ \quad \text{such that} \quad |\{u(\cdot,s) \leqslant \mu^+ - \frac{1}{4}\omega\} \cap K_\varrho \geqslant \frac{1}{2}\nu|K_\varrho|. \end{cases} \tag{4.10}$$

4.3.1 Dissemination of information through measure theory

Lemma 4.2. *Assume (4.9) as well as (4.10) are satisfied. There is a specific value for ε within the range* $(0, 1)$ *, which relies solely on ν, ξ, and the given data. This value is such that*

$$|\left\{u(\cdot,t) \le \mu^+ - \varepsilon\omega\right\} \cap K_\varrho| \ge \frac{1}{4}\nu|K_\varrho| \text{ for all } t \in (s, \bar{t}].$$

Proof. For convenient computational purpose assume $s = 0$. Utilizing the energy estimate from Proposition 2.5 within the cylinder denoted as $Q := K_\varrho \times (0, \delta\varepsilon^{2-p}\varrho^p]$, where $k = \mu^+ - \varepsilon\omega$, with $\delta > 0$ and $0 < \varepsilon \le \frac{1}{2}\xi$, which will be determined later. It is worth noting that in this context, we must emphasize that $k \ge \frac{1}{2}\xi\omega$. We will select a standard non-negative, time-independent cutoff function $\zeta(x, t) \equiv \zeta(x)$, which equals 1 within $K_{(1-\sigma)\varrho}$, where $\sigma(0, 1)$, and becomes zero on ∂K_ϱ, while also satisfying the condition $|D\zeta| \le (\sigma\varrho)^{-1}$. Under these conditions, for all $0 < t < \delta\varepsilon^{2-p}\varrho^p$, we can deduce that:

$$\int_{K_\varrho \times \{t\}} \int_k^u s^{p-2}(s-k)_+ \, ds \, \zeta^p \, dx \le \int_{K_\varrho \times \{0\}} \int_k^u s^{p-2}(s-k)_+ \, ds \, \zeta^p \, dx + \gamma \iint_Q (u-k)_+^p |D\zeta|^p \, dx \, dt.$$

Further estimation on the 1st term on the right can be made by using (4.10) yields

$$\int_{K_\varrho \times 0} \int_k^u s^{p-2}(s-k)_+ \, ds\zeta^p \, dx \le (1 - \frac{1}{2}\nu)|K_\varrho| \int_k^{\mu^+} s^{p-2}(s-k)_+ \, ds$$

The right-hand side's second term is limited by the following upper bound:

$$\iint_Q (u-k)_+^p |D\zeta|^p \, dx \, dt \le \frac{\gamma\delta}{\sigma^p}\varepsilon^{2-p}(\varepsilon\omega)^p|K_\varrho| \le \frac{\gamma\delta}{\sigma^p}\varepsilon^2\omega^p|K_\varrho|.$$

Regarding the left-hand side, we approximate it with a lower bound of

$$\int_{K_\varrho \times \{t\}} \int_k^u s^{p-2}(s-k)_+ \, ds\zeta^p \, dx \ge |\{u(\cdot,t) > k_{\tilde{\varepsilon}}\} \cap K_{(1-\sigma)\varrho}| \int_k^{k_{\tilde{\varepsilon}}} s^{p-2}(s-k)_+ \, ds,$$

where $k_{\tilde{\varepsilon}} = \mu^+ - \tilde{\varepsilon}\varepsilon\omega$ for some $\tilde{\varepsilon} \in (0, 1)$. Recalling $\xi\omega \le \mu^+ \le 2\omega$, we may get

$$\int_k^{k_{\tilde{\varepsilon}}} s^{p-2}(s-k)_+ \, ds \ge \gamma\omega^{p-2}(\varepsilon\omega)^2 = \gamma\varepsilon^2\omega^p.$$

A further and similar application of Lemma 3.1 then gives

$$|\{u(\cdot,t) > k_{\tilde{\varepsilon}}\} \cap K_{(1-\sigma)\varrho}| \le \frac{\int_k^{\mu^+} |s|^{p-2}(s-k)_+ \, ds}{\int_k^{k_{\tilde{\varepsilon}}} |s|^{p-2}(s-k)_+ \, ds}(1 - \frac{1}{2}\nu)|K_\varrho| + \frac{\gamma\delta}{\sigma^p}|K_\varrho|$$

The fractional integral term on the right hand can be revised as

$$1 + I_\varepsilon \text{ where } I_\varepsilon = \frac{\int_k^{\mu^+} |s|^{p-2}(s-k)_+ \, ds}{\int_k^{k_{\tilde{\varepsilon}}} |s|^{p-2}(s-k)_+ \, ds}.$$

We can further approximate the integral $I_\varepsilon \le \gamma\tilde{\varepsilon}$, with the aid of $\xi\omega \le \mu^+ \le 2\omega$ and $k \ge \frac{1}{2}\xi\omega$. Using the above estimate provide a route to

$$|\{u(\cdot,t) > k_{\tilde{\varepsilon}}\} \cap K_\varrho| \le (1 - \frac{1}{2}\nu)(1 + \gamma\tilde{\varepsilon})|K_\varrho| + \frac{\gamma\delta}{\sigma^p}|K_\varrho| + N\sigma|K_\varrho|.$$

Now we select $\tilde{\varepsilon}$ so that

$$(1-\frac{1}{2}\nu)(1+\gamma\tilde{\varepsilon}) \le 1-\frac{3}{8}\nu.$$

This determines $\tilde{\varepsilon}$ in relation to both p and ν. Next, we set σ to be $\frac{\nu}{16N}$ and pick a sufficiently small δ to ensure that $\frac{\gamma\delta}{\sigma^p} \le \frac{1}{16}\nu$. Lastly, we select the parameter ε such that $\delta\varepsilon^{2-p} \ge 1$. We reach the completion of the proof by replacing $\tilde{\varepsilon}\varepsilon$ as ε. □

As $\bar{t}$ is despotic in nature, we therefore acquire the measure theoretical information

$$|\{u(\cdot,t) \le \mu^+ - \varepsilon\omega\} \cap K_\varrho| \ge \frac{1}{4}\nu|K_\varrho| \text{ for all } t \in (-(A-1)\varrho^p, 0]. \tag{4.11}$$

4.3.2 Reducing the measure in the vicinity of the supremum

By $\varepsilon = \varepsilon(data) \in (0,1)$ in Lemma 4.2. We choose A in the form $A = 2^{j_*(p-2)} + 1$ with some j_* and define $Q_\varrho(\theta) = K_\varrho \times (-\theta\varrho^p, 0]$ with $\theta = 2^{j_*(p-2)}$.

Lemma 4.3. *For any positive integer j_*, there exists a positive constant $\gamma = \gamma(data)$ in such a way that the following holds:*

$$\left|\left\{u \ge \mu^+ - \frac{\varepsilon\omega}{2^{j_*}}\right\} \cap Q_\varrho(\theta)\right| \le \frac{\gamma}{j_*^{\frac{p-1}{p}}}|Q_\varrho(\theta)|$$

provided that (4.9) and (4.11) hold.

Proof. We utilize the energy estimate from Proposition 2.5 within the cylinder $K_{2\varrho} \times (-\theta\varrho^p, 0]$ using levels $k_j = \mu^+ - 2^{-j}\varepsilon\omega$ for $j = 0, 1, 2, \cdots, j_* - 1$. Additionally, we employ a time free cutoff function $\zeta(x,t) \equiv \zeta(x)$ with specific properties: it takes the value 1 in K_ϱ, becomes zero on the boundary of $K_{2\varrho}$, and satisfies $|D\zeta| \le 2\varrho^{-1}$. Consequently, we obtain:

$$\begin{aligned}&\iint_{Q_\varrho(\theta)} |D(u-k_j)_+|^p \, dx\, dt \\ &\le \int_{K_{2\varrho}\times\{-\theta\varrho^p\}} \zeta^p A^+(k_j,u)\, dx + \gamma \iint_{K_{2\varrho}\times(-\theta\varrho^p,0]} (u-k_j)_+^p |D\zeta|^p \, dx\, dt.\end{aligned}$$

Let's now analyze the components on the right-hand side individually, starting with the first one. Leveraging Lemma 2.3 and considering the inequality $\xi\omega \le \mu^+ \le 2\omega$, we can establish that:

$$\int_{K_{2\varrho}\times\{-\theta\varrho^p\}} \zeta^p A^+(k_j,u)\, dx \le \gamma\omega^{p-2}(\frac{\varepsilon\omega}{2^j})^2|K_{2\varrho}| \le \frac{\gamma}{\varrho^p\varepsilon^{p-2}}(\frac{\varepsilon\omega}{2^j})^p|Q_\varrho(\theta)| \le \frac{\gamma}{\varrho^p}(\frac{\varepsilon\omega}{2^j})^p|Q_\varrho(\theta)|.$$

The parameter $\varepsilon = \varepsilon(data)$ is already fixed earlier. Therefore, we can summarize the aforementioned manipulation as follows:

$$\iint_{Q_\varrho\theta} |D(u-k_j)_+|^p \, dx\, dt \le \frac{\gamma}{\varrho^p}(\frac{\varepsilon\omega}{2^j})^p|Q_\varrho(\theta)|$$

For each t within the interval $(-\theta\varrho^p, 0]$, we employ [4, Chapter I, Lemma 2.2] to analyze $u(\cdot,t)$ separately within the cube K_ϱ, considering ascending levels $k_{j+1} > k_j$. This allows us to explore the following details:

$$\left|\{u(\cdot,t) \le \mu^+ - \varepsilon\omega\} \cap K_\varrho\right| \ge \frac{1}{4}\nu|K_\varrho| \text{ for all } t \in (-\theta\varrho^p, 0].$$

This take us to

$$(k_{j+1}-k_j)|\{u(\cdot,t) < k_{j+1}\} \cap K_\varrho| \le \frac{\gamma\varrho^{N+1}}{|\{u(\cdot,t) < k_j\} \cap K_\varrho|} \int_{\{k_j<u(\cdot,t)<k_{j+1}\}\cap K_\varrho} |Du(\cdot,t)|\, dx$$

$$\leq \frac{\gamma\varrho}{\nu}\left[\int_{\{k_j<u(\cdot,t)<k_{j+1}\}\cap K_\varrho}|Du(\cdot,t)|^p\,dx\right]^{\frac{1}{p}}|\{k_j<u(\cdot,t)<k_{j+1}\}\cap K_\varrho|^{1-\frac{1}{p}}$$

$$=\frac{\gamma\varrho}{\nu}\left[\int_{\{k_j<u(\cdot,t)<k_{j+1}\}\cap K_\varrho}|D(\cdot,t)|^p\,dx\right]^{\frac{1}{p}}[|A_j(t)|-|A_{j+1}(t)|]^{1-\frac{1}{p}}.$$

In the final line, we employed the shorthand notation $A_j(t):=\{u(\cdot,t)<k_j\}\cap K_\varrho$. Moving forward, we integrate the last inequality with respect to t over the interval $(-\delta\varrho^p,0]$ and utilize Hölder's inequality in the time domain. Using the abbreviation $A_j=\{u<k_j\}\cap Q_\varrho(\theta)$, this process leads us to:

$$\frac{\varepsilon\omega}{2^{j+1}}|A_{j+1}|\leq\frac{\gamma\varrho}{\nu}\left[\int_{Q_\varrho(\theta)}|D(u-k_j)_-|^p\,dx\,dt\right]^{\frac{1}{p}}[|A_j|-|A_{j+1}|]^{1-\frac{1}{p}}$$

$$\leq\gamma\left(\frac{\varepsilon\omega}{2^j}\right)|Q_\varrho(\theta)|^{\frac{1}{p}}[|A_j|-|A_{j+1}|]^{1-\frac{1}{p}}.$$

Taking $\frac{p}{p-1}$th power on both sides implies

$$|A_{j+1}|^{\frac{p}{p-1}}\leq\gamma|Q_\varrho(\theta)|^{\frac{1}{p-1}}[|A_j|-|A_{j+1}|].$$

To wrap up the proof, we follow the same steps as we did in the proof of Lemma 3.3. Aggregating above estimates to j from 0 to j_*-1 and obtain

$$j_*|A_{j_*}|^{\frac{p}{p-1}}\leq\gamma|Q_\varrho(\theta)|^{\frac{p}{p-1}}.$$

This leads to the conclusion of the assertion, i.e.

$$|A_{j_*}|\leq\frac{\gamma}{j_*^{\frac{p-1}{p}}}|Q_\varrho(\theta)|.$$

This concludes the demonstration. □

4.3.3 A lemma similar to DiGiorgi

The symbol $\varepsilon\in(0,1)$ as earlier in Lemma 4.2.

Lemma 4.4. *Given that (4.9) and (4.10) are satisfied, let there be a constant $\nu_1=\nu_1(data)$ lies between 0 and 1 such that if, for a certain $j_*>1$, the measure condition*

$$|\{\mu^+-u\leq\frac{\varepsilon}{2^{j_*}}\}\cap Q_\varrho(\theta)|\leq\nu_1|Q_\varrho(\theta)|,$$

is met, where $\theta=2^{j_*(p-2)}$, then it follows that

$$\mu^+-u\geqslant\frac{\varepsilon\omega}{2^{j_*+1}}\ \textit{almost everywhere in}\ Q_{\frac{1}{2}\varrho}(\theta).$$

Proof. Assume $M:=2^{-j_*}\varrho\omega$ as well as

$$k_n:=\mu^+-\frac{M}{2}-\frac{M}{2^{n+1}}.$$

In a manner analogous to the proof of Lemma 3.4, we establish the definitions of $k_n,\tilde{k_n},\varrho_n,\tilde{\varrho_n},K_n,\tilde{K_n},Q_n$, and $\bar{Q}_n$. Define a cutoff function ζ that becomes zero at the boundary of Q_n and equals 1 within $\bar{Q}_n$, satisfying the condition:

$$|D\zeta|\leq\gamma\frac{2^n}{\varrho}\ \text{and}\ |\zeta_t|\leq\gamma\frac{2^{pn}}{\theta\varrho^p}.$$

The energy inequality can be computed using $\xi\omega \le \mu^+ \le 2\omega$ and similar to the way of proof to Lemma 3.4 and obtain

$$\omega^{p-2} \operatorname*{ess\,sup}_{-\theta\tilde{\varrho}^p<t<0} \int_{\tilde{K}_n} (u-\tilde{k}_n)_+^2 \, dx + \iint_{\bar{Q}_n} |D(u-\tilde{k}_n)_+|^p \, dx\, dt$$
$$\le \gamma \frac{2^{pn}}{\varrho^p} M^p \left(1+\frac{\omega^{p-2}}{\theta M^{p-2}}\right)|A_n| = \gamma \frac{2^{pn}}{\varrho^p} M^p (1+\varepsilon^{2-p})|A_n|,$$

where the following is used

$$A_n = \{u > k_n\} \cap Q_n.$$

Consider a cutoff function ζ that becomes zero on the parabolic boundary of $\bar{Q}_n$ and equals 1 in Q_{n+1}. Utilizing the Sobolev embedding from [4, Chapter I, Proposition 3.1] and the previous estimate, we can infer that:

$$\left(\frac{M}{2^{n+2}}\right)^p |A_{n+1}| \le \iint_{\bar{Q}_n} |(u-\tilde{k}_n)_+^p \zeta^p \, dx\, dt$$
$$\le \left[\iint_{\bar{Q}_n} [(u-\tilde{k}_n)_+\zeta]^{p\frac{N+2}{N}} \, dx\, dt\right]^{\frac{N}{N+2}} |A_n|^{\frac{2}{N+2}}$$
$$\le \gamma \left[\iint_{\bar{Q}_n} |D[(u-\tilde{k}_n)_+\zeta]|^p \, dx\, dt\right]^{\frac{N}{N+2}} \times \left[\operatorname*{ess\,sup}_{-\theta\tilde{\varrho}_n^p<t<0} \int_{\tilde{K}_n} (u-\tilde{k}_n)_-^2 \, dx\right]^{\frac{p}{N+2}} |A_n|^{\frac{2}{N+2}}$$
$$\le \gamma \omega^{\frac{p(2-p)}{N+2}} \left(\frac{2^{pn}M^p}{\varrho^p}\right)^{\frac{N+p}{N+2}} \left(1+\varepsilon^{2-p}\right)\right)^{\frac{N+p}{N+2}} |A_n|^{1+\frac{p}{N+2}}.$$

Setting $Y_n = \frac{|An|}{|Q_n|}$, we arrive at

$$Y_{n+1} \le \gamma b^n \left(\frac{\theta M^{p-2}}{\omega^{p-2}}\right)^{\frac{p}{N+2}} \left(1+\varepsilon^{2-p}\right)\right)^{\frac{N+p}{N+2}} |Y_n|^{1+\frac{p}{N+2}}$$
$$\le \gamma b^n \varepsilon^{\frac{p(p-2)}{N+2}} \left(1+\varepsilon^{2-p}\right)\right)^{\frac{N+p}{N+2}} |Y_n|^{1+\frac{p}{N+2}}$$

where $b = 4^p$ and $\gamma = \gamma(data)$. According to [4, Chapter I, Lemma 4.1], we can identify a constant $\nu_1 = \nu_1(data) \in (0,1)$, ensuring that Y_n converges to 0 when we impose the condition that $Y_0 \le \nu_1$. Now, we can proceed to conclude the reduction of oscillation near the supremum in the remaining scenario where (4.9) and (4.10) are met. We use the constants $\varepsilon \in (0,1)$, $\gamma > 0$, and $\nu_1 \in (0,1)$ as defined in Lemmas 4.2, 4.3, and 4.4. Subsequently, we select a positive integer j_* in the following manner:

$$\frac{\gamma}{j_*^{\frac{p-1}{p}}} \le \nu_1.$$

Successively applying Lemmas 4.2, 4.3, and 4.4 results in the following:

$$\mu^+ - u \ge \frac{\varepsilon\omega}{2^{j_*+1}} \text{ a.e. in } \bar{Q}_1,$$

where $\bar{Q}_1$ is defined in (4.3). Here we used the fact $\bar{Q}_1 \subset Q_{\frac{1}{2}\varrho}(\theta)$, since $\theta > 1$. This signifies a decrease in oscillation, That is to say, we possess

$$\operatorname*{ess\,osc}_{\bar{Q}_1} u \le \left(1-\frac{\varepsilon}{2^{j_*+1}}\right)\omega.$$

□

4.3.4 Ending of the reduction of oscillation around zero

Firstly, define the following quantities

$$\lambda = \min\left\{\frac{1}{4}, \frac{1}{2A^{\frac{1}{p}}}\right\}, \quad \bar{\eta} = \min\left\{\eta, \eta_1, \eta_2, \frac{\varepsilon}{2^{j_*+1}}\right\}.$$

We proceed by induction. For this, assume up to $i = 1, 2, \cdots, j-1$, we have constructed

$$\begin{cases} \varrho_i = \lambda \varrho_{i-1}, \ \omega_i = (1-\bar{\eta})\omega_{i-1}, \ Q_i = K_{\varrho_i} \times (-\varrho_i^p, 0], \\ \mu_i^+ = \operatorname{ess\,sup}_{Q_i} u, \ \mu_i^- = \operatorname{ess\,inf}_{Q_i} u, \ \operatorname{ess\,osc}_{Q_i} u = \leq \omega_i. \end{cases}$$

For $i = 1, 2, \cdots, j-1$, we considered the initial scenario in (4.1), i.e.

$$\mu_i^+ \geq \xi\omega_i \qquad \text{and } \mu_i^- \leq \xi\omega_i$$

where ξ is established in Proposition 3.1. By applying the reasoning presented in the preceding sections repeatedly, we obtain the following for every $i = 1, 2, \cdots, j$

$$\operatorname*{ess\,osc}_{Q_i} u \leq (1-\bar{\eta})\,\omega_{i-1} = \omega_i.$$

Hence, by repeating this recursive inequality, we derive the following for each $i = 1, 2, \cdots, j$,

$$\operatorname*{ess\,osc}_{Q_i} u \leq (1-\bar{\eta})^i\,\omega = \omega\left(\frac{\varrho_i}{\varrho}\right)^{\beta_0} \quad \text{where } \beta_0 = \frac{\ln(1-\bar{\eta})}{\ln\lambda}. \tag{4.12}$$

4.3.5 Reduction of oscillation off from zero

At first, suppose that j is the earliest index for which the second scenario in (4.1) is met, i.e.,

$$\text{either } \mu_j^- > \xi\omega_j \text{ or } \mu_j^+ < -\xi\omega_j.$$

We treat the either case, for instance, $\mu_j^- > \xi\omega_j$, The other scenario is similar or equivalent in nature. We notice that because j is the initial index at which this occurs, it implies that $\mu_{j-1} < \xi\omega_{j-1}$. Furthermore,

$$\mu_j^- \leq \mu_j^- + \omega_{j-1} - \omega_j \leq (1+\xi)\omega_{j-1} - \omega_j = \frac{\xi+\bar{\eta}}{1-\bar{\eta}}\omega_j.$$

Therefore, we obtain,

$$\xi\omega_j \leq \mu_j^- \leq \frac{\xi+\bar{\eta}}{1-\bar{\eta}}\omega_j. \tag{4.13}$$

The inclusion of condition (4.13) indicates that, starting from point j, equation (1.1) adopts the characteristics of a parabolic equation with a p-Laplacian type behavior in the region Q_j. Similar to the situation in the singular case when $1 < p < 2$, the subsequent effort to minimize oscillations closely follows the strategy detailed in [29, section 5.2] where we have used intrinsic scaling explained in [32].

5 Existence of infinite extinction

Proof of Theorem 1.2: In this section we will assume that the solution to (1.1) is non-negative. Assuming this we will prove that there are infinite time extinction to the equation (1.1). Now its turn to prove our second Theorem 1.2.

Proof. Multiplying (1.1) by u we have

$$(p-1)\int_\Omega u^{p-1}\partial_t u\,dx + \int_\Omega |\nabla u|^p\,dx = 0$$

Using Sobolev-Poincare inequality, we have

$$\frac{p-1}{p}\frac{d}{dt}\int_\Omega u^p\,dx + C_p^{-p}\int_\Omega u^p\,dx \leqslant 0$$

Assuming $\Psi(t) = \int_\Omega u^p\,dx$ we have,

$$\frac{\Psi'}{\Psi} \leqslant -\frac{p}{p-1}C_p^{-p}$$

Integrating the above differential inequality on $0 \leqslant s \leqslant t$

$$\int_0^t \frac{\Psi'}{\Psi}\,dt \leqslant -\int_0^t \frac{p}{p-1}C_p^{-p}\,dt$$

Then

$$\ln\Psi(s)\Big|_0^t \leqslant \frac{p}{p-1}C_p^{-p}t$$

Consequently, this suggests that

$$\Psi(t) \leqslant \Psi(0)e^{\frac{p}{p-1}C_p^{-p}t}$$

It is evident that

$$0 \leqslant \int_\Omega u^p(t)\,dx \leqslant \left(\int_\Omega u_0^p\,dx\right)e^{\frac{p}{p-1}C_p^{-p}t}$$

This indicates that extinction of u will be at infinity. □

6 Conclusion

This article provides an overview indicating that the Hölder continuity of a weak solution to Trudinger's equation has been confirmed within a confined region in a Sobolev space. Additionally, it is noted that the nonnegative weak solution to the mentioned equation experiences infinite-time extinction.

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